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ON AN AUTOMORPHISM OF Hilb[2] OF CERTAIN K3 SURFACES

EKATERINA AMERIK

Laboratoire de Mathématiques, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France (ekaterina.amerik@math.u-psud.fr) *and MPIM Bonn, Vivatsgasse 7, 53111 Bonn, Germany*

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Abstract Following some remarks made by O'Grady and Oguiso, the potential density of rational points on the second punctual Hilbert scheme of certain K3 surfaces is proved.

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1. Introduction

In a recent note, Oguiso [**9**] uses the structure of the cohomology of compact hyperkähler manifolds [**12**] to describe the behaviour of the dynamical degrees of an automorphism of such a manifold and makes an explicit computation in some particular cases. In one of his examples he considers a K3 surface S admitting two embeddings as a quartic in $\mathbb{P}^3$ (given by two different very ample line bundles H_1 and H_2). Each embedding induces an involution of the second punctual Hilbert scheme (that is, the Hilbert scheme parametrizing finite subschemes of length 2) $X = \mathrm{Hilb}^{[2]} S = S^{[2]}$, where a pair of general points p_1, p_2 is sent to the complement of $\{p_1, p_2\}$ in the intersection of S and the line $p_1 p_2$; it is shown in [**2**] that this involution is regular if and only if S does not contain lines. Oguiso considers the product of the two involutions and shows that this product is not induced from an automorphism of S, nor from any automorphism of a K3 surface S' such that $S'^{[2]} \cong S^{[2]}$.

On the other hand, in the recent past a few people have studied the potential density of rational points on K3 surfaces and their symmetric powers (see, for example, [**4**, **5**]). Recall that a variety X over a number field is called potentially dense if rational points on X become Zariski-dense after a finite field extension. In [**4**], it is proved that a K3 surface with an elliptic pencil or with infinitely many automorphisms is potentially dense. The proof proceeds by iterating rational curves by the automorphisms in the second case and by rational self-maps coming from the elliptic fibration (i.e. by fibrewise multiplication by a suitable number) in the first case. Several results concerning potential density on symmetric powers of K3 surfaces are given in [**5**]; the point to note here is that some of these symmetric products admit abelian fibrations with a suitable potentially dense multisection, which again can be iterated.

One might ask whether the example of Oguiso leads to a new potential density result. In fact, taken as it is in [**9**], it does not: indeed, Oguiso starts with a K3 surface S of Picard number 2 whose intersection form represents neither 0 nor -2 on the Neron–Severi group, and for such S the group of automorphisms is infinite (see the example in § 7 of [**10**]). So the potential density is known already for S, and *a fortiori* for its second punctual Hilbert scheme X. The purpose of this paper is to remark that an obvious modification of Oguiso's example gives K3 surfaces of Picard number 2 that carry (-2)-curves (and therefore have finite automorphism group by [**10**]) and no elliptic fibration but that still admit two different embeddings as quartics in $\mathbb{P}^3$. For such surfaces, potential density of the second punctual Hilbert scheme X can indeed be proved using the product of the two involutions, whereas the intersection form on the Neron–Severi group of X does not represent zero, and so there is no abelian fibration. In fact, there is not even any rational abelian fibration [**1**, § 3] and thus the argument of [**5**] does not apply.

In fact, Oguiso interprets a remark from an earlier work of O'Grady [**8**, § 4.4], where the author works in a much more general situation and proposes that the symplectic manifolds equipped with two involutions that satisfy certain properties are plausible candidates for a proof of potential density. However, in the explicit example given in [**8**], which is $S^{[2]}$ for S a general two-dimensional linear section of the Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^3$, $\mathrm{Aut}(S)$ is again infinite by [**10**], since the intersection form on the Neron–Severi lattice of S does not represent 0 or -2.

2. The example

We consider the binary quadratic form $b(x, y) = 4x^2 + 14xy + 4y^2$ (in Oguiso's note, the form is $b'(x, y) = 4x^2 + 16xy + 4y^2$, not representing -2; we have chosen ours so that it does, and the results below are also valid for other quadratic forms $b_a(x, y) = 4x^2 + 2axy + 4y^2$, $a \geqslant 7$, representing -2: see the remark at the end of this paper). This is an even indefinite form, so by [**7**] there are K3 surfaces for which b is the intersection form on the Neron–Severi lattice. Let S be such a K3 surface and let X be the second punctual Hilbert scheme of S, so that $\mathrm{NS}(X) \cong \mathrm{NS}(S) \oplus \mathbb{Z}E$, where E is one half of the class of the exceptional divisor of the projection $X \to S^{(2)}$, where $S^{(2)}$ denotes the symmetric square of S. On $\mathrm{NS}(X)$, we have the Beauville–Bogomolov quadratic form q, defined up to a constant. It is well known [**3**] that the direct sum decomposition is orthogonal with respect to q and that for a suitable choice of the constant, q restricts as the intersection form to $\mathrm{NS}(S)$ and $q(E) = -2$.

Proposition 2.1.

(1) *S does not carry an elliptic pencil but has (-2)-curves; in particular,* $\mathrm{Aut}(S)$ *is finite.*

(2) *X is not rationally fibred in abelian surfaces (or in other varieties of non-maximal Kodaira dimension).*

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Proof. The first part is immediate ($(3,-1)$ being an example of a (-2)-class) except for the finiteness of $\mathrm{Aut}(S)$, which is proved in [**10**, § 7]. For the second part, one remarks that if there is such a fibration, then the form q represents zero on $\mathrm{NS}(X)$ [**1**]. This is the same as saying that the equation $B(x,y)=2m^2$ has integer solutions. But $33y^2+8m^2$ would then be a square, which is impossible (as, for example, counting modulo 3 shows). □

Proposition 2.2. *On S, there are two classes of ample line bundles with self-intersection 4. Those classes are very ample and the surface S does not contain lines in the corresponding projective embeddings.*

Proof. Let h_1, h_2 be the classes of the line bundles corresponding to the vectors $(1,0)$ and $(0,1)$ of the base in which we have written the intersection form. We can immediately see that the intersection of a nodal class (i.e. a class with self-intersection -2) with h_1 or h_2 cannot be zero. Therefore, using Picard–Lefschetz reflections (that is, reflections associated to the (-2)-curves) if necessary, we may assume that h_1 is ample (the ample cone is a chamber of the positive cone with respect to Picard–Lefschetz reflections). To show that h_2 is also ample, it is enough to verify that the intersection of h_1 with every nodal class has the same sign as the intersection of h_2 with that class.

The nodal classes are (x,y) satisfying $2x^2+7xy+2y^2+1=0$, so one has $x=\frac{1}{4}(-7y\pm t)$, where $t^2=33y^2-8$, $t>0$. The intersection of $(\frac{1}{4}(-7y+t),y)$ with $H_1=(1,0)$ is now equal to t, so we must verify that $\frac{7}{4}(-7y+t)+4y>0$, or $7t-33y>0$. But it is immediate from $t^2=33y^2-8$, $t>0$, that $t\geqslant 5y$ (and the equality holds only for $t=5$, $y=1$), so the ampleness is proved.

The very ampleness is a consequence of the results of [**11**] (see § 2.7 for the absence of base components and Theorem 5.2 for very ampleness; here we use the fact that S does not carry an elliptic pencil). The non-existence of lines can be deduced from the same calculation as the ampleness. □

Corollary 2.3. *The second punctual Hilbert scheme $X=S^{[2]}$ has two regular involutions ι_1, ι_2, corresponding to the two embeddings of S by h_1 and h_2.*

(See [**2**].)

Proposition 2.4. *There exist K3 surfaces defined over a number field whose intersection form on the Neron–Severi lattice is as above.*

Proof. Such a K3 surface over $\mathbb{C}$ is a general member of a component of the Noether–Lefschetz locus of the family of quartics in $\mathbb{P}^3$. Those components (consisting of quartics containing a curve of genus 3 and degree 7) are algebraic subvarieties defined over a number field. The only problem now is that it could, *a priori*, happen that every quartic from this locus that is defined over a number field has higher Picard number; but this is ruled out by [**6**], in which it is shown that in any family (defined over a number field) of smooth projective varieties, there are members over a number field that have the same Neron–Severi group as the general member. □

3. Potential density

We now show that for S defined over a number field, the second punctual Hilbert scheme X is potentially dense. Observe that to each embedding of S as a quartic in $\mathbb{P}^3$, one can associate a covering of X by a family of surfaces birational to abelian ones: in the notation of [**5**], those are the surfaces of the form $C * C$, where the curve C runs through the family of hyperplane sections of S with one double point. It is enough to show that the iterations of one such surface, defined over a number field, by $\iota_2\iota_1$ are Zariski-dense in X.

Let H_1, H_2 be the elements of $\mathrm{NS}(X) \cong \mathrm{NS}(S) \oplus \mathbb{Z}E$ corresponding to $h_1, h_2 \in \mathrm{NS}(S)$ (geometrically, a divisor from the linear system $|H_i|$ parametrizes subschemes whose support meets a fixed divisor from $|h_i|$).

Recall from [**9**] that $\iota_k^* H_k = 3H_k - 4E$ and $\iota_k^* E = 2H_k - 3E$, where $k = 1, 2$. Moreover, the same computation as in [**9**] gives

$$\iota_1^* H_2 = 7H_1 - 7E - H_2, \qquad \iota_2^* H_1 = 7H_2 - 7E - H_1.$$

Therefore, in the basis $\{H_1, E, H_2\}$ of $\mathrm{NS}(X)$, the involutions ι_1^*, ι_2^* are given by the matrices

$$M_1 = \begin{pmatrix} 3 & 2 & 7 \\ -4 & -3 & -7 \\ 0 & 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -1 & 0 & 0 \\ -7 & -3 & -4 \\ 7 & 2 & 3 \end{pmatrix}.$$

The product $(\iota_2\iota_1)^*$ is thus represented by the matrix

$$M_1 M_2 = \begin{pmatrix} 32 & 8 & 13 \\ -24 & -5 & -9 \\ -7 & -2 & -3 \end{pmatrix}$$

on $\mathrm{NS}(X)$ and is the identity on its orthogonal complement in the second cohomologies of X (indeed, because $h^{2,0}(X) = 1$, this complement is an irreducible Hodge substructure, whereas $(\iota_2\iota_1)^*$ has to fix the holomorphic symplectic form).

Lemma 3.1. *No effective divisor on X is invariant under $\iota_2\iota_1$.*

Proof. The only divisor classes that are invariant under $(\iota_2\iota_1)^*$ are multiples of $L = 2H_1 - 11E + 2H_2$. These are not effective since, for instance, the class $A = H_1 - E$ is ample (this is the inverse image of the Plücker hyperplane section by the natural finite morphism $X \to G(1,3)$, which corresponds to the embedding of S in $\mathbb{P}^3$ by h_1 and which sends a subscheme $Z \subset S$ to the only line in $\mathbb{P}^3$ containing S), but its Beauville–Bogomolov intersection with L is zero. □

Now let C_1 be a hyperplane section of S with one double point in the projective embedding given by h_1, and let Δ_1 be the class of the surface $C_1 * C_1$ in the cohomology of X.

Proposition 3.2. *The surface $C_1 * C_1$ is not periodic by $\iota_2\iota_1$.*

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Proof. It suffices to prove that Δ_1 is not periodic. Since $H^4(X,\mathbb{Q}) \cong S^2H^2(X,\mathbb{Q})$ (from the description of the cohomologies of a symmetric square and the fact that the Hilbert–Chow morphism is just the blow-up of the diagonal in this case) and we know the eigenvalues of $(\iota_2\iota_1)^*$ on $H^2(X,\mathbb{Q})$, we see that periodicity actually means invariance, so we only need to show that $(\iota_2\iota_1)^*\Delta_1 \neq \Delta_1$, or, since Δ_1 is invariant by ι_1 and ι_1, ι_2 are involutions, that $\iota_2^*\Delta_1 \neq \Delta_1$. This in turn follows once we compute that

$$\Delta_1 \cdot E^2 \neq \Delta_1 \cdot \iota_2^* E^2 = \Delta_1 \cdot (2H_2 - 3E)^2.$$

Let $T_p \subset X$ be the surface parametrizing the length-2 subschemes of S containing a given point p, and let Σ be the class of T_p (obviously not depending on p). Since the class H_1 on X is the class of a divisor parametrizing subschemes having some support on the corresponding divisor on S, we have

$$\Delta_1 = H_1^2 - q(H_1)\Sigma = H_1^2 - 4\Sigma.$$

Furthermore, T_p is identified with the blow-up of S in p and E restricts to T_p as the exceptional divisor, thus $\Sigma \cdot E^2 = -1$. It is also clear from the geometry that

$$\Sigma \cdot H_1^2 = \Sigma \cdot H_2^2 = q(H_i) = 4.$$

Indeed, the intersection $\Sigma \cdot H_1^2$ is the sum of the cycles supported on p and an intersection point of two general divisors from $|h_1|$; as we know, there are four such intersection points, which give four points of X.

Moreover, for any two divisor classes α, β on X, one has [**3**]

$$\alpha^2 \cdot \beta^2 = q(\alpha)q(\beta) + 2q(\alpha,\beta)^2$$

(where, by abuse of notation, we denote by the same letter q the quadratic form and the associated bilinear form) and

$$E \cdot \alpha \cdot \beta^2 = 0.$$

Thus

$$\Delta_1 \cdot E^2 = (H_1^2 - 4\Sigma)E^2 = -8 + 4 = -4$$

and

$$\Delta_1 \cdot (2H_2 - 3E)^2 = (H_1^2 - 4\Sigma)(4H_2^2 - 12H_2E + 9E^2) > 0.$$

□

In summary, we have the following theorem.

Theorem 3.3. *If the K3 surface S as above is defined over a number field, rational points are potentially dense on $X = S^{[2]}$.*

Proof. Let $C = C_1$ be a curve as above, defined over a number field. Since $p_g(C) = 2$, $C * C$ is birational to an abelian surface and hence has potentially dense rational points. Thus it suffices to show that the union of surfaces $(\iota_2\iota_1)^k(C * C)$, $k \in \mathbb{Z}$, is Zariski-dense in X. By the preceding proposition, there is an infinite number of such surfaces, so if their union is not Zariski-dense in X, its Zariski closure is a divisor. But such a divisor would be invariant by $\iota_2\iota_1$, and this contradicts Lemma 3.1. □

Remark. At the start of this paper, we could have taken the binary quadratic form $b_a(x, y) = 4x^2 + 2axy + 4y^2$ with an arbitrary $a > 4$. And indeed, as soon as this form represents -2, we have exactly the same results as above, apart from one exceptional case where $a = 5$. In this case, $v = (1, -1)$ is a (-2)-class, and the basis vectors have intersection of different signs with v. Since either v or $-v$ is effective, these basis vectors cannot both represent ample (or anti-ample) classes, even up to Picard–Lefschetz reflections. On the other hand, for $a = 5$ (and only for $a = 5$) the form b_a represents zero as well, so that the corresponding K3 surface is elliptic and therefore potentially dense.

For a general $a \geqslant 7$, the numbers are as follows:

$$\iota_1^* H_2 = aH_1 - aE - H_2;$$

$(\iota_2\iota_1)^*$ on $\mathrm{NS}(X)$ has 1 as an eigenvalue of multiplicity 1, the corresponding eigenvector is $2H_1 - (a+4)E + 2H_2$ and it is not effective for the same reason as before. The other two eigenvalues are not roots of unity. Thus Δ_1 is invariant if periodic, and its non-invariance is checked in the same way as above.

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