Canad. Math. Bull. Vol. **48** (4), 2005 pp. 580–586

Exceptional Sets in Hartogs Domains

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Abstract. Assume that Ω is a Hartogs domain in $\mathbb{C}^{1+n}$, defined as $\Omega = \{(z,w) \in \mathbb{C}^{1+n} : |z| < \mu(w), w \in H\}$, where H is an open set in $\mathbb{C}^n$ and μ is a continuous function with positive values in H such that $-\ln\mu$ is a strongly plurisubharmonic function in H. Let $\Omega_w = \Omega \cap (\mathbb{C} \times \{w\})$. For a given set E contained in H of the type G_δ we construct a holomorphic function $f \in \mathbb{O}(\Omega)$ such that

$$E = \left\{ w \in \mathbb{C}^n : \int_{\Omega_w} |f(\cdot\,,w)|^2 \, d\mathfrak{L}^2 = \infty \right\}.$$

1 Preface

Assume that Ω is a Hartogs domain in $\mathbb{C}^{1+n}$ defined as

$$\Omega = \{(z,w) \in \mathbb{C}^{1+n} : |z| < \mu(w), w \in H\},$$

where H is an open set in $\mathbb{C}^n$ and μ is a continuous function with positive values in H such that $-\ln\mu$ is a strongly plurisubharmonic function in H. Let

$$\Omega_w = \Omega \cap (\mathbb{C} \times \{w\}).$$

We denote

$$E(\Omega, f) = \left\{ w \in \mathbb{C}^n : \int_{\Omega_w} |f(\cdot\,,w)|^2 \, d\mathfrak{L}^2 = \infty \right\}$$

the exceptional set for a holomorphic function f ($f \in \mathbb{O}(\Omega)$). By $\mathfrak{L}^2$ we denote the 2-dimensional Lebesgue measure.

Behaviour of the holomorphic functions on the slices has been investigated in different cases by many authors: [1, 2, 3, 4, 5, 6, 7]. In particular a construction of holomorphic function $f \in \mathbb{O}(\Omega)$ with a given closed or open exceptional set has been shown in the paper [3]. In this paper the author raises the question whether it is possible to create a holomorphic function with a given exceptional set of the type G_δ. In our paper we answer this question.

Theorem 2.4 *Let E be a subset of the type G_δ in H. Then there exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that $E(\Omega, f) = E$.*

Observe that this theorem solves completely the problem of description of exceptional sets because if E is an exceptional set for a holomorphic function $f \in \mathbb{O}(\Omega)$, then[1] E is a set of the type G_δ.

Received by the editors November 25, 2003; revised April 23, 2004.
AMS subject classification: 30B30.
Keywords: boundary behaviour of holomorphic functions, exceptional sets.

[1]Let $r \in (0,1)$. The function $u_r(w) := \int_{|z|<r\mu(w)} |f(z,w)|^2 \, d\mathfrak{L}^2(z)$ is continuous on H. Therefore $u(w) := \lim_{r\to 1} u_r(w)$ is a lower semi-continuous function on H. Now $E(\Omega, f) = u^{-1}(\infty)$, and therefore $E(\Omega, f)$ is a set of the type G_δ.

2 Exceptional Sets in Hartogs Domains

In the proof of Lemma 2.1 we use the idea based on [3, Theorem 1].

Lemma 2.1 *Assume that T, D are compact sets in H such that $D \cap T = \varnothing$. Let us set a natural number k and a real number $\varepsilon > 0$. There exists a natural number m and a function $f \in \mathbb{O}(\Omega)$ such that*

(1) $f(z,w) = \sum_{j=k}^{m} f_j(w)z^j$ *where* $f_j \in \mathbb{O}(H)$ *and* $\sqrt[j]{|f_j(w)|}\mu(w) \le 1+\varepsilon$ *for* $w \in T \cup D$,
(2) $\int_{\Omega_w} |f(\cdot\,,w)|^2 \, d\mathfrak{L}^2 \ge 1$ *for* $w \in T$,
(3) $|f(z,w)| \le \varepsilon$ *for* $(z,w) \in \Omega$ *and* $w \in D$.

Proof There exists a continuous function μ_T positive such that $\mu_T \ge \mu$, $-\ln \mu_T$ is a function strongly plurisubharmonic in H and $\mu_T(w) = \mu(w)$ iff $w \in T$. Let

$$\delta = \max\left\{ \sqrt{\sup_{w\in D} \frac{\mu(w)}{\mu_T(w)}}, \sqrt{\frac{1}{1+\varepsilon}} \right\}.$$

Obviously $\delta \in (0,1)$.

On the basis [8, Theorem 9, p. 214] there exist holomorphic functions $h_i \in \mathbb{O}(H)$ for $i = 1,\dots,s$ and natural numbers a_i for $i = 1,\dots,s$ such that

$$\ln\frac{1}{\mu_T(w)} \le \sup\left\{ \frac{1}{a_i}\ln|h_i(w)| : i = 1,\dots,s \right\} \le \ln\frac{1}{\delta\mu_T(w)} \tag{2.1}$$

for $w \in T \cup D$. Select natural numbers b_i with the following properties:

- $*$ $k < a_i b_i < a_{i+1}b_{i+1}$ for $i = 1,\dots,s-1$,
- $*$ $\sqrt{a_ib_i}\,\delta^{a_ib_i} \le \frac{\varepsilon}{2^i}\inf_{w\in D}\mu(w)$ for $i = 1,\dots,s$,
- $*$ $a_ib_i \le \pi(1+\varepsilon)^{a_ib_i-2}\inf_{w\in D}\mu^2(w)$ for $i = 1,\dots,s$.

We define function f as

$$f(z,w) = \sum_{i=1}^{s}\sqrt{\frac{a_ib_i}{\pi}}h_i^{b_i}(w)z^{a_ib_i-1} = \sum_{j=k}^{m} f_j(w)z^j,$$

where $m = a_sb_s - 1$, $f_{a_ib_i-1} = \sqrt{\frac{a_ib_i}{\pi}}h_i^{b_i}$ and $f_j = 0$ when $j \notin \{a_1b_1-1,\dots,a_sb_s-1\}$. Due to (2.1) and the inequality $a_ib_i \le \pi(1+\varepsilon)^{a_ib_i-2}\mu^2(w)$ for $w \in T \cup D$ we can estimate

$$\begin{aligned}|f_{a_ib_i-1}(w)|\mu^{a_ib_i-1}(w) &= \sqrt{\frac{a_ib_i}{\pi}}h_i^{b_i}(w)\mu^{a_ib_i-1}(w) \le \sqrt{\frac{a_ib_i}{\pi}}\frac{\mu^{a_ib_i-1}(w)}{\delta^{a_ib_i}\mu_T^{a_ib_i}(w)}\\ &\le \sqrt{\frac{a_ib_i}{\pi}}\frac{1}{\delta^{a_ib_i}\mu(w)} \le \sqrt{\frac{a_ib_i}{\pi}}\frac{(\sqrt{1+\varepsilon})^{a_ib_i}}{\mu(w)}\\ &\le (1+\varepsilon)^{a_ib_i-1}\end{aligned}$$

https://doi.org/10.4153/CMB-2005-053-0 Published online by Cambridge University Press

for $w \in T \cup D$. From this we obtain the property (1).

If $w \in T$, we can estimate:

$$\begin{aligned}\int_{\Omega_w} |f(\cdot\,,w)|^2\, d\mathfrak{L}^2 &= \sum_{i=1}^{s} \frac{a_i b_i}{\pi} \int_{|z|<\mu(w)} |h_i(w)|^{2b_i} |z|^{2a_i b_i - 2}\, d\mathfrak{L}^2(z)\\ &= \sum_{i=1}^{s} |h_i(w)|^{2b_i} \mu(w)^{2a_i b_i}\\ &\geq \sup_{i=1,\dots,s} \left(\frac{1}{\mu_T(w)}\right)^{2a_i b_i} \mu(w)^{2a_i b_i} = 1,\end{aligned}$$

from which property (2) results. Moreover

$$\begin{aligned}|f(z,w)| &\leq \sum_{i=1}^{s} \sqrt{\frac{a_i b_i}{\pi}} |h_i(w)|^{b_i} |z|^{a_i b_i - 1}\\ &\leq \sum_{i=1}^{s} \sqrt{a_i b_i} |h_i(w)|^{b_i} \mu(w)^{a_i b_i - 1}\\ &\leq \sum_{i=1}^{s} \sqrt{a_i b_i} \left(\frac{1}{\delta \mu_T(w)}\right)^{a_i b_i} \mu(w)^{a_i b_i - 1}\\ &\leq \sum_{i=1}^{s} \sqrt{a_i b_i} \left(\frac{\delta}{\mu(w)}\right)^{a_i b_i} \mu(w)^{a_i b_i - 1}\\ &\leq \sum_{i=1}^{s} \frac{\varepsilon \inf_{x\in D} \mu(x)}{2^i \mu(w)} \leq \varepsilon,\end{aligned}$$

for $w \in D$, which is property (3). ∎

Lemma 2.2 *Assume that T, D are compact sets in H such that $D \cap T = \varnothing$. Let also K be a compact set in Ω. Let us set a natural number k and a real number $\varepsilon > 0$. There exists a natural number m and a function $f \in \mathbb{O}(\Omega)$ such that*

(1) $f(z,w) = \sum_{j=k}^{m} f_j(w) z^j$, *where* $f_j \in \mathbb{O}(H)$,
(2) $\int_{\Omega_w} |f(\cdot\,,w)|^2\, d\mathfrak{L}^2 \geq 1$ *for* $w \in T$,
(3) $|f(z,w)| \leq \varepsilon$ *for* $(z,w) \in K$ *or* $w \in D$.

Proof Increasing a set K we can assume that

$$K = \{(z\lambda, w) : (z,w) \in K, w \in H, |\lambda| = 1\}.$$

Let $K_\delta := \{w \in \Omega : \inf_{z\in K} \|z - w\| \leq \delta\}$. Let us select number δ so small that K_δ is a compact subset of Ω.

There exists a sequence of compact sets D_i such that $D = D_1 \subset \cdots \subset D_i \subset D_{i+1} \subset \cdots \subset \bigcup_{i\in\mathbb{N}} D_i = H \setminus T$. On the basis of Lemma 2.1 there exist functions h_i and natural numbers a_i, b_i such that

(1) $k \le a_i < b_i < a_{i+1}$,
(2) $h_i(z,w) = \sum_{j=a_i}^{b_i} h_{i,j}(w)z^j$, where $h_{i,j} \in \mathbb{O}(H)$ and

$$\sqrt[j]{|h_{i,j}(w)|}\mu(w) \le 1 + \frac{\varepsilon}{2^i} \quad \text{for } w \in T \cup D_i,$$

(3) $\int_{\Omega_w} |h_i(\cdot\,, w)|^2 \, d\mathfrak{L}^2 \ge 1$ for $w \in T$,
(4) $|h_i(z,w)| \le \frac{\varepsilon}{2^i}$ for $w \in D_i$.

We define

$$g(z,w) = \sum_{i\in\mathbb{N}} h_i(z,w) = \sum_{i\in\mathbb{N}} \sum_{j=a_i}^{b_i} h_{i,j}(w)z^j = \sum_{j\in\mathbb{N}} a_j(w)z^j.$$

Observe that

$$\limsup_{j\to\infty} \sqrt[j]{|a_j(w)|}\mu(w) = \limsup_{j\to\infty} \sqrt[j]{|h_{i,j}(w)|}\mu(w) \le 1$$

for $w \in H$. Therefore, due to Hadamard's test, the function g is holomorphic on Ω. In particular

$$\sum_{i\in\mathbb{N}} \int_{K_\delta} |h_i|^2 \, d\mathfrak{L}^{2n+2} = \int_{K_\delta} |g|^2 \, d\mathfrak{L}^{2n+2} < \infty. \tag{2.2}$$

It is obvious that there exists a constant $C_K > 0$ such that

$$|h_i(z,w)| \le C_K \int_{K_\delta} |h_i|^2 \, d\mathfrak{L}^{2n+2}$$

for $(z,w) \in K$. Due to (2.2) there exists $i_0 \in \mathbb{N}$ such that

$$\int_{K_\delta} |h_{i_0}|^2 \, d\mathfrak{L}^{2n+2} \le \frac{\varepsilon}{C_K}.$$

Observe that $|h_{i_0}(z,w)| \le \varepsilon$ for $(z,w) \in K$. Therefore it is enough to set $f = h_{i_0}$. ∎

Lemma 2.3 *If $E = \bigcap_{i\in\mathbb{N}} U_i \subset H$, where $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of open sets, then there exists a sequence of closed sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ and closed sets $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ in H such that $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$, $T_{i,j} \cap D_{i,j} = \varnothing$, $H \setminus U_i \subset D_{i,j}$ and $T_{i,j} \subset D_{i,k}$, when $|j-k| \ge 2$.*

Proof We denote:

$$T_{i,j} := \left\{ z \in U_i : \frac{1}{j+2} \le \inf_{w\in\partial U_i} \|z-w\| \le \frac{1}{j+1} \right\},$$

$$D_{i,j} := \left\{ z \in H : \frac{1}{(j+3)^2} \le \inf_{w\in T_{i,j}} \|z-w\| \right\}.$$

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It is a direct result of the definition that $H \setminus U_i \subset D_{i,j}$.

Observe that $\|z-w\| \geq \frac{1}{(j+3)^2}$ for $z \in T_{i,j}$ and $w \in T_{i,k}$ when $k-j \geq 2$. In fact suppose that $z \in T_{i,j}$, $w \in T_{i,k}$ and $\|z-w\| < \frac{1}{(j+3)^2}$. In this case there exists $u \in \partial U_i$ such that $\|u-w\| \leq \frac{1}{k+1} \leq \frac{1}{j+3}$. We can estimate

$$\frac{1}{j+2} \leq \|u-z\| \leq \|u-w\| + \|w-z\| < \frac{1}{j+3} + \frac{1}{(j+3)^2} \leq \frac{1}{j+2},$$

which is impossible.

Let $k \geq j+2$. If $x \in T_{i,j} \setminus D_{i,k}$, then there exists a point $y \in T_{i,k}$ such that $\|x-y\| < \frac{1}{(k+3)^2} \leq \frac{1}{(j+3)^2}$, which is impossible due to the first part of the proof.

Let $k \leq j-2$. If $x \in T_{i,j} \setminus D_{i,k}$, then there exists $y \in T_{i,k}$ such that $\|x-y\| < \frac{1}{(k+3)^2}$, which is also impossible due to the first part of the proof. ∎

Theorem 2.4 *Let E be a subset of the type G_δ in H. Then there exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that $E(\Omega, f) = E$.*

Proof There exists a sequence of open sets $\{U_i\}_{i\in\mathbb{N}}$ such that

$$E = \bigcap_{j\in\mathbb{N}} U_j \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots$$

We define an increasing sequence of compact sets K_i such that $K_i \subset V_{i+1} \subset K_{i+1} \subset \cdots \subset \bigcup_{i\in\mathbb{N}} K_i = \Omega$, where V_i denotes the interior of the set K_i. On the basis of Lemma 2.3 there exists a sequence of closed sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ and closed sets $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ in H such that $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$, $T_{i,j} \cap D_{i,j} = \varnothing$, $H \setminus U_i \subset D_{i,j}$ and $T_{i,j} \subset D_{i,k}$, when $|j-k| \geq 2$. We define also a sequence of compact sets $\{S_i\}_{i\in\mathbb{N}}$ such that $S_i \subset S_{i+1} \subset \cdots \subset \bigcup_{j\in\mathbb{N}} S_j = H$.

As $D_{i,j} \cap T_{i,j} = \varnothing$, therefore on the basis of Lemma 2.2 we can select functions $f_{i,j} \in \mathbb{O}(H)$ and natural numbers $a_{i,j}, b_{i,j}$ such that

(1) $a_{i,j} < b_{i,j}$ and[2] $[a_{i,j}, b_{i,j}] \cap [a_{k,l}, b_{k,l}] = \varnothing$ for $(i,j) \neq (k,l)$,
(2) $f_{i,j}(z,w) = \sum_{m=a_{i,j}}^{b_{i,j}} h_{i,j,m}(w) z^m$, where $h_{i,j,m} \in \mathbb{O}(H)$,
(3) $\int_{\Omega_w} |f_{i,j}(\cdot, w)|^2 \, d\mathfrak{L}^2 \geq 1$ for $w \in T_{i,j} \cap S_{i+j}$,
(4) $|f_{i,j}(z,w)|^2 \leq \frac{1}{2^{i+j}}$ when $(z,w) \in K_{i+j}$ or $w \in D_{i,j} \cap S_{i+j}$.

Observe that

$$\int_{\Omega_w} |f_{i,j}(\cdot, w)|^2 \, d\mathfrak{L}^2 = \sum_{m=a_{i,j}}^{b_{i,j}} \int_{\Omega_w} |h_{i,j,m}(w)|^2 |z|^{2m} \, d\mathfrak{L}^2(z)$$

$$= \sum_{m=a_{i,j}}^{b_{i,j}} 2\pi |h_{i,j,m}(w)|^2 \frac{\mu(w)^{2m+2}}{2m+2} < \infty$$

[2] We denote $[a,b] := \{t : a \leq t \leq b\}$.

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for $w \in H$. We define

$$f = \sum_{i,j=1}^{\infty} f_{i,j}.$$

Due to the property (4) function f is holomorphic on Ω. If $w \in E$, then

$$\sum_{i,j \in \mathbb{N}\colon\, w \in T_{i,j} \cap S_{i+j}} 1 = \infty.$$

Therefore using properties (1)–(3), we can estimate:

$$\begin{aligned}
\int_{\Omega_w} |f(\cdot\,, w)|^2 \, d\mathfrak{L}^2 &= \sum_{i,j=1}^{\infty} \int_{\Omega_w} |f_{i,j}(\cdot\,, w)|^2 \, d\mathfrak{L}^2 \\
&\geq \sum_{i,j \in \mathbb{N}\colon\, w \in T_{i,j} \cap S_{i+j}} \int_{\Omega_w} |f_{i,j}(\cdot\,, w)|^2 \, d\mathfrak{L}^2 \\
&\geq \sum_{i,j \in \mathbb{N}\colon\, w \in T_{i,j} \cap S_{i+j}} 1 = \infty.
\end{aligned}$$

If $w \in H \setminus E$, then there exists $i_0 \in \mathbb{N}$ such that $w \in S_{i+j}$ for $i + j > i_0$ and $w \in H \setminus U_i \subset D_{i,j}$ for $i > i_0$. Moreover there exist natural numbers k_i for $i = 1, \dots, i_0$ such that $w \in D_{i,j}$ for $|j - k_i| \geq 2$. Denote

$$A := \{(i, j) : i \leq i_0, |j - k_i| \leq 1 \text{ or } i + j \leq i_0\}.$$

Observe that $\# A < \infty$. Moreover if $(i, j) \notin A$ then $w \in D_{i,j} \cap S_{i+,j}$. We can estimate

$$\begin{aligned}
\sum_{i \leq i_0, j \in \mathbb{N}} \int_{\Omega_w} |f_{i,j}(\cdot\,, w)|^2 \, d\mathfrak{L}^2 &\leq \sum_{(i,j) \in A} \int_{\Omega_w} |f_{i,j}(\cdot\,, w)|^2 \, d\mathfrak{L}^2 \\
&\quad + \sum_{(i,j) \notin A} \int_{\Omega_w} |f_{i,j}(\cdot\,, w)|^2 \, d\mathfrak{L}^2 \\
&\leq \sum_{(i,j) \in A} \sum_{m=a_{i,j}}^{b_{i,j}} 2\pi |h_{i,j,m}(w)|^2 \frac{\mu^{2m+2}(w)}{2m+2} \\
&\quad + \sum_{i,j \in \mathbb{N}} \int_{\Omega_w} \frac{1}{2^{i+j}} \, d\mathfrak{L}^2 < \infty.
\end{aligned}$$

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In particular, on the basis of properties (1),(2), and (4) we obtain the following estimation

$$\begin{aligned}\int_{\Omega_w} |f(\cdot\,,w)|^2\,d\mathfrak{L}^2 &= \sum_{i,j=1}^{\infty}\int_{\Omega_w}|f_{i,j}(\cdot\,,w)|^2\,d\mathfrak{L}^2\\ &\le \sum_{i\le i_0,\,j\in\mathbb{N}}\int_{\Omega_w}|f_{i,j}(\cdot\,,w)|^2\,d\mathfrak{L}^2+\sum_{i>i_0,\,j\in\mathbb{N}}\int_{\Omega_w}\frac{1}{2^{i+j}}\,d\mathfrak{L}^2\\ &\le \sum_{i\le i_0,\,j\in\mathbb{N}}\int_{\Omega_w}|f_{i,j}(\cdot\,,w)|^2\,d\mathfrak{L}^2+2\pi\mu(w)^2<\infty.\end{aligned}$$

It follows that $E(\Omega, f) = E$. ■

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