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PAIR CORRELATION OF LOW-LYING ZEROS OF QUADRATIC L-FUNCTIONS

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Abstract. In this paper, we investigate the nontrivial zeros of quadratic L-functions near the real axis. Assuming the generalized Riemann hypothesis, we give an asymptotic formula for the weighted pair correlation function of quadratic L-functions associated to the Kronecker symbols. From this formula, we obtain several results on the rate of simple zeros of quadratic L-functions and on the average distance of such nontrivial zeros.

§1. Introduction

In the early 1970s, Montgomery [11] published his famous paper titled "The pair correlation of zeros of the zeta function". In this paper, assuming the Riemann hypothesis (RH), he investigated the function

$$F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma\leqslant T \\ 0<\gamma'\leqslant T}} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where $w(u) = 4/(4+u^2)$ and γ, γ' run over the set of the imaginary parts of the nontrivial zeros of the Riemann zeta-function $\zeta(s)$ in $0 < \mathrm{Im}(s) \leqslant T$. He obtained an asymptotic formula for $F(\alpha, T)$ $(0 < \alpha \leqslant 1-\epsilon)$, and using this formula, he obtained several results on the distances of the nontrivial zeros. For example, under the assumption of the RH, he proved that at least $2/3 - \epsilon$ of the nontrivial zeros are simple, and that

$$\liminf_{n\to\infty} \frac{(\gamma_{n+1}-\gamma_n)\log\gamma_n}{2\pi} \leqslant \lambda < 1$$

holds for specific λ, where γ_n denotes the imaginary part of the nth nontrivial zero of $\zeta(s)$ in the upper half-plane.

Later, Montgomery's idea, combined with new conceptions or improvements, was extended to many types of L-functions or other situations.

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For example, Özlük [12] investigated the nontrivial zeros of the Dirichlet L-functions near the real axis. Assuming the generalized Riemann hypothesis (GRH), he proved that at least $11/12$ of such zeros are simple in some sense. One of the other interesting generalizations is the work of Hejhal [7]. From the explicit formula of the Riemann zeta-function introduced in another of his papers [6], he constructed a certain asymptotic formula for the function involving the pairs of three distinct zeros of $\zeta(s)$. Further, the result of Hejhal was generalized by Rudnick and Sarnak [16], and the n-level correlation of the zeros of principal L-functions was obtained. In particular, their results agree with the prediction for the Gaussian unitary ensemble of random matrix theory.

Our aim in this paper is to investigate the pair correlation of the zeros of the quadratic L-functions near the real axis. As prior research, Özlük and Snyder [13] investigated such zeros. Under the assumption of the GRH, they studied the asymptotic behavior of the function

$$G_K(\alpha, D) = \left(\frac{1}{2}K\left(\frac{1}{2}\right)D\right)^{-1}\sum_{d\neq 0} e^{-\pi d^2/D^2}\sum_{\rho\in Z_d} K(\rho)D^{i\alpha\gamma}$$

as $D\to\infty$ for $|\alpha|<2$, where $\rho = 1/2 + i\gamma$ runs over the set of all nontrivial zeros of $L(s,\chi_d)$, the quadratic L-function associated to the Kronecker symbol $\chi_d = (d/\cdot)$, and $K(s)$ is some weight function. From their asymptotic formula, they proved that assuming the GRH, not more than 6.25% of all integers d have the property that $s = 1/2$ is a zero of $L(s,\chi_d)$. Slightly later, by a completely different method, Soundararajan [18] unconditionally proved that $L(1/2,\chi_d)\neq 0$ for at least 87.5% of all fundamental discriminants d. On the other hand, there are several researches on the 'n-level density' of the low-lying zeros of quadratic L-functions. For an odd, square-free integer $d>0$, $\chi_{8d} = \left(\frac{8d}{\cdot}\right)$ becomes a primitive character. Assuming the GRH, we denote the nontrivial zeros of $L(s,\chi_{8d})$ by

$$\tfrac{1}{2} + i\gamma_{8d,j}\quad (j=\pm 1,\pm 2,\ldots),$$

where $0\leqslant\gamma_{8d,1}\leqslant\gamma_{8d,2}\leqslant\cdots$ and $\gamma_{8d,-j} = -\gamma_{8d,j}$. For $X>0$, we put

$$\mathcal{D}(X) := \{X\leqslant d\leqslant 2X \mid d: \text{odd},\ \text{square free}\},$$

and for a Schwartz function $f\in\mathcal{S}(\mathbf{R}^n)$, we put

$$W_f^{(n)}(d) := \sum_{\substack{j_1,\ldots,j_n=\pm1,\pm2,\ldots\\ |j_k|:\text{distinct}}} f\left(\frac{\gamma_{8d,j_1}\log X}{2\pi},\ldots,\frac{\gamma_{8d,j_n}\log X}{2\pi}\right).$$

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Then, the Katz–Sarnak density conjecture [9] asserts that

$$\langle W_f^{(n)}\rangle_{\mathcal{D}(X)} := \frac{1}{\sharp\mathcal{D}(X)} \sum_{d\in\mathcal{D}(X)} W_f^{(n)}(d)\Phi\left(\frac{d}{X}\right) \to \int_{\mathbf{R}^n} f(x)W_{\mathrm{USp}}^{(n)}(x)\,dx \tag{1.1}$$

as $X\to\infty$, where Φ is a nonnegative smooth function supported in $(1,2)$ satisfying $\int \Phi(x)\,dx = 1$, and

$$W_{\mathrm{USp}}^{(n)}(x) = \det(K(x_i,x_j))_{i,j=1,\ldots,n},$$

with

$$K(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)} - \frac{\sin\pi(x+y)}{\pi(x+y)}.$$

Katz and Sarnak [9], assuming the GRH, proved that (1.1) holds if $n = 1$ and $\hat{f}(u) = \int_{\mathbf{R}} f(x)e^{-2\pi iux}\,dx$ has a support in $|u| < 2$. Rubinstein [14], [15], assuming the GRH, established (1.1) under the condition that $\hat{f}(u) = \int_{\mathbf{R}^n} f(u)e^{-2\pi iu\cdot x}\,dx$ has a support in $\sum_{j=1}^n |u_j| < 1$, and later Gao [3], [4], under the GRH, proved that if f is of the form $f(x_1,\ldots,x_n) = \prod_{j=1}^n f_j(x_j)$ and each $\hat{f}_j$ is supported in $|u_j| < s_j$ with $\sum s_j < 2$, then

$$\lim_{X\to\infty}\langle W_f^{(n)}\rangle_{\mathcal{D}(X)} = A(f),$$

where $A(f)$ is some complicated expression involving $f_1,\ldots,f_n$. Moreover, he confirmed that $A(f)$ is equal to the right-hand side of (1.1) if $n = 2, 3$. More recently, Levinson and Miller [10] proved that this fact is also valid for $4 \leqslant n \leqslant 7$. Finally, Entin, Roditty-Gershon and Rudnick [2] proved that assuming the GRH, (1.1) is valid for all n if $\hat{f}$ is supported in $\sum_{j=1}^n |u_j| < 2$.

If anything, our approach to investigate the pair correlation of low-lying zeros is similar to that of Özlük and Snyder [13], in which they investigated the 1-level density of these zeros. In this paper, assuming the GRH (including the RH), we investigate the function $F_K(\alpha, D)$, defined as follows. Let $K(s)$ be analytic in $-1 < \mathrm{Re}(s) < 2$ and satisfy $K(1/2 - it) = K(1/2 + it)$ for any $t \in \mathbf{R}$. Moreover, we assume that its Mellin inverse transform

$$a(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}\,ds \tag{1.2}$$

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converges absolutely for any $-1<c<2$, $x>0$, and that $a(x)$ is real, nonnegative, belongs to the C^1-class, and has a support in $[A, B]$ for some $0<A<B<\infty$. Then, $K(s)$ is given by

$$K(s)=\int_0^\infty a(t)t^s\,\frac{dt}{t}. \tag{1.3}$$

For $d\in\mathbf{Z}$, let $\chi_d=(d/\cdot)$ be the Kronecker symbol, and let $L(s, \chi_d)$ be the L-function associated to χ_d. We denote the set of nontrivial zeros of $L(s, \chi_d)$ by Z_d. For $x>0$, $D>0$, we put

$$f_K(x, D)=\sum_d e^{-\pi d^2/D^2}\sum_{\rho_1,\rho_2\in Z_d}K(\rho_1)\overline{K(\rho_2)}x^{\rho_1+\overline{\rho_2}},$$

and for $\alpha\in\mathbf{R}$, we put

$$\begin{aligned}F_K(\alpha, D)&=\left[\frac{1}{xD\log D}f_K(x, D)\right]_{x=D^\alpha}\\ &=\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\rho_1,\rho_2\in Z_d}K(\rho_1)\overline{K(\rho_2)}D^{i\alpha(\gamma_1-\gamma_2)},\end{aligned} \tag{1.4}$$

where $\rho_j=1/2+i\gamma_j$ for $j=1, 2$. Then, the main theorem is stated as follows.

THEOREM 1.1. *Assuming the GRH, for any small $\delta>0$, we have*

$$\begin{aligned}F_K(\alpha, D)&=L(1)\alpha+a(D^{-\alpha})^2D^{-\alpha}\log D+a(D^{-\alpha})\cdot O(\alpha D^{-\alpha/2}\log D)\\ &\quad+a(D^{-\alpha})^2\cdot O(D^{-\alpha})+O(\min\{1, \alpha D^{-\alpha}\log^2 D\})\\ &\quad+O(\min\{\alpha\log D, \alpha D^{-1/2}\log D+\alpha(1+\alpha^2)D^{-\alpha/2}\log^3 D\})\\ &\quad+O(\min\{D^\alpha(\log D)^{-1}, \alpha^2D^{-\alpha}\log^3 D\})+o(1)\end{aligned} \tag{1.5}$$

uniformly for $0<\alpha<1-\delta$ as $D\to\infty$, where

$$L(1)=\int_0^\infty a(x)^2\,dx.$$

The implied constants depend only on $K(s)$ and $\delta>0$.

It should be noticed that the result on the 1-level density by Katz and Sarnak [9] is stronger than that of Özlük and Snyder [13] in some sense. However, the author believes that the asymptotic formula of Theorem 1.1

is more useful than the limit formula (1.1) ($n = 2$) if we restrict the target only to the study of the pair correlation of low-lying zeros of $L(s, \chi_d)$, since the factor $\gamma_1 - \gamma_2$ is contained in the definition of $F_K(\alpha, D)$, and both the left-hand side and the main terms of the right-hand side of (1.5) are very simple; hence, we can easily compute the integrals involving these terms. In fact, several concrete results on the average gaps of the nontrivial zeros are obtained. Section 4 of this paper is devoted to their study. In Corollary 4.2, we give a certain upper bound for the number of pairs of 'close zeros' near the real axis. In Corollary 4.3, we give a lower bound for the weighted sum involving simple zeros of $L(s, \chi_d)$. Finally, in Corollary 4.4, we prove that there are quite a few pairs of zeros $(1/2 + i\gamma_1, 1/2 + i\gamma_2)$ of $L(s, \chi_d)$ ($d \in \mathbf{Z}\backslash\{0\}$) satisfying $0 < |\gamma_1 - \gamma_2| \leqslant (2\pi\lambda)/\log D$, if λ is large to a certain extent.

§2. Preliminaries

To prove the main theorem, we prepare several lemmas. The following nine lemmas (Lemmas 2.1–2.9) are all found in [13].

LEMMA 2.1. *We have*

$$\sum_{d} e^{-\pi d^2/y^2} = y + o(1) \quad (y \to \infty). \tag{2.1}$$

Here, $\sum_d$ denotes the sum over all nonzero integers d.

LEMMA 2.2.

$$\sum_{d} e^{-\pi d^2/y^2} \log|d| = y\log y + O(y) \quad (y \to \infty). \tag{2.2}$$

LEMMA 2.3. *We have*

$$\sum_{d=\square} e^{-\pi d^2/y^2} = Iy^{1/2} - \frac{1}{2} + O(y^{-1/2}) \tag{2.3}$$

as $y \to \infty$. Here, $I = (1/4)\pi^{-1/4}\Gamma(1/4)$, and $\sum_{d=\square}$ denotes the sum over all positive square integers.

Instead of (2.3), sometimes we use

$$\sum_{d=\square} e^{-\pi d^2/y^2} = Iy^{1/2} - \frac{1}{2} + O(\min\{1, y^{-1/2}\}) \tag{2.4}$$

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or

$$\sum_{d=\square} e^{-\pi d^2/y^2} = I y^{1/2} + O(1). \tag{2.5}$$

LEMMA 2.4. *Assuming the RH, we have*

$$\sum_{p} a\left(\frac{p^2}{x}\right) \log p = \frac{1}{2} K\left(\frac{1}{2}\right) x^{1/2} + O(x^{1/4} \log^2 x) \quad (x \to \infty). \tag{2.6}$$

Here, $\sum_p$ denotes the sum over all primes p.

It should be noticed that the implied constant in (2.6) is dependent on $a(x)$ (and hence on $K(s)$). Hereafter, this fact will be valid for almost all asymptotic formulas, although we will not comment again.

LEMMA 2.5. *Assuming the RH, we have*

$$\sum_{p} a\left(\frac{p}{x}\right) \log p = K\,(1)\, x + O(x^{1/2} \log^2 x) \quad (x \to \infty). \tag{2.7}$$

LEMMA 2.6. *Assuming the RH, we have*

$$\sum_{p} a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} = K\left(\frac{1}{2}\right) x^{1/2} + O(\log^2 x) \quad (x \to \infty). \tag{2.8}$$

LEMMA 2.7. *Assuming the RH, we have*

$$\sum_{p} a\left(\frac{p}{x}\right) \frac{\log p}{p} = K\,(0) + O(x^{-1/2} \log^2 x) \quad (x \to \infty). \tag{2.9}$$

LEMMA 2.8. *Assuming the RH, we have*

$$\sum_{p} a\left(\frac{p^2}{x}\right) \frac{\log p}{p} = \frac{1}{2} K\,(0) + O(x^{-1/4} \log^2 x) \quad (x \to \infty). \tag{2.10}$$

LEMMA 2.9. *We have*

$$\sum_{p} a\left(\frac{p}{x}\right) p^{1/2} \log^2 p \ll x^{3/2} \log x \quad (x \to \infty). \tag{2.11}$$

In addition to these nine lemmas quoted from [13], we need several asymptotic formulas.

LEMMA 2.10. *We have*

$$\sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log^2 d = \frac{1}{2} y \log^2 y + O(y \log y) \quad (y \to \infty). \tag{2.12}$$

Proof. The left-hand side is

$$\begin{aligned}\sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log^2 d = \sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log^2 \left(\frac{d}{y}\right) + 2(\log y) \sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log d \\ - (\log y)^2 \sum_{d=1}^{\infty} e^{-\pi d^2/y^2}.\end{aligned} \tag{2.13}$$

The first term on the right-hand side of (2.13) is

$$\begin{aligned}\sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log^2 \left(\frac{d}{y}\right) &= \int_{1-}^{\infty} e^{-\pi u^2/y^2} \log^2 \left(\frac{u}{y}\right) d[u] \\ &= \int_{1}^{\infty} e^{-\pi u^2/y^2} \log^2 \left(\frac{u}{y}\right) du \\ &\quad - \int_{1-}^{\infty} e^{-\pi u^2/y^2} \log^2 \left(\frac{u}{y}\right) d\{u\},\end{aligned} \tag{2.14}$$

where $[u]$ denotes the integer part of u, and $\{u\} := u - [u]$. By the change of parameters $u/y = v$, we can easily see that

$$\int_{1}^{\infty} e^{-\pi u^2/y^2} \log^2 \left(\frac{u}{y}\right) du \ll y,$$

and by partial integration, we have

$$\int_{1-}^{\infty} e^{-\pi u^2/y^2} \log^2 \left(\frac{u}{y}\right) d\{u\} \ll \log^2 y.$$

Hence, the first term on the right-hand side of (2.13) is $O(y)$. Moreover, by Lemma 2.1, we have

$$\sum_{d=1}^{\infty} e^{-\pi d^2/y^2} = \frac{1}{2} y + o(1),$$

and by Lemma 2.2, we have

$$\sum_{d=1}^{\infty} e^{-\pi d^2/y^2} \log d = \frac{1}{2} y \log y + O(y).$$

By inserting these into (2.13), we obtain the result. ∎

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LEMMA 2.11. *We have*

$$\sum_p a\left(\frac{p}{x}\right) p\log p \ll x^2 \quad (x\to\infty). \tag{2.15}$$

Proof. It should be recalled that the function $a(u)$ is bounded and has a support in $[A, B]$. By the prime number theorem (PNT), the number of primes p satisfying the condition $p/x \leqslant B$ is $O(x/\log x)$. Therefore,

$$\sum_p a\left(\frac{p}{x}\right) p\log p \ll x\log x\cdot\frac{x}{\log x} = x^2.$$

□

By a similar argument, we obtain the following.

LEMMA 2.12. *We have*

$$\sum_p a\left(\frac{p^2}{x}\right) p\log p \ll x \quad (x\to\infty). \tag{2.16}$$

Hereafter, we put

$$L(s) := \int_0^\infty a(t)^2 t^s\,\frac{dt}{t}. \tag{2.17}$$

LEMMA 2.13. *Assuming the RH, we have*

$$\sum_p a\left(\frac{p}{x}\right)^2 (\log p)^2 = L(1)x\log x + L'(1)x + O(\sqrt{x}\log^3 x) \quad (x\to\infty). \tag{2.18}$$

Proof. We write

$$\theta(u) := \sum_{p\leqslant u}\log p = u + E(u).$$

Then, it is well known that $E(u)$ is evaluated by $O(u^{1/2}\log^2 u)$ under the assumption of the RH. Now, we have

$$\begin{aligned}\sum_p a\left(\frac{p}{x}\right)^2(\log p)^2 &= \int_0^\infty a\left(\frac{u}{x}\right)^2\log u\,d\theta(u)\\ &= \int_0^\infty a\left(\frac{u}{x}\right)^2\log u\,du + \int_0^\infty a\left(\frac{u}{x}\right)^2\log u\,dE(u).\end{aligned} \tag{2.19}$$

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By the change of parameters $u/x = v$, we have

$$\begin{aligned}\int_0^\infty a\left(\frac{u}{x}\right)^2 \log u\, du &= x\log x \int_0^\infty a(v)^2\, dv + x\int_0^\infty a(v)^2 \log v\, dv\\ &= L(1)x\log x + L'(1)x. \end{aligned} \tag{2.20}$$

On the other hand, by partial integration, we have

$$\begin{aligned}&\int_0^\infty a\left(\frac{u}{x}\right)^2 \log u\, dE(u)\\ &\quad = -\int_0^\infty E(u)\left\{\frac{2a\left(\frac{u}{x}\right)a'\left(\frac{u}{x}\right)\log u}{x} + \frac{a\left(\frac{u}{x}\right)^2}{u}\right\} du,\end{aligned}$$

and by the change of parameters $u/x = v$, combining with the estimate $E(u) \ll \sqrt{u}\log^2 u$, we easily find that

$$\int_0^\infty E(u)\frac{a\left(\frac{u}{x}\right)a'\left(\frac{u}{x}\right)\log u}{x}\, du \ll \sqrt{x}\log^3 x,$$

$$\int_0^\infty E(u)\frac{a\left(\frac{u}{x}\right)^2}{u}\, du \ll \sqrt{x}\log^2 x.$$

Hence, we get

$$\int_0^\infty a\left(\frac{u}{x}\right)^2 \log u\, dE(u) \ll \sqrt{x}\log^3 x. \tag{2.21}$$

By inserting (2.20), (2.21) into (2.19), we obtain (2.18). ∎

LEMMA 2.14. *Assuming the RH, we have*

$$\sum_p a\left(\frac{p}{x}\right)^2 \frac{(\log p)^2}{p} = L(0)\log x + L'(0) + O(x^{-1/2}\log^3 x) \quad (x\to\infty). \tag{2.22}$$

The proof of Lemma 2.14 is almost the same as that of Lemma 2.13; hence, we omit it.

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Lemma 2.15. *We assume the RH. Then, when $x^{\frac{1}{l}} \gg 1$, we have*

$$\sum_p \sum_{q \neq p} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q)$$

$$= l^{-1} K(1) K\left(\frac{1}{l}\right) x^{1+\frac{1}{l}} + O(l^{-1} x^{1+\frac{1}{2l}} \log^2 x). \tag{2.23}$$

The implied constant is independent of l.

Proof. The left-hand side is

$$\sum_p \sum_{q \neq p} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q)$$

$$= - \sum_p a\left(\frac{p}{x}\right) a\left(\frac{p^l}{x}\right) \log^2 p + \sum_p a\left(\frac{p}{x}\right) \log p \sum_q a\left(\frac{q^l}{x}\right) \log q. \tag{2.24}$$

By the PNT, the number of primes p satisfying $p^l/x \leqslant B$ is $O(lx^{1/l}/\log x)$. Therefore,

$$\sum_p a\left(\frac{p}{x}\right) a\left(\frac{p^l}{x}\right) \log^2 p \ll l^{-1} x^{\frac{1}{l}} \log x. \tag{2.25}$$

By Lemma 2.5,

$$\sum_p a\left(\frac{p}{x}\right) \log p = K(1)x + O(x^{1/2} \log^2 x). \tag{2.26}$$

Further,

$$\sum_q a\left(\frac{q^l}{x}\right) \log q = \int_0^\infty a\left(\frac{u^l}{x}\right) d\theta(u)$$

$$= \int_0^\infty a\left(\frac{u^l}{x}\right) du + \int_0^\infty a\left(\frac{u^l}{x}\right) dE(u).$$

By the change of parameters $u^l/x = v$, the first term on the right-hand side becomes

$$\int_0^\infty a\left(\frac{u^l}{x}\right) du = l^{-1} x^{\frac{1}{l}} K\left(\frac{1}{l}\right).$$

Since $E(u) \ll u^{1/2} \log^2 u$, and since $a'(v)$ is bounded and has a support in $[A, B]$, the second integral is

$$\begin{aligned}\int_0^\infty a\left(\frac{u^l}{x}\right) dE(u) &= -\int_0^\infty E(u)\frac{lu^{l-1}}{x} a'\left(\frac{u^l}{x}\right) du \\ &\ll lx^{-1}\int_0^\infty u^{l-\frac{1}{2}}\left|a'\left(\frac{u^l}{x}\right)\right| \log^2 u\, du \\ &\ll lx^{-1}(x^{\frac{1}{l}})^{l-\frac{1}{2}} \log^2(x^{\frac{1}{l}}) \cdot x^{\frac{1}{l}} \\ &\ll l^{-1}x^{\frac{1}{2l}} \log^2 x.\end{aligned}$$

Combining these, we have

$$\sum_q a\left(\frac{q^l}{x}\right) \log q = l^{-1}x^{\frac{1}{l}} K\left(\frac{1}{l}\right) + O(l^{-1}x^{\frac{1}{2l}} \log^2 x). \tag{2.27}$$

By inserting (2.25)–(2.27) into (2.24), we obtain the result. ∎

LEMMA 2.16. *We assume the RH. Then, when $x^{\frac{1}{l}} \gg 1$, we have*

$$\sum_p \sum_{q\neq p} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) \frac{(\log p)(\log q)}{\sqrt{pq}} = l^{-1} K\left(\frac{1}{2}\right) K\left(\frac{1}{2l}\right) x^{1/2+\frac{1}{2l}} + O(l^{-1}x^{1/2} \log^3 x). \tag{2.28}$$

The implied constant is independent of l.

The proof of Lemma 2.16 is almost the same as that of Lemma 2.15; hence, we omit it. It should be noticed that the asymptotic formulas of Lemmas 2.15, 2.16 are still valid if we replace the sum with $p, q \geqslant 3$, $p \neq q$.

To obtain the asymptotic formula of the main theorem, we need the following translation formula for the theta function.

LEMMA 2.17. *Let $p, q \geqslant 3$ be distinct primes, and let $D > 0$. Then,*

$$\sum_d \left(\frac{d}{pq}\right) e^{-\pi d^2/D^2} = \frac{D}{\sqrt{pq}} \sum_m \left(\frac{m}{pq}\right) e^{-\pi m^2 D^2/p^2q^2}. \tag{2.29}$$

Here, d and m above run over the set of all nonzero integers.

Proof. If $(-1/pq) = -1$, the identity (2.29) clearly holds, since both sides become 0. We assume $(-1/pq) = 1$, and put

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$$\chi_{pq}(n) = \left(\frac{n}{pq}\right).$$

Then, χ_{pq} becomes a primitive character of modulo pq. We define the theta function $\psi(x, \chi_{pq})$ by

$$\psi(x, \chi_{pq}) = \sum_{n=-\infty}^{\infty} \chi_{pq}(n) e^{-n^2\pi x/pq}$$

for $x > 0$. Then, $\psi(x, \chi_{pq})$ satisfies

$$\tau(\chi_{pq})\psi(x, \chi_{pq}) = \left(\frac{pq}{x}\right)^{1/2} \psi(x^{-1}, \chi_{pq}),$$

where $\tau(\chi_{pq})$ is the Gaussian sum associated to χ_{pq} (for example, see [1, p. 67]). Moreover, we have $\tau(\chi_{pq}) = \sqrt{pq}$. Hence, we get

$$\sum_{n=-\infty}^{\infty} \left(\frac{n}{pq}\right) e^{-n^2\pi x/pq} = x^{-1/2} \sum_{m=-\infty}^{\infty} \left(\frac{m}{pq}\right) e^{-\pi m^2/pqx}.$$

By putting $x = pq/D^2$, we obtain (2.29). ∎

§3. The proof of Theorem 1.1

We start from the explicit formula

$$\sum_{\rho\in Z_d} K(\rho)x^\rho = K(1)E(\chi_d)x - \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) + a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right) + O(\min\{x, \log|d|\log x\}) \quad (x \geqslant 1),$$

introduced in [13]. Here, $E(\chi) = 1$ if χ is a principal character, and otherwise $E(\chi) = 0$. The error term is interpreted as $O(1)$ if $x = 1$. Since the main terms on the right-hand side are real, we have

$$\sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)}x^{\rho_1+\overline{\rho_2}} = \left\{K(1)E(\chi_d)x - \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) + a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right) + O(\min\{x, \log|d|\log x\})\right\}^2$$

$$
\begin{aligned}
&= K(1)^2E(\chi_d)^2x^2 + \sum_{n,m=1}^{\infty} a\left(\frac{n}{x}\right) a\left(\frac{m}{x}\right) \Lambda(n)\Lambda(m) \left(\frac{d}{n}\right)\left(\frac{d}{m}\right) \\
&\quad + a\left(\frac{1}{x}\right)^2 \log^2\left(\frac{|d|}{\pi}\right) - 2K(1)E(\chi_d)x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) \\
&\quad + 2K(1)E(\chi_d)xa\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right) \\
&\quad - 2a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) \\
&\quad + O\left(\max\left\{K(1)E(\chi_d)x, \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right), a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)\right\}\right) \\
&\quad \times O\left(\min\{x, \log|d|\log x\}\right) \\
&\quad + O\left(\min\{x^2, \log^2|d|\log^2 x\}\right).
\end{aligned}
$$

By multiplying both sides by $e^{-\pi d^2/D^2}$ and taking the sum over d, we have

$$
f_K(x, D) = M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + O_1 + O_2 + O_3 + O_4, \tag{3.1}
$$

where

$$
M_1 = K(1)^2x^2\sum_d E(\chi_d)^2e^{-\pi d^2/D^2},
$$

$$
M_2 = \sum_{n,m=1}^{\infty} a\left(\frac{n}{x}\right) a\left(\frac{m}{x}\right)\Lambda(n)\Lambda(m)\sum_d e^{-\pi d^2/D^2}\left(\frac{d}{n}\right)\left(\frac{d}{m}\right),
$$

$$
M_3 = a\left(\frac{1}{x}\right)^2\sum_d e^{-\pi d^2/D^2}\log^2\left(\frac{|d|}{\pi}\right),
$$

$$
M_4 = -2K(1)x\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\sum_d e^{-\pi d^2/D^2}E(\chi_d)\left(\frac{d}{n}\right),
$$

$$
M_5 = 2K(1)xa\left(\frac{1}{x}\right)\sum_d e^{-\pi d^2/D^2}E(\chi_d)\log\left(\frac{|d|}{\pi}\right),
$$

$$
M_6 = -2a\left(\frac{1}{x}\right)\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\sum_d e^{-\pi d^2/D^2}\left(\frac{d}{n}\right)\log\left(\frac{|d|}{\pi}\right),
$$

$$
O_1 = O\left(\min\{O_{11}, O_{12}\}\right), \qquad O_2 = O\left(\min\{O_{21}, O_{22}\}\right),
$$

$$
O_3 = O\left(\min\{O_{31}, O_{32}\}\right), \qquad O_4 = \left(\min\{O_{41}, O_{42}\}\right),
$$

with

$$O_{11} = K(1)x^2 \sum_d e^{-\pi d^2/D^2} E(\chi_d),$$

$$O_{12} = K(1)x \log x \sum_d e^{-\pi d^2/D^2} E(\chi_d) \log |d|,$$

$$O_{21} = x \sum_d e^{-\pi d^2/D^2} \left| \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \right|,$$

$$O_{22} = \log x \sum_d e^{-\pi d^2/D^2} \log |d| \left| \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \right|,$$

$$O_{31} = xa\left(\frac{1}{x}\right) \sum_d e^{-\pi d^2/D^2} \left| \log\left(\frac{|d|}{\pi}\right) \right|,$$

$$O_{32} = a\left(\frac{1}{x}\right) \log x \sum_d e^{-\pi d^2/D^2} \log |d| \left| \log\left(\frac{|d|}{\pi}\right) \right|,$$

$$O_{41} = x^2 \sum_d e^{-\pi d^2/D^2}, \quad O_{42} = \log^2 x \sum_d e^{-\pi d^2/D^2} \log^2 |d|.$$

3.1 Evaluations of O_1, O_2, O_3, O_4

First, we evaluate the error terms O_i $(i = 1, 2, 3, 4)$. By Lemma 2.3,

$$\sum_d e^{-\pi d^2/D^2} E(\chi_d) = \sum_{d=\square} e^{-\pi d^2/D^2} \ll D^{1/2}.$$

Combining this with abelian summation, we find that

$$\sum_d e^{-\pi d^2/D^2} E(\chi_d) \log |d| \ll D^{1/2} \log D.$$

Therefore, we have

$$O_1 \ll \min\{x^2 D^{1/2}, xD^{1/2} \log x \log D\}. \tag{3.2}$$

By Lemmas 2.2 and 2.10, we have

$$\sum_d e^{-\pi d^2/D^2} \left| \log\left(\frac{|d|}{\pi}\right) \right| \ll D \log D,$$

$$\sum_d e^{-\pi d^2/D^2} \log |d| \left| \log\left(\frac{|d|}{\pi}\right) \right| \ll D \log^2 D.$$

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Hence, we obtain

$$O_3 \ll \min\{xD\log D, D\log x \log^2 D\}. \tag{3.3}$$

Next, we evaluate O_2. In particular, we evaluate O_{22} in two ways. First, since

$$\left|\sum_n a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\right| \leqslant \sum_n a\left(\frac{n}{x}\right)\Lambda(n) \ll x,$$

by Lemma 2.2, we have

$$O_{22} \ll x\log x \sum_d e^{-\pi d^2/D^2}\log|d| \ll xD\log x\log D. \tag{3.4}$$

On the other hand, we decompose

$$\begin{aligned} O_{22} = \log x \Bigg\{ &\sum_{d=\square} e^{-\pi d^2/D^2}\log|d| \left|\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\right| \\ &+ \sum_{d\neq\square} e^{-\pi d^2/D^2}\log|d| \left|\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\right| \Bigg\}. \end{aligned} \tag{3.5}$$

Then, the first term is evaluated by

$$\begin{aligned} \log x &\sum_{d=\square} e^{-\pi d^2/D^2}\log|d| \left|\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\right| \\ &\ll x\log x \sum_{d=\square} e^{-\pi d^2/D^2}\log|d| \\ &\ll xD^{1/2}\log x\log D. \end{aligned} \tag{3.6}$$

We evaluate the second term on the right-hand side of (3.5). It is known that, assuming the GRH, for $d \neq \square$, the estimate

$$\sum_{p\leqslant x}\left(\frac{d}{p}\right)\log p \ll x^{1/2}\log^2|d|x$$

holds (see [13, p. 221]). We decompose

$$\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) = \sum_p a\left(\frac{p}{x}\right)\left(\frac{d}{p}\right)\log p + \sum_{l\geqslant 2}\sum_p a\left(\frac{p^l}{x}\right)\left(\frac{d}{p^l}\right)\log p. \tag{3.7}$$

Using the estimate above, the first term is evaluated by

$$\begin{aligned}\sum_p a\left(\frac{p}{x}\right)\left(\frac{d}{p}\right)\log p &= \int_0^\infty a\left(\frac{u}{x}\right) d\left(\sum_{p\leqslant u}\left(\frac{d}{p}\right)\log p\right)\\ &= -x^{-1}\int_0^\infty \left(\sum_{p\leqslant u}\left(\frac{d}{p}\right)\log p\right) a'\left(\frac{u}{x}\right) du\\ &\ll x^{1/2}\log^2|d|x. \end{aligned}\tag{3.8}$$

Next, we evaluate the second term on the right-hand side of (3.7). Since $a(x)$ has a support in $[A, B]$ for some $0 < A < B < \infty$, we may assume that the range of l is restricted to $2 \leqslant l \ll \log x$. Therefore,

$$\begin{aligned}\sum_{l\geqslant 2}\sum_p a\left(\frac{p^l}{x}\right)\left(\frac{d}{p^l}\right)\log p &\ll \sum_{l\geqslant 2}\sum_p a\left(\frac{p^l}{x}\right)\log p\\ &\ll \sum_{2\leqslant l\ll \log x} x^{1/2}\log x \ll x^{1/2}\log^2 x. \end{aligned}\tag{3.9}$$

Combining (3.7)–(3.9), we obtain

$$\sum_{n=1}^\infty a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) \ll x^{1/2}\log^2|d|x \tag{3.10}$$

for $d \neq \square$, assuming the GRH. Therefore, the second term on the right-hand side of (3.5) is

$$\begin{aligned}&\log x\sum_{d\neq\square} e^{-\pi d^2/D^2}\log|d|\left|\sum_{n=1}^\infty a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\right|\\ &\ll x^{1/2}\log x\sum_{d\neq 0} e^{-\pi d^2/D^2}\log|d|\log^2|d|x\\ &\ll x^{1/2}D\log x\log D(\log^2 D+\log^2 x). \end{aligned}\tag{3.11}$$

Combining (3.6) and (3.11), we obtain

$$O_{22} \ll xD^{1/2}\log x\log D + x^{1/2}D\log x\log D(\log^2 D+\log^2 x). \tag{3.12}$$

By combining (3.4), (3.12), and the evaluation $O_2 \ll O_{22}$, we obtain

$$O_2 \ll \min\Big\{xD\log x\log D, xD^{1/2}\log x\log D + x^{1/2}D\log x\log D(\log^2 D+\log^2 x)\Big\}. \tag{3.13}$$

Finally, since

$$x^2\sum_d e^{-\pi d^2/D^2} \ll x^2 D,$$

$$\log^2 x\sum_d e^{-\pi d^2/D^2}\log^2|d| \ll D\log^2 x\log^2 D,$$

we have

$$O_4 \ll \min\{x^2 D, D\log^2 x\log^2 D\}. \tag{3.14}$$

3.2 The computations of M_1, M_3 and the evaluation of M_5

Next, we compute M_1, M_3, M_5. These terms do not involve Kronecker symbols. First, by Lemma 2.3,

$$M_1 = K(1)^2x^2\sum_{d=\square} e^{-\pi d^2/D^2} = IK(1)^2x^2D^{1/2} - \frac{1}{2}K(1)^2x^2 + O(x^2D^{-1/2}). \tag{3.15}$$

Next, we compute M_3. The sum with respect to d is

$$\sum_d e^{-\pi d^2/D^2}\log^2\left(\frac{|d|}{\pi}\right) = 2\sum_{d=1}^{\infty} e^{-\pi d^2/D^2}\log^2 d - 4\log\pi\sum_{d=1}^{\infty} e^{-\pi d^2/D^2}\log d + 2(\log\pi)^2\sum_{d=1}^{\infty} e^{-\pi d^2/D^2}. \tag{3.16}$$

By Lemmas 2.1, 2.2, 2.10, we have

$$\sum_{d=1}^{\infty} e^{-\pi d^2/D^2}\log^2 d = \frac{1}{2}D\log^2 D + O(D\log D),$$

$$\sum_{d=1}^{\infty} e^{-\pi d^2/D^2}\log d = \frac{1}{2}D\log D + O(D), \qquad \sum_{d=1}^{\infty} e^{-\pi d^2/D^2} = \frac{1}{2}D + O(1).$$

By inserting these into (3.16), we obtain

$$\sum_d e^{-\pi d^2/D^2}\log^2\left(\frac{|d|}{\pi}\right) = D\log^2 D + O(D\log D).$$

Hence,

$$M_3 = a\left(\frac{1}{x}\right)^2 \{D\log^2 D + O(D\log D)\}. \tag{3.17}$$

Finally, we evaluate M_5. Since

$$\begin{aligned}\sum_d e^{-\pi d^2/D^2} E(\chi_d)\log\left(\frac{|d|}{\pi}\right) &\ll \sum_{d=1}^{\infty} e^{-\pi d^4/D^2}\log d\\ &\ll D^{1/2}\log D,\end{aligned}$$

we obtain

$$M_5 \ll xa\left(\frac{1}{x}\right) D^{1/2}\log D. \tag{3.18}$$

3.3 The computation of M_4 and the evaluation of M_6

Next, we compute M_4. We decompose this by

$$\begin{aligned} M_4 &= -2K(1)x\left(\sum_{n=p} + \sum_{n=p^2} + \sum_{n=p^k, k\geqslant 3}\right) a\left(\frac{n}{x}\right)\Lambda(n)\\ &\qquad \times \sum_d e^{-\pi d^2/D^2} E(\chi_d)\left(\frac{d}{n}\right)\\ &=: -2K(1)x\left(M_4^{(1)} + M_4^{(2)} + M_4^{(3)}\right), \end{aligned} \tag{3.19}$$

say. First,

$$\begin{aligned} M_4^{(1)} &= \sum_p a\left(\frac{p}{x}\right)\log p \sum_{d=1}^{\infty} e^{-\pi d^4/D^2}\left(\frac{d^2}{p}\right)\\ &= \sum_p a\left(\frac{p}{x}\right)\log p\left\{\sum_{d=1}^{\infty} e^{-\pi d^4/D^2} - \sum_{\substack{d=1\\ p\mid d}}^{\infty} e^{-\pi d^4/D^2}\right\}\\ &= \sum_p a\left(\frac{p}{x}\right)\log p\left\{\sum_{d=1}^{\infty} e^{-\pi d^4/D^2} - \sum_{d=1}^{\infty} e^{-\pi p^4 d^4/D^2}\right\}. \end{aligned} \tag{3.20}$$

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By Lemmas 2.3 and 2.5,

$$\begin{aligned}
&\sum_p a\left(\frac{p}{x}\right)\log p\sum_{d=1}^{\infty} e^{-\pi d^4/D^2}\\
&\quad = \left(K(1)x+O(x^{1/2}\log^2 x)\right)\left(ID^{1/2}-\frac{1}{2}+O(D^{-1/2})\right)\\
&\quad = IK(1)xD^{1/2}-\frac{1}{2}K(1)x+O((Dx)^{1/2}\log^2 x)+O(xD^{-1/2}). \qquad (3.21)
\end{aligned}$$

By Lemma 2.3,

$$\sum_{d=1}^{\infty} e^{-\pi p^4 d^4/D^2}=\frac{ID^{1/2}}{p}-\frac{1}{2}+O(\min\{1,pD^{-1/2}\}).$$

Therefore, by Lemmas 2.5 and 2.7, we have

$$\begin{aligned}
&\sum_p a\left(\frac{p}{x}\right)\log p\sum_{d=1}^{\infty} e^{-\pi p^4 d^4/D^2}\\
&\quad = \sum_p a\left(\frac{p}{x}\right)\log p\left\{I\cdot\frac{D^{1/2}}{p}-\frac{1}{2}+O(\min\{1,pD^{-1/2}\})\right\}\\
&\quad = IK(0)D^{1/2}+O(D^{1/2}x^{-1/2}\log^2 x)-\frac{1}{2}K(1)x+O(x^{1/2}\log^2 x)\\
&\qquad +O\left(\sum_p a\left(\frac{p}{x}\right)(\log p)\min\{1,pD^{-1/2}\}\right). \qquad (3.22)
\end{aligned}$$

By Lemmas 2.5 and 2.11, we have

$$\sum_p a\left(\frac{p}{x}\right)\log p\ll x,\qquad D^{-1/2}\sum_p a\left(\frac{p}{x}\right)p\log p\ll x^2D^{-1/2}.$$

Therefore, the last line of (3.22) is evaluated by $O(\min\{x,x^2D^{-1/2}\})$. Hence, we have

$$\begin{aligned}
&\sum_p a\left(\frac{p}{x}\right)\log p\sum_{d=1}^{\infty} e^{-\pi p^4 d^4/D^2}\\
&\quad = IK(0)D^{1/2}-\frac{1}{2}K(1)x\\
&\qquad +O(D^{1/2}x^{-1/2}\log^2 x+x^{1/2}\log^2 x+\min\{x,x^2D^{-1/2}\}). \qquad (3.23)
\end{aligned}$$

By inserting (3.21), (3.23) into (3.20), we obtain

$$M_4^{(1)} = IK(1)D^{1/2}x + O((Dx)^{1/2}\log^2 x + \min\{x, x^2D^{-1/2}\}). \tag{3.24}$$

Next,

$$\begin{aligned} M_4^{(2)} &= \sum_p a\left(\frac{p^2}{x}\right)\log p \sum_{d=1}^{\infty} e^{-\pi d^4/D^2}\left(\frac{d^2}{p^2}\right) \\ &= \sum_p a\left(\frac{p^2}{x}\right)\log p \left\{\sum_{d=1}^{\infty} e^{-\pi d^4/D^2} - \sum_{\substack{d=1 \\ p\mid d}}^{\infty} e^{-\pi d^4/D^2}\right\}. \end{aligned} \tag{3.25}$$

By Lemmas 2.3 and 2.4,

$$\begin{aligned} &\sum_p a\left(\frac{p^2}{x}\right)\log p \sum_{d=1}^{\infty} e^{-\pi d^4/D^2} \\ &\quad = \left\{\frac{1}{2}K\left(\frac{1}{2}\right)x^{1/2} + O(x^{1/4}\log^2 x)\right\}\left\{ID^{1/2} - \frac{1}{2} + O(D^{-1/2})\right\} \\ &\quad = \frac{1}{2}IK\left(\frac{1}{2}\right)D^{1/2}x^{1/2} - \frac{1}{4}K\left(\frac{1}{2}\right)x^{1/2} \\ &\quad\quad + O(D^{1/2}x^{1/4}\log^2 x + x^{1/2}D^{-1/2}). \end{aligned} \tag{3.26}$$

On the other hand, since

$$\sum_{\substack{d=1 \\ p\mid d}}^{\infty} e^{-\pi d^4/D^2} = I\cdot\frac{D^{1/2}}{p} - \frac{1}{2} + O(\min\{1, pD^{-1/2}\}),$$

by Lemmas 2.4 and 2.8, we have

$$\begin{aligned} &\sum_p a\left(\frac{p^2}{x}\right)\log p \sum_{\substack{d=1 \\ p\mid d}}^{\infty} e^{-\pi d^4/D^2} \\ &\quad = \sum_p a\left(\frac{p^2}{x}\right)\log p \left\{I\cdot\frac{D^{1/2}}{p} - \frac{1}{2} + O(\min\{1, pD^{-1/2}\})\right\} \\ &\quad = ID^{1/2}\left\{\frac{1}{2}K(0) + O(x^{-1/4}\log^2 x)\right\} \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\left\{\frac{1}{2}K\left(\frac{1}{2}\right)x^{1/2} + O(x^{1/4}\log^2 x)\right\} \\
&\quad + O\left(\min\left\{\sum_p a\left(\frac{p^2}{x}\right)\log p,\, D^{-1/2}\sum_p a\left(\frac{p^2}{x}\right)p\log p\right\}\right) \\
&= \frac{1}{2}IK(0)D^{1/2} - \frac{1}{4}K\left(\frac{1}{2}\right)x^{1/2} \\
&\quad + O(x^{-1/4}D^{1/2}\log^2 x) + O(x^{1/4}\log^2 x) \\
&\quad + O\left(\min\left\{\sum_p a\left(\frac{p^2}{x}\right)\log p,\, D^{-1/2}\sum_p a\left(\frac{p^2}{x}\right)p\log p\right\}\right). \qquad (3.27)
\end{aligned}
$$

By Lemmas 2.4 and 2.12, we have

$$
\sum_p a\left(\frac{p^2}{x}\right)\log p \ll x^{1/2}, \qquad \sum_p a\left(\frac{p^2}{x}\right)p\log p \ll x.
$$

Hence, the last line of (3.27) becomes $O(\min\{x^{1/2}, xD^{-1/2}\})$. Therefore,

$$
\begin{aligned}
&\sum_p a\left(\frac{p^2}{x}\right)\log p \sum_{\substack{d=1\\ p\mid d}}^{\infty} e^{-\pi d^4/D^2} \\
&\quad = \frac{1}{2}IK(0)D^{1/2} - \frac{1}{4}K\left(\frac{1}{2}\right)x^{1/2} + O(x^{-1/4}D^{1/2}\log^2 x) \\
&\qquad + O(x^{1/4}\log^2 x) + O(\min\{x^{1/2}, xD^{-1/2}\}). \qquad (3.28)
\end{aligned}
$$

By inserting (3.26), (3.28) into (3.25), we obtain

$$
M_4^{(2)} = \frac{1}{2}IK\left(\frac{1}{2}\right)D^{1/2}x^{1/2} + O(D^{1/2}x^{1/4}\log^2 x) + O(\min\{x^{1/2}, xD^{-1/2}\}). \qquad (3.29)
$$

Next, we evaluate

$$
M_4^{(3)} = \sum_{k\geqslant 3}\sum_p a\left(\frac{p^k}{x}\right)\log p \sum_d e^{-\pi d^2/D^2} E(\chi_d)\left(\frac{d}{p^k}\right).
$$

Since $a(u) = 0$ for $u > B$, we only have to compute the part $p^k/x \leqslant B$. The number of such k is $O(\log x)$. Since p must satisfy $p \leqslant (Bx)^{1/k} \leqslant (Bx)^{1/3}$, and

$$\left|\sum_d e^{-\pi d^2/D^2} E(\chi_d) \left(\frac{d}{p^k}\right)\right| \leqslant \sum_{d=1}^{\infty} e^{-\pi d^4/D^2} \ll D^{1/2},$$

using the PNT, we find that

$$\begin{aligned} M_4^{(3)} &\ll (\log x) \cdot \left(\sum_{p \leqslant (Bx)^{1/3}} \log p \right) \cdot D^{1/2} \\ &\ll x^{1/3} D^{1/2} \log x. \end{aligned} \tag{3.30}$$

By inserting (3.24), (3.29), (3.30) into (3.19), we obtain

$$M_4 = -2IK(1)^2 D^{1/2} x^2 + O(D^{1/2} x^{3/2} \log^2 x) + O(\min\{x^2, x^3 D^{-1/2}\}). \tag{3.31}$$

Next, we evaluate M_6. We decompose this by

$$\begin{aligned} M_6 &= -2a\left(\frac{1}{x}\right) \left\{ \sum_{n=p} + \sum_{n=p^k, k \geqslant 2} \right\} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\pi d^2/D^2} \left(\frac{d}{n}\right) \log\left(\frac{|d|}{\pi}\right) \\ &=: M_6^{(1)} + M_6^{(2)}, \end{aligned}$$

say. By the Pólya–Vinogradov inequality, we find that

$$\sum_d e^{-\pi d^2/D^2} \left(\frac{d}{p}\right) \log\left(\frac{|d|}{\pi}\right) \ll \sqrt{p} \log p \log D.$$

Hence, by Lemma 2.9,

$$\begin{aligned} M_6^{(1)} &\ll a\left(\frac{1}{x}\right) \sum_p a\left(\frac{p}{x}\right) \log p \cdot \sqrt{p} \log p \log D \\ &\ll a\left(\frac{1}{x}\right) \cdot x^{3/2} \log x \log D. \end{aligned} \tag{3.32}$$

Next, we evaluate $M_6^{(2)}$. By Lemmas 2.1, 2.2 and the conditions $k \ll \log x$, $p \leqslant \sqrt{Bx}$, we have

$$\begin{aligned} M_6^{(2)} &\ll a\left(\frac{1}{x}\right) \log x \sum_{p \leqslant \sqrt{Bx}} \log p \sum_d e^{-\pi d^2/D^2} \log\left(\frac{|d|}{\pi}\right) \\ &\ll a\left(\frac{1}{x}\right) D x^{1/2} \log x \log D. \end{aligned} \tag{3.33}$$

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Combining (3.32), (3.33), we obtain

$$M_6 \ll a\left(\frac{1}{x}\right)\{x^{3/2}\log x\log D + Dx^{1/2}\log x\log D\}. \tag{3.34}$$

3.4 The computation of M_2

Finally, we compute

$$M_2 = \sum_{k,l=1}^{\infty}\sum_{p,q\in P} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_d e^{-\pi d^2/D^2}\left(\frac{d}{p^k}\right)\left(\frac{d}{q^l}\right),$$

where P denotes the set of all prime numbers. It will be convenient to keep in mind that only k, l satisfying $k, l \ll \log x$ contribute to the sum above. First, we evaluate the contribution of the part $p = 2$ to M_2. The contribution of the part $p = q = 2$ is

$$\ll \sum_{k,l=1}^{\infty} a\left(\frac{2^k}{x}\right) a\left(\frac{2^l}{x}\right)\sum_d e^{-\pi d^2/D^2} \ll D\log^2 x. \tag{3.35}$$

The contribution of the part $p = 2$, $q \geqslant 3$, $l \geqslant 2$ is

$$\begin{aligned}
&\ll \sum_k\sum_{l\geqslant 2}\sum_{q\in P} a\left(\frac{2^k}{x}\right) a\left(\frac{q^l}{x}\right)(\log q)\sum_d e^{-\pi d^2/D^2}\\
&\ll Dx^{1/2}\log^2 x. \qquad (3.36)
\end{aligned}$$

Since $(\cdot/2^k q)$ is a nonprincipal character whose conductor is at most $2q$, by the Pólya–Vinogradov inequality, we have

$$\sum_d e^{-\pi d^2/D^2}\left(\frac{d}{2^k}\right)\left(\frac{d}{q}\right) \ll q^{1/2}\log q$$

for primes $q \geqslant 3$. Therefore, the contribution of the part $p = 2$, $q \geqslant 3$, $l = 1$ is

$$\begin{aligned}
&\sum_k\sum_{q\in P_{\geqslant 3}} a\left(\frac{2^k}{x}\right) a\left(\frac{q}{x}\right)(\log 2)(\log q)\sum_d e^{-\pi d^2/D^2}\left(\frac{d}{2^k}\right)\left(\frac{d}{q}\right)\\
&\quad \ll (\log x)\sum_{q\in P} a\left(\frac{q}{x}\right)(\log q)\cdot q^{1/2}\log q\\
&\quad \ll x^{3/2}\log^2 x, \qquad (3.37)
\end{aligned}$$

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where $P_{\geqslant 3}$ denotes the set of all prime numbers greater than 2. By (3.35), (3.36) and (3.37), the contribution of the part $p=2$ to M_2 is at most $O(Dx^{1/2}\log^2 x + x^{3/2}\log^2 x)$. The contribution of the part $q=2$ is the same. Therefore, we conclude that

$$\begin{aligned} M_2 &= \sum_{k,l=1}^{\infty} \sum_{p,q\in P_{\geqslant 3}} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q) \\ &\qquad \times \sum_{d} e^{-\pi d^2/D^2}\left(\frac{d}{p^k}\right)\left(\frac{d}{q^l}\right) \\ &\quad + O(Dx^{1/2}\log^2 x + x^{3/2}\log^2 x) \\ &=: \sum_{k,l=1}^{\infty} M_2^{(k,l)} + O(Dx^{1/2}\log^2 x + x^{3/2}\log^2 x), \end{aligned} \tag{3.38}$$

say. Moreover, by the PNT and Lemma 2.1, we have

$$\begin{aligned} M_2^{(k,l)} &\ll \sum_{p,q} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q) \sum_{d} e^{-\pi d^2/D^2} \\ &\ll klDx^{1/k+\frac{1}{l}} \end{aligned}$$

for each k,l. Hence, the total contribution of the part $k\geqslant 3, l\geqslant 2$ or $k\geqslant 2, l\geqslant 3$ is at most $O(Dx^{5/6}\log^4 x)$. Therefore,

$$\begin{aligned} M_2 &= M_2^{(1,1)} + M_2^{(2,2)} + 2\sum_{l\geqslant 2} M_2^{(1,l)} \\ &\qquad + O(Dx^{5/6}\log^4 x + x^{3/2}\log^2 x). \end{aligned} \tag{3.39}$$

By the computation above, $M_2^{(2,2)}$ is evaluated by

$$M_2^{(2,2)} \ll Dx. \tag{3.40}$$

Next, we compute $M_2^{(1,l)}$ for $l\geqslant 1$, $1\ll x\ll D^{1-\delta}$.

(A) First, we consider the case that l is odd. In this case, we decompose

$$M_2^{(1,l)} = \left(\sum_{\substack{p,q\in P_{\geqslant 3}\\ p=q}} + \sum_{\substack{p,q\in P_{\geqslant 3}\\ p\neq q}}\right) a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)$$

$$\times \sum_{d} e^{-\pi d^2/D^2} \left(\frac{d}{p}\right) \left(\frac{d}{q}\right)$$

$$=: M_{2,1}^{(1,l)} + M_{2,2}^{(1,l)}, \tag{3.41}$$

say. First, we compute $M_{2,1}^{(1,1)}$. This term is given by

$$\begin{aligned} M_{2,1}^{(1,1)} &= \sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \sum_{d} e^{-\pi d^2/D^2} \\ &\quad - \sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \sum_{\substack{d \\ p|d}} e^{-\pi d^2/D^2}. \end{aligned} \tag{3.42}$$

By Lemmas 2.1, 2.13, the first term on the right-hand side of (3.42) is

$$\begin{aligned} &\sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \sum_{d} e^{-\pi d^2/D^2} \\ &= \{D + O(1)\}\{L(1)x\log x + L'(1)x + O(x^{1/2}\log^3 x)\} \\ &= L(1)Dx\log x + O(Dx + x\log x). \end{aligned} \tag{3.43}$$

By Lemmas 2.1, 2.13, 2.14, the second term on the right-hand side of (3.42) is

$$\begin{aligned} &\sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \sum_{\substack{d \\ p|d}} e^{-\pi d^2/D^2} \\ &= \sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \left\{\frac{D}{p} + O(1)\right\} \\ &\ll D\sum_{p} a\left(\frac{p}{x}\right)^2 \frac{(\log p)^2}{p} + \sum_{p} a\left(\frac{p}{x}\right)^2 (\log p)^2 \\ &\ll D\log x + x\log x. \end{aligned} \tag{3.44}$$

By inserting (3.43), (3.44) into (3.42), we obtain

$$M_{2,1}^{(1,1)} = L(1)Dx\log x + O(Dx + x\log x). \tag{3.45}$$

If $l \geqslant 2$ (including the case that l is even), by the PNT and Lemma 2.1,

$$M_{2,1}^{(1,l)} \ll \sum_{p} a\left(\frac{p}{x}\right) a\left(\frac{p^l}{x}\right) \log^2 p \sum_{d} e^{-\pi d^2/D^2}$$

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$$\ll D \cdot \frac{lx^{\frac{1}{l}}}{\log x} \cdot \log^2 x$$

$$\ll lDx^{\frac{1}{l}} \log x. \tag{3.46}$$

Next, we compute $M_{2,2}^{(1,l)}$.

(a) Let $x = o(D^{1/2})$. By the Pólya–Vinogradov inequality, we obtain

$$\sum_d e^{-\pi d^2/D^2} \left(\frac{d}{p}\right) \left(\frac{d}{q}\right) \ll (pq)^{1/2} \log(pq).$$

Hence, by the PNT, we have

$$\begin{aligned} M_{2,2}^{(1,l)} &\ll \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q)(pq)^{1/2} \log(pq) \\ &\ll \sum_p a\left(\frac{p}{x}\right) p^{1/2} \log^2 p \sum_q a\left(\frac{q^l}{x}\right) q^{1/2} \log^2 q \\ &\ll \frac{x}{\log x} x^{1/2} \log^2 x \cdot \frac{lx^{\frac{1}{l}}}{\log x} x^{\frac{1}{2l}} \log^2 x \\ &\ll lx^{3/2(1+\frac{1}{l})} \log^2 x. \end{aligned} \tag{3.47}$$

(b) If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, by the translation formula of the theta function (Lemma 2.17), we have

$$M_{2,2}^{(1,l)} = D \sum_{p \geqslant 3} \sum_{\substack{q \geqslant 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_m \left(\frac{m}{pq}\right) e^{-\pi m^2 D^2/p^2 q^2}.$$

We decompose this by

$$M_{2,2}^{(1,l)} = M_s^{(1,l)} - M_p^{(1,l)} - M_q^{(1,l)} + M_{pq}^{(1,l)} + E, \tag{3.48}$$

where

$$M_s^{(1,l)} = D \sum_{p \geqslant 3} \sum_{\substack{q \geqslant 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{m=\square} e^{-\pi m^2 D^2/p^2 q^2},$$

$$M_p^{(1,l)} = D \sum_{p \geqslant 3} \sum_{\substack{q \geqslant 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{\substack{m=\square \\ p \mid m}} e^{-\pi m^2 D^2/p^2 q^2},$$

$$M_q^{(1,l)} = D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{\substack{m=\square\\ q\mid m}} e^{-\pi m^2D^2/p^2q^2},$$

$$M_{pq}^{(1,l)} = D\sum_{p\geqslant 3}\sum_{\substack{q\neq p\\ q\geqslant 3}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{\substack{m=\square\\ pq\mid m}} e^{-\pi m^2D^2/p^2q^2},$$

and

$$E = D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m\neq\square} e^{-\pi m^2D^2/p^2q^2}.$$

First, by Lemma 2.3,

$$\begin{aligned} M_s^{(1,l)} &= D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\left(I\sqrt{\frac{pq}{D}}+O(1)\right)\\ &= ID^{1/2}\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\\ &\quad + O\left(D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)\frac{(\log p)(\log q)}{\sqrt{pq}}\right). \end{aligned} \tag{3.49}$$

By applying Lemmas 2.15, 2.16 to (3.49), we obtain

$$M_s^{(1,l)} = ID^{1/2}l^{-1}K(1)K\left(\frac{1}{l}\right)x^{1+\frac{1}{l}} + O(l^{-1}D^{1/2}x^{1+\frac{1}{2l}}\log^2 x) + O(Dx^{1/2+\frac{1}{2l}}). \tag{3.50}$$

In the computation above, we used $|K\,(1/2l)| \ll \int_A^B t^{1/2l}\,dt/t \ll l$. Next, by Lemma 2.3,

$$\begin{aligned} M_p^{(1,l)} &= D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m=1}^{\infty} e^{-\pi p^2D^2m^4/q^2}\\ &\ll D\sum_{p}\sum_{q} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)\frac{(\log p)(\log q)}{\sqrt{pq}}\left(\sqrt{\frac{q}{pD}}+O(1)\right) \end{aligned}$$

$$\ll D^{1/2} \sum_p a\left(\frac{p}{x}\right) \frac{\log p}{p} \sum_q a\left(\frac{q^l}{x}\right) (\log q)$$
$$+ D \sum_p a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} \sum_q a\left(\frac{q^l}{x}\right) \frac{\log q}{\sqrt{q}}.$$

By Lemma 2.7,

$$\sum_p a\left(\frac{p}{x}\right) \frac{\log p}{p} \ll 1.$$

By the PNT,

$$\sum_q a\left(\frac{q^l}{x}\right) \log q \ll \frac{l x^{\frac{1}{l}}}{\log x} \cdot \log x \ll l x^{\frac{1}{l}}.$$

By Lemma 2.6,

$$\sum_p a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} \ll x^{1/2}.$$

By the PNT and the Stieltjes integral,

$$\sum_q a\left(\frac{q^l}{x}\right) \frac{\log q}{\sqrt{q}} \ll \int_1^\infty a\left(\frac{u^l}{x}\right) \frac{du}{\sqrt{u}} \ll \int_1^{(Bx)^{\frac{1}{l}}} \frac{du}{\sqrt{u}} \ll x^{\frac{1}{2l}}.$$

Combining these, we obtain

$$M_p^{(1,l)} \ll l D^{1/2} x^{\frac{1}{l}} + D x^{1/2+\frac{1}{2l}}. \tag{3.51}$$

Next, by Lemma 2.3,

$$\begin{aligned} M_q^{(1,l)} &= D \sum_{p\geqslant 3} \sum_{\substack{q\geqslant 3 \\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{m=1}^\infty e^{-\pi q^2 D^2 m^4/p^2} \\ &\ll D \sum_p \sum_q a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) \frac{(\log p)(\log q)}{\sqrt{pq}} \left(\sqrt{\frac{p}{qD}} + O(1)\right) \\ &\ll D^{1/2} \sum_p a\left(\frac{p}{x}\right) \log p \sum_q a\left(\frac{q^l}{x}\right) \frac{\log q}{q} \\ &\quad + D \sum_p a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} \sum_q a\left(\frac{q^l}{x}\right) \frac{\log q}{\sqrt{q}}. \end{aligned} \tag{3.52}$$

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By the PNT and the Stieltjes integral,

$$\sum_p a\left(\frac{p}{x}\right)\log p \ll x,$$

$$\sum_q a\left(\frac{q^l}{x}\right)\frac{\log q}{q} \ll \int_1^\infty a\left(\frac{u^l}{x}\right)\frac{du}{u} \ll \int_1^{(Bx)^{\frac{1}{l}}}\frac{du}{u} \ll l^{-1}\log x.$$

The last line of (3.52) is $O(Dx^{1/2+1/2l})$, as we computed in the evaluation of $M_p^{(1,l)}$. Combining these, we obtain

$$M_q^{(1,l)} \ll l^{-1}D^{1/2}x\log x + Dx^{1/2+\frac{1}{2l}}. \tag{3.53}$$

The term $M_{pq}^{(1,l)}$ is evaluated by

$$\begin{aligned} M_{pq}^{(1,l)} &= D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m=1}^\infty e^{-\pi p^2q^2D^2m^4} \\ &\ll D\sum_p a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\sum_q a\left(\frac{q^l}{x}\right)\frac{\log q}{\sqrt{q}} \\ &\ll Dx^{1/2+\frac{1}{2l}}. \end{aligned} \tag{3.54}$$

Finally, we evaluate E. First,

$$\begin{aligned} E &= D\sum_{p\geqslant 3}\sum_{\substack{q\geqslant 3\\ q\neq p}} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m\neq\square}\left(\frac{m}{pq}\right)e^{-\pi D^2m^2/p^2q^2} \\ &\ll D\left|\sum_{p\geqslant 3}\sum_{q\geqslant 3}a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m\neq\square}\left(\frac{m}{p}\right)\left(\frac{m}{q}\right)e^{-\pi D^2m^2/p^2q^2}\right| \\ &\quad + D\sum_{p\geqslant 3}a\left(\frac{p}{x}\right)a\left(\frac{p^l}{x}\right)\log^2 p\cdot\frac{1}{p}\sum_{m=1}^\infty e^{-\pi m^2D^2/p^4}. \end{aligned} \tag{3.55}$$

By the PNT and Lemma 2.1, the second term on the right-hand side of (3.55) is

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$$D\sum_{p\geqslant 3} a\left(\frac{p}{x}\right) a\left(\frac{p^l}{x}\right) \log^2 p\cdot\frac{1}{p}\sum_{m=1}^{\infty} e^{-\pi m^2D^2/p^4}$$
$$\ll D\sum_{p} a\left(\frac{p}{x}\right) a\left(\frac{p^l}{x}\right) \frac{\log^2 p}{p}\left(\frac{p^2}{D}+O(1)\right)$$
$$\ll l^{-1}x^{\frac{2}{l}}\log x + Dx^{\frac{1}{l}}. \tag{3.56}$$

Next, since

$$e^{-\pi m^2D^2/p^2q^2} \leqslant \exp(-\pi m^2D^2(Bx)^{-2-\frac{2}{l}})$$

holds for $p \leqslant Bx$, $q \leqslant (Bx)^{1/l}$, the first term on the right-hand side of (3.55) is evaluated by

$$D\left|\sum_{p\geqslant 3}\sum_{q\geqslant 3} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m\neq\square}\left(\frac{m}{p}\right)\left(\frac{m}{q}\right)e^{-\pi D^2m^4/p^2q^2}\right|$$
$$\ll D\sum_{m\neq\square} e^{-\pi m^2D^2/(Bx)^{2+\frac{2}{l}}}\left|\sum_{p} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{m}{p}\right)\right|$$
$$\times\left|\sum_{q} a\left(\frac{q^l}{x}\right)\frac{\log q}{\sqrt{q}}\left(\frac{m}{q}\right)\right|. \tag{3.57}$$

According to [13, p. 221], under the assumption of the GRH,

$$\sum_{p} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{m}{p}\right) \ll \log^2 mx$$

holds uniformly for $m \ll x$ $m\neq\square$. Moreover, since the GRH implies

$$\sum_{q\leqslant x^{\frac{1}{l}}}\left(\frac{m}{q}\right)\log q \ll x^{\frac{1}{2l}}\log^2(mx^{\frac{1}{l}})$$

uniformly for $m \ll x^{1/l}$ $m\neq\square$, we have

$$\sum_{q} a\left(\frac{q^l}{x}\right)\frac{\log q}{\sqrt{q}}\left(\frac{m}{q}\right)$$
$$= \int_0^{\infty} a\left(\frac{u^l}{x}\right)\frac{1}{\sqrt{u}}\, d\left(\sum_{q\leqslant u}\left(\frac{m}{q}\right)\log q\right)$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty \left\{\sum_{q\leqslant u}\left(\frac{m}{q}\right)\log q\right\} a\left(\frac{u^l}{x}\right) u^{-3/2}\,du\\
&\quad -\frac{l}{x}\int_0^\infty \left\{\sum_{q\leqslant u}\left(\frac{m}{q}\right)\log q\right\} a'\left(\frac{u^l}{x}\right) u^{l-\frac{3}{2}}\,du\\
&= \frac{1}{2}\int_0^\infty \left\{\sum_{q\leqslant (xv)^{\frac{1}{l}}}\left(\frac{m}{q}\right)\log q\right\} a(v)(xv)^{-\frac{3}{2l}}\cdot x^{\frac{1}{l}}l^{-1}v^{\frac{1}{l}-1}\,dv\\
&\quad -\frac{l}{x}\int_0^\infty \left\{\sum_{q\leqslant (xv)^{\frac{1}{l}}}\left(\frac{m}{q}\right)\log q\right\} a'(v)(xv)^{\frac{1}{l}(l-\frac{3}{2})}\cdot x^{\frac{1}{l}}l^{-1}v^{\frac{1}{l}-1}\,dv\\
&\ll \frac{1}{2}\int_A^B (xv)^{\frac{1}{2l}}\log^2(m(xv)^{\frac{1}{l}})a(v)(xv)^{-\frac{3}{2l}}x^{\frac{1}{l}}l^{-1}v^{\frac{1}{l}-1}\,dv\\
&\quad + lx^{-1}\int_A^B (xv)^{\frac{1}{2l}}\log^2(m(xv)^{\frac{1}{l}})a'(v)(xv)^{\frac{1}{l}(l-\frac{3}{2})}x^{\frac{1}{l}}l^{-1}v^{\frac{1}{l}-1}\,dv\\
&\ll \log^2 mx
\end{aligned}
$$

uniformly for $m \ll x^{1/l}$. On the other hand, by forgetting the Kronecker symbols, we have

$$\sum_p a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{m}{p}\right) \ll x^{1/2}, \qquad \sum_q a\left(\frac{q^l}{x}\right)\frac{\log q}{\sqrt{q}}\left(\frac{m}{q}\right) \ll x^{\frac{1}{2l}}.$$

Hence, the first term on the right-hand side of (3.55) is

$$
\begin{aligned}
&\ll D\sum_{m\leqslant x^{\frac{1}{l}}} e^{-\pi m^2D^2/(Bx)^{2+\frac{2}{l}}}\log^4 mx\\
&\quad + D\sum_{m\geqslant x^{\frac{1}{l}}} e^{-\pi m^2D^2/(Bx)^{2+\frac{2}{l}}}x^{1/2+\frac{1}{2l}}\\
&\ll x^{1+\frac{1}{l}}\log^4 x + D\log^4 x \qquad (3.58)
\end{aligned}
$$

for $D^{1/2-\delta} \ll x \ll D^{1-\delta}$. By inserting (3.56), (3.58) into (3.55), we obtain

$$E \ll Dx^{\frac{1}{l}} + x^{1+\frac{1}{l}}\log^4 x. \qquad (3.59)$$

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By combining (3.50), (3.51), (3.53), (3.54) and (3.59), for odd l and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, we have

$$M_{2,2}^{(1,l)} = ID^{1/2}l^{-1}K(1)K\left(\frac{1}{l}\right)x^{1+\frac{1}{l}} + O(l^{-1}D^{1/2}x^{1+\frac{1}{2l}}\log^2 x) + O(Dx^{1/2+\frac{1}{2l}}) + O(x^{1+\frac{1}{l}}\log^4 x). \tag{3.60}$$

By combining this and (3.46), for odd $l \geqslant 3$ and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, we have

$$M_{2}^{(1,l)} = ID^{1/2}l^{-1}K(1)K\left(\frac{1}{l}\right)x^{1+\frac{1}{l}} + O(l^{-1}D^{1/2}x^{1+\frac{1}{2l}}\log^2 x) + O(lDx^{\frac{1}{l}}\log x) + O(Dx^{1/2+\frac{1}{2l}}) + O(x^{1+\frac{1}{l}}\log^4 x). \tag{3.61}$$

If $l = 1$, $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, by (3.45) and (3.60), we have

$$M_{2}^{(1,1)} = L(1)Dx\log x + IK(1)^2D^{1/2}x^2 + O(Dx + D^{1/2}x^{3/2}\log^2 x + x^2\log^4 x). \tag{3.62}$$

On the other hand, for odd $l \geqslant 3$ and $x = o(D^{1/2})$, by (3.46) and (3.47), we have

$$M_{2}^{(1,l)} \ll lDx^{\frac{1}{l}}\log x + lx^{3/2(1+\frac{1}{l})}\log^2 x, \tag{3.63}$$

and for $l = 1$, $x = o(D^{1/2})$, by (3.45) and (3.47), we have

$$M_{2}^{(1,1)} = L(1)Dx\log x + O(Dx + x^3\log^2 x). \tag{3.64}$$

(B) Next, we consider the case that l is even. In this case, we have

$$M_{2}^{(1,l)} = \sum_{p,q} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2} - \sum_{p,q} a\left(\frac{p}{x}\right)a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{q|d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2}. \tag{3.65}$$

Since

$$\sum_{d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2} \ll \sqrt{p}\log p,$$

the first term on the right-hand side of (3.65) is

$$\begin{aligned}
&\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2}\\
&\quad\ll \sum_{p} a\left(\frac{p}{x}\right)\sqrt{p}\log^2 p\sum_{q} a\left(\frac{q^l}{x}\right)\log q\\
&\quad\ll x^{3/2}\log x\cdot lx^{\frac{1}{l}}\\
&\quad\ll lx^{3/2+\frac{1}{l}}\log x. \qquad (3.66)
\end{aligned}$$

The second term on the right-hand side of (3.65) is

$$\begin{aligned}
&\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{q|d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2}\\
&\quad= \sum_{p} a\left(\frac{p}{x}\right)(\log p)\sum_{q}\left(\frac{q}{p}\right) a\left(\frac{q^l}{x}\right)(\log q)\sum_{d}\left(\frac{d}{p}\right)e^{-\pi q^2d^2/D^2}\\
&\quad\ll \sum_{p} a\left(\frac{p}{x}\right)(\log p)\sum_{q} a\left(\frac{q^l}{x}\right)(\log q)\left|\sum_{d}\left(\frac{d}{p}\right)e^{-\pi q^2d^2/D^2}\right|. \qquad (3.67)
\end{aligned}$$

Now, since q satisfies $q\leqslant (Bx)^{1/l}\ll D^{1-\delta}\ll D$, $D/q\gg 1$ holds. Therefore, by the Pólya–Vinogradov inequality, we have

$$\sum_{d}\left(\frac{d}{p}\right)e^{-\pi q^2d^2/D^2}\ll\sqrt{p}\log p. \qquad (3.68)$$

Therefore,

$$\begin{aligned}
&\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{q|d}\left(\frac{d}{p}\right)e^{-\pi d^2/D^2}\\
&\quad\ll \sum_{p} a\left(\frac{p}{x}\right)\sqrt{p}\log^2 p\sum_{q} a\left(\frac{q^l}{x}\right)\log q\\
&\quad\ll lx^{3/2+\frac{1}{l}}\log x. \qquad (3.69)
\end{aligned}$$

By combining (3.66) and (3.69), we obtain

$$M_2^{(1,l)}\ll lx^{3/2+\frac{1}{l}}\log x \qquad (3.70)$$

for even l.

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Now, we have computed or evaluated $M_2^{(1,l)}$ for all l. If $x = o(D^{1/2})$, by (3.63) and (3.70), we have

$$\sum_{l\geqslant 2} M_2^{(1,l)} \ll Dx^{1/3}\log x + x^2\log^2 x. \tag{3.71}$$

By inserting (3.40), (3.64), (3.71) into (3.39), we obtain

$$M_2 = L(1)Dx\log x + O(Dx + x^3\log^2 x). \tag{3.72}$$

If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, by (3.61),

$$\sum_{l\geqslant 3,\mathrm{odd}} M_2^{(1,l)} \ll D^{1/2}x^{4/3} + Dx^{2/3} + x^{4/3}\log^4 x. \tag{3.73}$$

(It should be noticed that $K(1/l) \ll l$.) On the other hand, by (3.70),

$$\sum_{l\geqslant 2,\mathrm{even}} M_2^{(1,l)} \ll x^2\log x. \tag{3.74}$$

By inserting (3.40), (3.62), (3.73), (3.74) into (3.39), we obtain

$$\begin{aligned} M_2 = {} & L(1)Dx\log x + IK(1)^2D^{1/2}x^2 \\ & + O(Dx + D^{1/2}x^{3/2}\log^2 x + x^2\log^4 x) \end{aligned} \tag{3.75}$$

for $D^{1/2-\delta} \ll x \ll D^{1-\delta}$.

3.5 Conclusion

Now, we have computed or evaluated all terms appearing on the right-hand side of (3.1). We fix $\delta > 0$ sufficiently small. Then, we have the following.

(1) If $1 < x \ll D^{1/2-\delta}$, by inserting (3.2), (3.3), (3.13), (3.14), (3.15), (3.17), (3.18), (3.31), (3.34) and (3.72) into (3.1), we obtain

$$\begin{aligned} & f_K(x, D) \\ & = L(1)Dx\log x - IK(1)^2D^{1/2}x^2 - \frac{1}{2}K(1)^2x^2 + a\left(\frac{1}{x}\right)^2 D\log^2 D \\ & \quad + a\left(\frac{1}{x}\right) O(xD^{1/2}\log D + x^{3/2}\log x\log D + Dx^{1/2}\log x\log D) \\ & \quad + a\left(\frac{1}{x}\right)^2 O(D\log D) + O(x^2D^{-1/2}) + O(Dx + x^3\log^2 x) \end{aligned}$$

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$$
\begin{aligned}
&+ O(D^{1/2}x^{3/2}\log^2 x) + O(\min\{x^2, x^3D^{-1/2}\}) \\
&+ O(\min\{x^2D^{1/2}, xD^{1/2}\log x\log D\}) \\
&+ O\big(\min\big\{xD\log x\log D, xD^{1/2}\log x\log D \\
&+ x^{1/2}D\log x\log D(\log^2 D + \log^2 x)\big\}\big) \\
&+ O(\min\{xD\log D, D\log x\log^2 D\}) \\
&+ O(\min\{x^2D, D\log^2 x\log^2 D\}).
\end{aligned}
$$

(2) If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, since $a(1/x) = 0$, by inserting (3.2), (3.3), (3.13), (3.14), (3.15), (3.17), (3.18), (3.31), (3.34) and (3.75) into (3.1), we obtain

$$
\begin{aligned}
&f_K(x, D) \\
&\quad = L(1)Dx\log x - \frac{1}{2}K(1)^2x^2 \\
&\qquad + O(x^2D^{-1/2}) + O(Dx + D^{1/2}x^{3/2}\log^2 x + x^2\log^4 x) \\
&\qquad + O(D^{1/2}x^{3/2}\log^2 x) + O(\min\{x^2, x^3D^{-1/2}\}) \\
&\qquad + O(\min\{x^2D^{1/2}, xD^{1/2}\log x\log D\}) \\
&\qquad + O\big(\min\big\{xD\log x\log D, xD^{1/2}\log x\log D \\
&\qquad + x^{1/2}D\log x\log D(\log^2 D + \log^2 x)\}\big) \\
&\qquad + O(\min\{xD\log D, D\log x\log^2 D\}) \\
&\qquad + O(\min\{x^2D, D\log^2 x\log^2 D\}).
\end{aligned}
$$

By dividing both sides by $xD\log D$ and putting $x = D^\alpha$ ($\alpha > 0$), we obtain the following formulas.

(1) If $0 < \alpha \leqslant 1/2 - \delta$, we have

$$
\begin{aligned}
&F_K(\alpha, D) \\
&\quad = L(1)\alpha + a(D^{-\alpha})^2D^{-\alpha}\log D + a(D^{-\alpha})\cdot O(\alpha D^{-\alpha/2}\log D) \\
&\qquad + a(D^{-\alpha})^2\cdot O(D^{-\alpha}) + O(\min\{1, \alpha D^{-\alpha}\log^2 D\}) \\
&\qquad + O(\min\{\alpha\log D, \alpha D^{-1/2}\log D + \alpha(1+\alpha^2)D^{-\alpha/2}\log^3 D\}) \\
&\qquad + O(\min\{D^\alpha(\log D)^{-1}, \alpha^2D^{-\alpha}\log^3 D\}) + o(1).
\end{aligned}
\tag{3.76}
$$

(2) If $1/2 - \delta < \alpha \leqslant 1 - \delta$, we have

$$
F_K(\alpha, D) = L(1)\alpha + o(1). \tag{3.77}
$$

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Here, the implied constants depend only on $K(s)$ and $\delta > 0$. However, the identity (3.76) is still valid for $1/2 - \delta < \alpha \leqslant 1 - \delta$, since the terms except for the first term on the right-hand side of (3.76) are all $o(1)$ if α is large. Hence, we obtain (1.5). ∎

§4. Applications of Theorem 1.1

We extend $F_K(\alpha, D)$ to the whole of $\alpha \in \mathbf{R}$ by $F_K(\alpha, D) := F_K(-\alpha, D)$ for $\alpha < 0$. Then, the identity (1.5) holds for $0 < |\alpha| \leqslant 1 - \delta$, by replacing α on the right-hand side with $|\alpha|$. To investigate the low-lying zeros of quadratic L-functions, we consider the integral of $F_K(\alpha, D)$ multiplied by some bounded function. First, we prove that in this integral, the contribution of the error terms in (1.5) is small.

LEMMA 4.1. *We have*

$$\int_{-1}^{1} |\alpha| D^{-|\alpha|/2} \log D \, d\alpha \ll \frac{1}{\log D}, \tag{4.1}$$

$$\int_{-1}^{1} D^{-|\alpha|} \, d\alpha \ll \frac{1}{\log D}, \tag{4.2}$$

$$\int_{-1}^{1} \min\{1, |\alpha| D^{-|\alpha|} \log^2 D\} \, d\alpha \ll \frac{1}{\sqrt{\log D}}, \tag{4.3}$$

$$\int_{-1}^{1} \min\{D^{|\alpha|} (\log D)^{-1}, \alpha^2 D^{-|\alpha|} \log^3 D\} \, d\alpha \ll \frac{(\log\log D)^2}{\log D}, \tag{4.4}$$

$$\begin{aligned}\int_{-1}^{1} \min\{|\alpha| \log D, |\alpha| D^{-1/2} \log D + |\alpha|(1+\alpha^2) D^{-|\alpha|/2} \log^3 D\} \, d\alpha \\ \ll (\log D)^{-1/3}.\end{aligned} \tag{4.5}$$

Proof. The estimates (4.1), (4.2) follow from direct computations of integrals. The estimate (4.3) follows from

$$\begin{aligned}&\int_{-1}^{1} \min\{1, |\alpha| D^{-|\alpha|} \log^2 D\} \, d\alpha \\ &\quad \ll \int_{0}^{1/\sqrt{\log D}} 1 \, d\alpha + \int_{\frac{1}{\sqrt{\log D}}}^{1} \alpha D^{-\alpha} \log^2 D \, d\alpha\end{aligned}$$

and

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$$\int_0^{1/\sqrt{\log D}} 1\, d\alpha \ll \frac{1}{\sqrt{\log D}},$$
$$\int_{\frac{1}{\sqrt{\log D}}}^1 \alpha D^{-\alpha} \log^2 D\, d\alpha \ll D^{-1/\sqrt{\log D}} \log^{1/2} D.$$

The estimate (4.4) follows from

$$\int_{-1}^{1} \min\{D^{|\alpha|}(\log D)^{-1}, \alpha^2 D^{-|\alpha|} \log^3 D\}\, d\alpha$$
$$\ll \int_0^{\log\log D/\log D} D^{\alpha}(\log D)^{-1}\, d\alpha + (\log^3 D) \int_{\frac{\log\log D}{\log D}}^1 \alpha^2 D^{-\alpha}\, d\alpha$$

and

$$\int_0^{\log\log D/\log D} D^{\alpha}(\log D)^{-1}\, d\alpha \ll \frac{\log\log D}{\log D},$$
$$(\log^3 D) \int_{\frac{\log\log D}{\log D}}^1 \alpha^2 D^{-\alpha}\, d\alpha \ll \frac{(\log\log D)^2}{\log D}.$$

Finally, the left-hand side of (4.5) is at most

$$2\int_0^{(\log D)^{-2/3}} \alpha \log D\, d\alpha$$
$$+ 2\int_{(\log D)^{-2/3}}^1 \{\alpha D^{-1/2}\log D + \alpha(1+\alpha^2)D^{-\alpha/2}\log^3 D\}\, d\alpha,$$

and by direct computations, we easily find that these integrals are at most $O((\log D)^{-1/3})$. □

Next, we mention the L-functions associated to Kronecker symbols. If $d \not\equiv 3 \pmod 4$, $\chi_d = (d/\cdot)$ becomes a Dirichlet character modulo $4|d|$ or $|d|$. In this case, we denote the conductor of χ_d by d^*. If $d \equiv 3 \pmod 4$, by the reciprocity law of Kronecker symbols, we find that

$$\chi_d(p^k) = \begin{cases} \left(\dfrac{p}{d}\right)^k & (p \equiv 1 \pmod 4 \text{ or } p = 2) \\ (-1)^k \left(\dfrac{p}{d}\right)^k & (p \equiv 3 \pmod 4) \end{cases}$$

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for primes p. Hence, the L-function associated to χ_d is expressed by

$$L(s,\chi_d)=\frac{1}{1-\left(\frac{2}{d}\right)2^{-s}}\prod_{p\geqslant 3}\frac{1}{1-\eta_4(p)\left(\frac{p}{d}\right)p^{-s}},$$

where η_4 is the nonprincipal character of modulo 4. In this case, we denote the conductor of $\eta_4(\cdot)(\cdot/d)$ by d^*. Let $N(\chi_d,T)$ denote the number of zeros of $L(s,\chi_d)$ in the rectangle $0<\sigma<1$, $-T\leqslant t\leqslant T$. Then, it is well known that

$$N(\chi_d,T)=\frac{T}{\pi}\log\frac{d^*T}{2\pi e}+O\left(\frac{\log(d^*T)}{\log\log(d^*T+3)}\right)$$

holds uniformly for $d^*T>1$ (for example, see [17]). Then, by partial integration, we have

$$\begin{aligned}\sum_{\rho\in Z_d}|K(\rho)|^2&=\int_0^\infty\left|K\left(\frac{1}{2}+it\right)\right|^2\,dN(\chi_d,t)\\ &=\frac{\log d^*}{\pi}\int_0^\infty\left|K\left(\frac{1}{2}+it\right)\right|^2\,dt+O_K\left(\frac{\log d^*}{\log\log d^*}\right).\end{aligned}\tag{4.6}$$

We define the constants A_+^*, A_-^* by

$$\begin{aligned}A_-^*&=\liminf_{D\to\infty}\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\log d^*,\\ A_+^*&=\limsup_{D\to\infty}\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\log d^*.\end{aligned}\tag{4.7}$$

It should be noticed that since $\log d^*\leqslant\log d+O(1)$, we have $A_+^*\leqslant 1$.

Corollary 4.2. *Assuming the GRH, for any $C>0$, $0\leqslant\mu<2\pi$, $\epsilon>0$, we have*

$$\begin{aligned}&\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\#\left\{\rho_1,\rho_2\in Z_d\,\middle|\,|\mathrm{Im}(\rho_i)|\leqslant C\ (i=1,2),\right.\\ &\qquad\qquad\left.0<|\gamma_1-\gamma_2|\leqslant\frac{\mu}{\log D}\right\}\\ &\leqslant\frac{1}{\sin^4 1}\left\{\left(\frac{4}{9}C+\epsilon\right)g(\mu)-B_-^*C+\epsilon\right\}.\end{aligned}\tag{4.8}$$

Here,

$$g(\mu)=\left(\frac{\frac{\mu}{2}}{\sin\frac{\mu}{2}}\right)^2,\qquad B_-^*=\frac{1}{3}A_-^*,$$

and A_-^* is the constant given by (4.7).

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Proof. We fix λ such that $1/2 < \lambda \leqslant 1-\delta$, and put

$$r(u)=\left(\frac{\sin \pi\lambda u}{\pi\lambda u}\right)^2.$$

Then, its Fourier transform is given by

$$\hat{r}(\alpha)=\lambda^{-2}\max\{\lambda-|\alpha|, 0\}.$$

In particular, $\hat{r}(\alpha)$ is bounded and has a support in $[-1+\delta, 1-\delta]$. Therefore, by (1.5) and Lemma 4.1, we have

$$\int_{-\infty}^{\infty} F_K(\alpha, D)\hat{r}(\alpha)\,d\alpha$$

$$= L(1)\int_{-\infty}^{\infty}|\alpha|\hat{r}(\alpha)\,d\alpha+(\log D)\int_{-\infty}^{\infty}a(D^{-|\alpha|})^2D^{-|\alpha|}\hat{r}(\alpha)\,d\alpha+o(1). \tag{4.9}$$

The implied constant is dependent only on $K(s)$ and λ. By a direct computation, the first integral on the right-hand side of (4.9) is

$$\int_{-\infty}^{\infty}|\alpha|\hat{r}(\alpha)\,d\alpha=\frac{\lambda}{3}. \tag{4.10}$$

On the other hand, the second integral of the right hand side of (4.9) is

$$\begin{aligned}
&\int_{-\infty}^{\infty}a(D^{-|\alpha|})^2D^{-|\alpha|}\hat{r}(\alpha)\,d\alpha\\
&\quad=2\int_{0}^{\infty}a(D^{-\alpha})^2D^{-\alpha}\hat{r}(\alpha)\,d\alpha\\
&\quad=2\lambda^{-2}\int_{0}^{\lambda}(\lambda-\alpha)a(D^{-\alpha})^2D^{-\alpha}\,d\alpha\\
&\quad=2\lambda^{-1}\int_{0}^{\lambda}a(D^{-\alpha})^2D^{-\alpha}\,d\alpha-2\lambda^{-2}\int_{0}^{\lambda}a(D^{-\alpha})^2\alpha D^{-\alpha}\,d\alpha. \qquad (4.11)
\end{aligned}$$

By the change of parameters $t=D^{-\alpha}$, the first integral on the right-hand side of (4.11) is

$$\begin{aligned}
\int_{0}^{\lambda}a(D^{-\alpha})^2D^{-\alpha}\,d\alpha&=\int_{0}^{\infty}a(D^{-\alpha})^2D^{-\alpha}\,d\alpha\\
&=\frac{1}{\log D}\int_{0}^{1}a(t)^2\,dt\\
&=\frac{M}{\log D}, \qquad (4.12)
\end{aligned}$$

where

$$M = M_K := \int_0^1 a(t)^2\,dt. \tag{4.13}$$

On the other hand, since $a(D^{-\alpha})^2$ is bounded, we have

$$\int_0^\lambda a(D^{-\alpha})^2 \alpha D^{-\alpha} d\alpha \ll \int_0^\infty \alpha D^{-\alpha}\,d\alpha \ll \frac{1}{\log^2 D}. \tag{4.14}$$

By inserting (4.12), (4.14) into (4.11), we have

$$\int_{-\infty}^\infty a(D^{-|\alpha|})^2 D^{-|\alpha|}\hat{r}(\alpha)\,d\alpha = \frac{2M}{\lambda \log D} + O\left(\frac{1}{\log^2 D}\right). \tag{4.15}$$

By inserting (4.10), (4.15) into (4.9), we obtain

$$\int_{-\infty}^\infty F_K(\alpha, D)\hat{r}(\alpha)\,d\alpha = \frac{\lambda}{3}L(1) + \frac{2}{\lambda}M + o(1). \tag{4.16}$$

For $C > 0$, we take

$$K(s) = K_C(s) = C^2\left(\frac{e^{(s-(1/2))/C} - e^{-(s-(1/2))/C}}{2s-1}\right)^2.$$

Then, since $K(1/2+it) = C^2(t^{-1}\sin(t/C))^2$, $K(s)$ is real on the line $\mathrm{Re}(s) = 1/2$ and satisfies $K(1/2+it_1)\overline{K(1/2+it_2)} \geqslant 0$ for all $t_1, t_2 \in \mathbf{R}$. Moreover, by an easy computation, its Mellin inverse transform $a(x) = a_C(x)$, defined in (1.2), is given by

$$a(x) = \begin{cases} 0 & (x < e^{-2/C} \text{ or } x \geqslant e^{2/C}) \\ \dfrac{C^2}{2}x^{-1/2}\left(\dfrac{1}{C} + \dfrac{1}{2}\log x\right) & (e^{-2/C} \leqslant x < 1) \\ \dfrac{C^2}{2}x^{-1/2}\left(\dfrac{1}{C} - \dfrac{1}{2}\log x\right) & (1 \leqslant x < e^{2/C}). \end{cases}$$

This function surely satisfies the conditions given in Section 1. By the definition of $F_K(\alpha, D)$, we have

$$\begin{aligned}&\int_{-\infty}^\infty F_K(\alpha, D)\hat{r}(\alpha)\,d\alpha \\ &= \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2} \sum_{\rho_1,\rho_2 \in Z_d} K(\rho_1)\overline{K(\rho_2)} \int_{-\infty}^\infty D^{i\alpha(\gamma_1-\gamma_2)}\hat{r}(\alpha)\,d\alpha\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)}\, r\left(\frac{(\gamma_1-\gamma_2)\log D}{2\pi}\right)\\
&= \frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)} \left(\frac{\sin\left(\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}\right)}{\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}}\right)^2 .
\end{aligned}
\tag{4.17}
$$

We fix a real number μ satisfying $0\leqslant\mu<2\pi$. Since the function $f(x):=((\sin x)/x)^2$ is even and decreasing in $[0,\pi]$, the inequality

$$
f\left(\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}\right)\geqslant f\left(\frac{\mu}{2}\right)
$$

holds if γ_1, γ_2 satisfy $|\gamma_1-\gamma_2|\leqslant\mu/(\lambda\log D)$. Therefore, by (4.17), we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha,D)\hat{r}(\alpha)\,d\alpha\\
&\quad\geqslant \frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\substack{\rho_1,\rho_2\in Z_d\\ |\gamma_1-\gamma_2|\leqslant\frac{\mu}{\lambda\log D}}} K(\rho_1)\overline{K(\rho_2)} f\left(\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}\right)\\
&\quad\geqslant \frac{f\left(\frac{\mu}{2}\right)}{D\log D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\substack{\rho_1,\rho_2\in Z_d\\ |\gamma_1-\gamma_2|\leqslant\frac{\mu}{\lambda\log D}}} K(\rho_1)\overline{K(\rho_2)}.
\end{aligned}
\tag{4.18}
$$

Now, we have

$$
\begin{aligned}
&\sum_{\substack{\rho_1,\rho_2\in Z_d\\ |\gamma_1-\gamma_2|\leqslant\frac{\mu}{\lambda\log D}}} K(\rho_1)\overline{K(\rho_2)}\\
&\qquad\geqslant \sum_{\substack{\rho_1,\rho_2\in Z_d\\ 0<|\gamma_1-\gamma_2|\leqslant\frac{\mu}{\lambda\log D}\\ |\mathrm{Im}(\rho_1)|,|\mathrm{Im}(\rho_2)|\leqslant C}} K(\rho_1)\overline{K(\rho_2)} + \sum_{\rho\in Z_d}|K(\rho)|^2.
\end{aligned}
\tag{4.19}
$$

By (4.6), the second term on the right-hand side of (4.19) is given by

$$
\sum_{\rho\in Z_d}|K(\rho)|^2 = \frac{\log d^*}{\pi}C^4\int_0^\infty \left(\frac{\sin\frac{t}{C}}{t}\right)^4 dt + O_C\left(\frac{\log d^*}{\log\log d^*}\right)
$$

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$$= \frac{C \log d^*}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt + O_C\left(\frac{\log d^*}{\log\log d^*}\right)$$

$$= \frac{C}{3} \log d^* + O_C\left(\frac{\log d^*}{\log\log d^*}\right). \tag{4.20}$$

On the other hand, since

$$\min_{|y|\leqslant C} \left|K\left(\frac{1}{2}+iy\right)\right|^2 = \sin^4 1,$$

the first term on the right-hand side of (4.19) is

$$\sum_{\substack{\rho_1,\rho_2\in Z_d\\ 0<|\gamma_1-\gamma_2|\leqslant \frac{\mu}{\lambda\log D}\\ |\mathrm{Im}(\rho_1)|,|\mathrm{Im}(\rho_2)|\leqslant C}} K(\rho_1)\overline{K(\rho_2)}$$
$$\geqslant (\sin^4 1)\#\left\{\rho_1,\rho_2\in Z_d \,\middle|\, |\mathrm{Im}(\rho_i)|\leqslant C\ (i=1,2),\ 0<|\gamma_1-\gamma_2|\leqslant \frac{\mu}{\lambda\log D}\right\}$$
$$\geqslant (\sin^4 1)\#\left\{\rho_1,\rho_2\in Z_d \,\middle|\, |\mathrm{Im}(\rho_i)|\leqslant C\ (i=1,2),\ 0<|\gamma_1-\gamma_2|\leqslant \frac{\mu}{\log D}\right\}.$$

Hence, by (4.18), we obtain

$$\int_{-\infty}^{\infty} F_K(\alpha, D)\hat{r}(\alpha)\,d\alpha$$
$$\geqslant \frac{f\left(\frac{\mu}{2}\right)}{D\log D}\sum_d e^{-\pi d^2/D^2}$$
$$\times\left\{(\sin^4 1)\#\left\{\rho_1,\rho_2\in Z_d \,\middle|\, |\mathrm{Im}(\rho_i)|\leqslant C\ (i=1,2),\right.\right.$$
$$\left.\left. 0<|\gamma_1-\gamma_2|\leqslant\frac{\mu}{\log D}\right\} + \frac{C}{3}\log d^* + O_C\left(\frac{\log d^*}{\log\log d^*}\right)\right\}. \tag{4.21}$$

Combining (4.16) and (4.21), we obtain

$$\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\#\left\{\rho_1,\rho_2\in Z_d \,\middle|\, |\mathrm{Im}(\rho_i)|\leqslant C\ (i=1,2),\right.$$
$$\left. 0<|\gamma_1-\gamma_2|\leqslant\frac{\mu}{\log D}\right\}$$

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$$\leqslant \frac{1}{f\left(\frac{\mu}{2}\right)\sin^4 1}\left\{\frac{\lambda}{3}L(1)+\frac{2}{\lambda}M\right.$$

$$\left. -\frac{f\left(\frac{\mu}{2}\right)}{D\log D}\cdot\frac{C}{3}\sum_d e^{-\pi d^2/D^2}\log d^* + o(1)\right\}. \tag{4.22}$$

By direct computation, we find that

$$L(1)=\int_0^\infty a(x)^2\,dx=\frac{C}{3}, \qquad M=\int_0^1 a(x)^2\,dx=\frac{C}{6}.$$

Moreover, by our assumption,

$$\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\log d^* \geqslant A_-^*.$$

By inserting these results into (4.22) and putting $\lambda = 1-\delta$ with sufficiently small $\delta > 0$, we obtain (4.8). ∎

Next, we give a certain lower bound for the rate of simple zeros of quadratic L-functions.

Corollary 4.3. *We assume the GRH. For any $\epsilon > 0$, we have*

$$\sum_d e^{-\pi d^2/D^2}\sum_{\substack{\rho\in Z_d\\ \text{simple}}} K(\rho)^2$$

$$\geqslant \left(2B_-^* - \frac{4}{9} - \epsilon\right)(B_+^*)^{-1}\sum_d e^{-\pi d^2/D^2}\sum_{\rho\in Z_d} K(\rho)^2 \tag{4.23}$$

when $D > 1$ is sufficiently large. Here,

$$K(s)=K_1(s)=\left(\frac{e^{s-\frac{1}{2}}-e^{-s+\frac{1}{2}}}{2s-1}\right)^2, \qquad B_\pm^* = \frac{1}{3}A_\pm^*,$$

and $A_\pm^*$ are the constants defined by (4.7).

Proof. In the proof of Corollary 4.2, we showed that

$$\frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)}\left(\frac{\sin\left(\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}\right)}{\frac{(\gamma_1-\gamma_2)\lambda\log D}{2}}\right)^2$$

$$= \int_{-\infty}^{\infty} F_K(\alpha, D)\hat{r}(\alpha)\, d\alpha$$

$$= \frac{\lambda}{3}L(1) + \frac{2}{\lambda}M + o(1) \tag{4.24}$$

holds for $\lambda = 1 - \delta$. Moreover, in this case, we have

$$L(1) = \int_0^{\infty} a(x)^2\, dx = \frac{1}{3}, \qquad M = \int_0^{1} a(x)^2\, dx = \frac{1}{6},$$

since $C = 1$. Therefore,

$$\frac{1}{D \log D} \sum_d e^{-\pi d^2/D^2} \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1)\overline{K(\rho_2)} \left(\frac{\sin\left(\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}\right)}{\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}} \right)^2$$

$$= \frac{\lambda}{9} + \frac{1}{3\lambda} + o(1). \tag{4.25}$$

Let $m_{\rho,d}$ be the multiplicity of the zero of $L(s, \chi_d)$ at $\rho = 1/2 + i\gamma$. Then,

$$\sum_{\substack{\rho \in Z_d \\ \text{simple}}} K(\rho)^2 \geqslant \sum_{\rho \in Z_d} (2 - m_{\rho,d}) K(\rho)^2$$

$$\geqslant 2 \sum_{\rho \in Z_d} K(\rho)^2 - \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1)K(\rho_2) \left(\frac{\sin\left(\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}\right)}{\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}} \right)^2 .$$

Hence, we obtain

$$\frac{1}{D \log D} \sum_d e^{-\pi d^2/D^2} \sum_{\substack{\rho \in Z_d \\ \text{simple}}} K(\rho)^2$$

$$\geqslant \frac{2}{D \log D} \sum_d e^{-\pi d^2/D^2} \sum_{\rho \in Z_d} K(\rho)^2$$

$$- \frac{1}{D \log D} \sum_d e^{-\pi d^2/D^2}$$

$$\times \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1)K(\rho_2) \left(\frac{\sin\left(\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}\right)}{\frac{(\gamma_1 - \gamma_2)\lambda \log D}{2}} \right)^2 . \tag{4.26}$$

The second term on the right-hand side is given by (4.25), since $K(\rho_2)$ is real. On the other hand, the first term is

$$\begin{aligned}
&\frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2}\sum_{\rho\in Z_d} K(\rho)^2\\
&= \frac{1}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt \cdot \frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2}\log d^* + o(1)\\
&\geqslant \frac{1}{3}A_-^* + o(1)\\
&= B_-^* + o(1). \qquad (4.27)
\end{aligned}$$

Therefore, by inserting (4.25), (4.27) into (4.26) and taking $\delta > 0$ sufficiently small, we obtain

$$\frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2}\sum_{\substack{\rho\in Z_d\\ \text{simple}}} K(\rho)^2 \geqslant 2B_-^* - \frac{4}{9} - \epsilon.$$

By combining this and

$$\frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2}\sum_{\rho\in Z_d} K(\rho)^2 \leqslant B_+^* + o(1),$$

we obtain (4.23). ∎

COROLLARY 4.4. *We assume the GRH, and that all zeros of $L(s,\chi_d)$ are simple. For $0<\lambda<1$, we have*

$$\begin{aligned}
&\frac{1}{D\log D}\sum_{d} e^{-\pi d^2/D^2}\sum_{\substack{\rho_1,\rho_2\in Z_d\\ 0<|\gamma_1-\gamma_2|\leqslant \frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2)\\
&\geqslant \frac{2}{3}\lambda - \frac{2}{9}\lambda^2 - \frac{\cos 2\pi\lambda}{6\pi^2} + \frac{\sin 2\pi\lambda}{12\pi^3\lambda} - B_+^* + o(1) \qquad (4.28)
\end{aligned}$$

as $D\to\infty$, where $B_+^* = A_+^*/3$ and

$$K(s) = K_1(s) = \left(\frac{e^{s-(1/2)} - e^{-s+(1/2)}}{2s-1}\right)^2.$$

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In particular, if $\lambda > \lambda_0 = 0.6073$*, we have*

$$\sum_{d} e^{-\pi d^2/D^2} \sum_{\substack{\rho_1,\rho_2 \in Z_d \\ 0<|\gamma_1-\gamma_2|\leqslant \frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2) \gg D\log D \tag{4.29}$$

as $D \to \infty$.

Proof. Instead of the function $r(u)$ used in the proof of Corollary 4.2, we use the Selberg minorant

$$h(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2 \frac{1}{1-u^2}.$$

This function is bounded and satisfies $h(u) \leqslant 1$, $h(u) < 0$ if $|u| > 1$. The Fourier transform of $h(u)$ is given by

$$\hat{h}(\alpha) = \begin{cases} 1 - |\alpha| + \dfrac{\sin 2\pi|\alpha|}{2\pi} & (|\alpha| \leqslant 1) \\ 0 & (|\alpha| > 1) \end{cases}$$

(for example, see [5]). For $0 < \lambda < 1$, we give lower and upper bounds for the integral

$$\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda\hat{h}(\lambda\alpha)\, d\alpha.$$

First, since the integrant is nonnegative and $1/\lambda > 1$, by (1.5), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda\hat{h}(\lambda\alpha)\, d\alpha \\
&\quad = \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} F_K(\alpha, D) \cdot \lambda\hat{h}(\lambda\alpha)\, d\alpha \\
&\quad \geqslant \int_{-1}^{1} F_K(\alpha, D) \cdot \lambda\hat{h}(\lambda\alpha)\, d\alpha \\
&\quad = \lambda L(1) \int_{-1}^{1} |\alpha| \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha \\
&\qquad + \lambda \log D \int_{-1}^{1} a(D^{-|\alpha|})^2 D^{-|\alpha|} \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha + o(1).
\end{aligned} \tag{4.30}$$

The first term on the right-hand side of (4.30) is

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$$
\begin{aligned}
&\lambda L(1) \int_{-1}^{1} |\alpha| \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha \\
&= 2\lambda L(1) \int_{0}^{1} \left(\alpha - \lambda\alpha^2 + \frac{\alpha \sin 2\pi\lambda\alpha}{2\pi}\right) d\alpha \\
&= 2\lambda L(1) \left\{\frac{1}{2} - \frac{\lambda}{3} - \frac{\cos 2\pi\lambda}{4\pi^2\lambda} + \frac{\sin 2\pi\lambda}{8\pi^3\lambda^2}\right\}. \qquad (4.31)
\end{aligned}
$$

Next, we compute the second term on the right-hand side of (4.30). Since

$$
1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi} = 1 + O\left(\frac{1}{\log^3 D}\right)
$$

for $|\alpha| \ll 1/(\log D)$, we have

$$
\begin{aligned}
&\lambda \log D \int_{-1}^{1} a(D^{-|\alpha|})^2 D^{-|\alpha|} \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha \\
&= 2\lambda \log D \int_{0}^{\infty} a(D^{-\alpha})^2 D^{-\alpha} \left\{1 + O\left(\frac{1}{\log^3 D}\right)\right\} d\alpha.
\end{aligned}
$$

By the change of parameters $D^{-\alpha} = v$, we have

$$
\int_{0}^{\infty} a(D^{-\alpha})^2 D^{-\alpha}\, d\alpha = \frac{1}{\log D} \int_{0}^{1} a(v)^2\, dv = \frac{M}{\log D},
$$

where $M = \int_0^1 a(v)^2\, dv$. Therefore,

$$
\lambda \log D \int_{-1}^{1} a(D^{-|\alpha|})^2 D^{-|\alpha|} \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha = 2M\lambda + o(1). \qquad (4.32)
$$

By inserting (4.31), (4.32) into (4.30), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda\hat{h}(\lambda\alpha)\, d\alpha \\
&\geqslant 2\lambda L(1) \left\{\frac{1}{2} - \frac{\lambda}{3} - \frac{\cos 2\pi\lambda}{4\pi^2\lambda} + \frac{\sin 2\pi\lambda}{8\pi^3\lambda^2}\right\} + 2M\lambda + o(1) \\
&= \frac{2}{3}\lambda - \frac{2}{9}\lambda^2 - \frac{\cos 2\pi\lambda}{6\pi^2} + \frac{\sin 2\pi\lambda}{12\pi^3\lambda} + o(1), \qquad (4.33)
\end{aligned}
$$

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since $L(1)=1/3$, $M=1/6$. On the other hand, we have

$$\int_{-\infty}^{\infty} F_K(\alpha,D)\cdot\lambda\hat{h}(\lambda\alpha)\,d\alpha$$
$$= \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)K(\rho_2)h\left(\frac{(\gamma_1-\gamma_2)\log D}{2\pi\lambda}\right). \tag{4.34}$$

Now, since $h((\gamma_1-\gamma_2)\log D/(2\pi\lambda))$ is negative if $|\gamma_1-\gamma_2|>(2\pi\lambda)/\log D$, by (4.34), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha,D)\cdot\lambda\hat{h}(\lambda\alpha)\,d\alpha\\
&\leqslant \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\substack{\rho_1,\rho_2\in Z_d\\ |\gamma_1-\gamma_2|\leqslant\frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2)h\left(\frac{(\gamma_1-\gamma_2)\log D}{2\pi\lambda}\right)\\
&= \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\rho\in Z_d} K(\rho)^2\\
&\quad + \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\substack{\rho_1,\rho_2\in Z_d\\ 0<|\gamma_1-\gamma_2|\leqslant\frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2)h\left(\frac{(\gamma_1-\gamma_2)\log D}{2\pi\lambda}\right)\\
&\leqslant B_+^* + \frac{1}{D\log D}\sum_d e^{-\pi d^2/D^2}\sum_{\substack{\rho_1,\rho_2\in Z_d\\ 0<|\gamma_1-\gamma_2|\leqslant\frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2)+o(1).
\end{aligned}$$

By combining (4.33) and the above, we obtain (4.28). Since $B_+^*=A_+^*/3\leqslant 1/3$, the right-hand side of (4.28) becomes positive if $\lambda>\lambda_0=0.6073$. Therefore, we obtain (4.29). ∎

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