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VANISHING OF COHOMOLOGY OVER COMPLETE INTERSECTION RINGS

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Abstract. Let R be a complete intersection ring, and let M and N be R-modules. It is shown that the vanishing of $\operatorname{Ext}^i_R(M, N)$ for a certain number of consecutive values of i starting at n forces the complete intersection dimension of M to be at most $n-1$. We also estimate the complete intersection dimension of M^*, the dual of M, in terms of vanishing of cohomology modules, $\operatorname{Ext}^i_R(M, N)$.

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1. Introduction. In this paper, we study the relationship between the vanishing of $\operatorname{Ext}^i_R(M, N)$ for various consecutive values of i, and the complete intersection dimensions of M and M^*, the dual of M. The vanishing of homology was first studied by Auslander [**3**]. For two finitely generated modules M and N over an unramified regular local ring R, he proved that if $\operatorname{Tor}^R_i(M, N) = 0$ for some $i > 0$, then $\operatorname{Tor}^R_n(M, N) = 0$ for all $n \geq i$. In [**17**], Lichtenbaum settled the ramified case. It is easy to see that a similar statement is not true in general, with Tor replaced by Ext. In [**15**], Jothilingam studied the vanishing of cohomology by using the Rigidity Theorem of Auslander. For two non-zero modules M and N over a regular local ring R, he proved that if M satisfies (S_n) for some $n \geq 0$ and $\operatorname{Ext}^i_R(M, N) = 0$ for some positive integer i such that $i \geq \operatorname{depth}_R(N) - n$, then $\operatorname{Ext}^j_R(M, N) = 0$ for all $j \geq i$. In [**16**], Jothilingam and Duraivel studied the relationship between the vanishing of $\operatorname{Ext}^i_R(M, N)$ and the freeness of M^*. For two non-zero modules M and N over a regular local ring R, they proved that if $\operatorname{Ext}^i_R(M, N) = 0$ for all $1 \leq i \leq \max\{1, \operatorname{depth}_R(N) - 2\}$, then M^* is free. In this paper we are going to generalize these results.

An R-module M is said to be *c-rigid* if for all R-modules N, $\operatorname{Tor}^R_{i+1}(M, N) = \operatorname{Tor}^R_{i+2}(M, N) = \cdots = \operatorname{Tor}^R_{i+c}(M, N) = 0$ for some $i \geq 0$ implies that $\operatorname{Tor}^R_n(M, N) = 0$ for all $n > i$. If $c = 1$, then we simply say that M is rigid.

The aim of this paper is to study the following question.

Question 1.1. Let R be a Gorenstein local ring, and let M and N be R-modules. Assume that $n \geq 0$, $c > 0$ are integers and that N is c-rigid. If $\operatorname{Ext}^i_R(M, N) = 0$ for all i, $1 \leq i \leq \max\{c, \operatorname{depth}_R(N) - n\}$, then what can we say about the Gorenstein dimensions of M and M^*?

In Section 2, we collect necessary notations, definitions and some known results which will be used in this paper.

In Section 3, we study Question 1.1 for rigid modules. Over the Gorenstein local ring R, given non-zero R-modules M and N such that N has reducible complexity, we show that if N is rigid and $\operatorname{Ext}^i_R(M, N) = 0$ for all i, $1 \leq i \leq \max\{1, \operatorname{depth}_R(N) - n\}$ and some $n \geq 2$, then $\text{G-dim}_R(M^*) \leq n - 2$, which is a generalization of [**16**, Theorem 1]. In particular, if M satisfies (S_n), then $\text{G-dim}_R(M) = 0$ (see Theorem 3.2). As a consequence, for two non-zero modules M and N over a complete intersection ring R, it is shown that if N is rigid and $\operatorname{Ext}^i_R(M, N) = 0$ for some positive integer $i \geq \operatorname{depth}_R(N)$, then $\text{CI-dim}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\} < i$ (see Theorem 3.5).

In Section 4, we generalize [**15**, Corollary 1] for modules over a complete intersection ring. For two modules M and N over a complete intersection ring R with codimension c, it is shown that if M satisfies (S_t) for some $t \geq 0$, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i, n \leq i \leq n + c$ and some $n > 0$ and $\operatorname{depth}_R(N) \leq n + c + t$, then $\text{CI-dim}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\} < n$ (see Corollary 4.3).

2. Preliminaries. Throughout the paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring and all modules are finite (i.e. finitely generated) R-modules. The codimension of R is defined to be the non-negative integer $\operatorname{embdim}(R) - \dim(R)$, where $\operatorname{embdim}(R)$, the embedding dimension of R, is the minimal number of generators of $\mathfrak{m}$. Recall that R is said to be a complete intersection if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has the form $Q/(f)$, where f is a regular sequence of Q, and Q is a regular local ring. A complete intersection of codimension one is called a hypersurface. A local ring R is said to be an admissible complete intersection if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has the form $Q/(f)$, where f is a regular sequence of Q and Q is a power series ring over a field or a discrete valuation ring. Let

$$\cdots \to F_{n+1} \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

be the minimal free resolution of M. Recall that the nth syzygy of an R-module M is the co-kernel of the $F_{n+1} \to F_n$ and denoted by $\Omega^n M$, and it is unique up to isomorphism. The nth Betti number, denoted as $\beta^R_n(M)$, is the rank of the free R-module F_n. The complexity of M is defined as follows:

$$\operatorname{cx}_R(M) = \inf\{i \in \mathbb{N} \cup \{0\} \mid \exists \gamma \in \mathbb{R} \text{ such that } \beta^R_n(M) \leq \gamma n^{i-1} \text{ for } n \gg 0\}.$$

Note that $\operatorname{cx}_R(M) = \operatorname{cx}_R(\Omega^i M)$ for every $i \geq 0$. It follows from the definition that $\operatorname{cx}_R(M) = 0$ if and only if $\operatorname{pd}_R(M) < \infty$. If R is a complete intersection, then the complexity of M is less than or equal to the codimension of R (see [**12**]). The complete intersection dimension was introduced by Avramov et al. [**6**]. A module of finite complete intersection dimension behaves homologically like a module over a complete intersection. Recall that a quasi-deformation of R is a diagram $R \to A \twoheadleftarrow Q$ of local homomorphisms, in which $R \to A$ is faithfully flat, and $A \twoheadleftarrow Q$ is surjective with kernel generated by a regular sequence. The module M has finite complete intersection dimension if there exists such a quasi-deformation for which $\operatorname{pd}_Q(M \otimes_R A)$ is finite. The complete intersection dimension of M, denoted as $\text{CI-dim}_R(M)$, is defined as follows:

$$\text{CI-dim}_R(M) = \inf\{\operatorname{pd}_Q(M \otimes_R A) - \operatorname{pd}_Q(A) \mid R \to A \twoheadleftarrow Q \text{ is a quasi-deformation }\}.$$

The complete intersection dimension of M is bounded above by the projective dimension, $\operatorname{pd}_R(M)$, of M, and if $\operatorname{pd}_R(M) < \infty$, then the equality holds (see [**6**,

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Theorem 1.4]). Every module of finite complete intersection dimension has finite complexity (see [**6**, Theorem 5.3]).

The concept of modules with reducible complexity was introduced by Bergh [**7**].

Let M and N be R-modules, and consider a homogeneous element η in the graded R-module $\operatorname{Ext}^*_R(M, N) = \bigoplus_{i=0}^{\infty} \operatorname{Ext}^i_R(M, N)$. Choose a map $f_\eta : \Omega^{|\eta|}_R(M) \to N$ representing η, and denote by K_η the pushout of this map and the inclusion $\Omega^{|\eta|}_R(M) \hookrightarrow F_{|\eta|-1}$. Therefore, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{|\eta|}M & \longrightarrow & F_{|\eta|-1} & \longrightarrow & \Omega^{|\eta|-1}M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\eta} & & \downarrow & & \downarrow{\scriptstyle \|} & & \\ 0 & \longrightarrow & N & \longrightarrow & K_\eta & \longrightarrow & \Omega^{|\eta|-1}M & \longrightarrow & 0. \end{array}$$

with exact rows. Note that the module K_η is independent, up to isomorphism, of the map f_η chosen to represent η.

Definition 2.1. The full subcategory of R-modules comprising the modules having reducible complexity is defined inductively as follows:

(i) Every R-module of finite projective dimension has reducible complexity.

(ii) An R-module M of finite positive complexity has reducible complexity if there exists a homogeneous element $\eta \in \operatorname{Ext}^*_R(M, M)$ of positive degree such that $\operatorname{cx}_R(K_\eta) < \operatorname{cx}_R(M)$, $\operatorname{depth}_R(M) = \operatorname{depth}_R(K_\eta)$ and K_η has reducible complexity.

By [**7**, Proposition 2.2(i)], every module of finite complete intersection dimension has reducible complexity. In particular, every module over a local complete intersection ring has reducible complexity. On the other hand, there are modules having reducible complexity but whose complete intersection dimension is infinite (see, for example, [**9**, Corollary 4.7]).

The notion of the Gorenstein(or G-) dimension was introduced by Auslander [**2**], and developed by Auslander and Bridger in [**4**].

Definition 2.2. An R-module M is said to be of G-dimension zero whenever

(i) *the biduality map* $M \to M^{**}$ *is an isomorphism.*

(ii) $\operatorname{Ext}^i_R(M, R) = 0$ *for all* $i > 0$.

(iii) $\operatorname{Ext}^i_R(M^*, R) = 0$ *for all* $i > 0$.

The Gorenstein dimension of M, denoted as $\operatorname{G-dim}_R(M)$, is defined to be the infimum of all non-negative integers n such that there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

in which all G_i have G-dimension zero. By [**4**, Theorem 4.13], if M has finite Gorenstein dimension, then $\operatorname{G-dim}_R(M) = \operatorname{depth} R - \operatorname{depth}_R(M)$. By [**6**, Theorem 1.4], $\operatorname{G-dim}_R(M)$ is bounded above by the complete intersection dimension, $\operatorname{CI-dim}_R(M)$, of M, and if $\operatorname{CI-dim}_R(M) < \infty$, then the equality holds.

Let R be a local ring, and let M and N be finite non-zero R-modules. We say the pair (M, N) satisfies the depth formula provided:

$$\operatorname{depth}_R(M \otimes_R N) + \operatorname{depth} R = \operatorname{depth}_R(M) + \operatorname{depth}_R(N).$$

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The depth formula was first studied by Auslander [**3**] for finite modules of finite projective dimension. In [**13**], Huneke and Wiegand proved that the depth formula holds for M and N over complete intersection rings R, provided $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. In [**9**], Bergh and Jorgensen generalize this result for modules with reducible complexity over a local Gorenstein ring. More precisely, they proved the following result.

THEOREM 2.3 [**9**, Corollary 3.4]. *Let R be a Gorenstein local ring, and let M and N be non-zero R-modules. If M has reducible complexity and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $\operatorname{depth}_R(M \otimes_R N) + \operatorname{depth} R = \operatorname{depth}_R(M) + \operatorname{depth}_R(N)$.*

We denote by $G(R)$ the Grothendieck group of finite modules over R, that is, the quotient of free abelian group of all isomorphism classes of finite R-modules by the subgroup generated by the relations coming from short exact sequences of finite R-modules. We also denote by $\overline{G}(R) = G(R)/[R]$ the reduced Grothendieck group. For an abelian group G, we set $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a finite projective presentation of M. Applying the functor $(-)^* := \operatorname{Hom}_R(-, R)$, the coker f^*, which is unique up to projective equivalence, is called the transpose of M and denoted by $\operatorname{Tr} M$. Hence, there exists the exact sequence:

$$0 \to M^* \to P_0^* \to P_1^* \to \operatorname{Tr} M \to 0. \tag{2.1}$$

Note that the minimal projective presentations of M represent isomorphic transposes of M. Two modules M and N are called *stably isomorphic*, and we write $M \approx N$ if $M \oplus P \cong N \oplus Q$ for some projective modules P and Q. Note that $M^* \approx \Omega^2 \operatorname{Tr} M$ by the exact sequence (2.1).

The compositions $\mathcal{T}_k := \operatorname{Tr} \Omega^{k-1}$ for $k > 0$ were introduced by Auslander and Bridger in [**4**]. If $\operatorname{Ext}_R^i(M, R) = 0$ for some $i > 0$, then it is easy to see that $\mathcal{T}_i M \approx \Omega \mathcal{T}_{i+1} M$.

We frequently use the following theorem by Auslander and Bridger.

THEOREM 2.4 [**4**, Theorem 2.8]. *Let M be an R-module, and $n \geq 0$ be an integer. Then there are exact sequences of functors:*

$$0 \to \operatorname{Ext}_R^1(\mathcal{T}_{n+1} M, -) \to \operatorname{Tor}_n^R(M, -) \to \operatorname{Hom}_R(\operatorname{Ext}_R^n(M, R), -) \to \operatorname{Ext}_R^2(\mathcal{T}_{n+1} M, -), \tag{2.2}$$

$$\operatorname{Tor}_2^R(\mathcal{T}_{n+1} M, -) \to (\operatorname{Ext}_R^n(M, R) \otimes_R -) \to \operatorname{Ext}_R^n(M, -) \to \operatorname{Tor}_1^R(\mathcal{T}_{n+1} M, -) \to 0. \tag{2.3}$$

For an integer $n \geq 0$, we say M satisfies Serre's condition (S_n) if $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{n, \dim(R_{\mathfrak{p}})\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. If R is Gorenstein, then M satisfies (S_n) if and only if $\operatorname{Ext}_R^i(\operatorname{Tr} M, R) = 0$ for all $1 \leq i \leq n$ (see [**4**, Theorem 4.25]). In particular, M satisfies (S_2) if and only if it is reflexive, i.e. the natural map $M \to M^{**}$ is bijective, where $M^* = \operatorname{Hom}_R(M, R)$ (see [**11**, Theorem 3.6]).

The following results will be used throughout the paper.

THEOREM 2.5. *Let R be a local complete intersection ring, and let M and N be R-modules. Then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Moreover, if R is a hypersurface and $\operatorname{Tor}_i^R(M, N) = 0$ for $i \gg 0$, then either $\operatorname{pd}_R(M) < \infty$ or $\operatorname{pd}_R(N) < \infty$.*

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Proof. See [**5**, Theorem 6.1 and Proposition 5.12]. □

THEOREM 2.6. *Let R be a local ring, and let M and N be non-zero R-modules. If* $\mathrm{Ext}^i_R(M,N)=0$ *for all* $i \gg 0$*, then the following statements hold true.*

(i) If $\text{CI-dim}_R(M)<\infty$*, then* $\text{CI-dim}_R(M)=\sup\{i \mid \mathrm{Ext}^i_R(M,N)\neq 0\}$.
(ii) If $\text{G-dim}_R(M)<\infty$ *and* $\text{CI-dim}_R(N)<\infty$*, then*

$$\text{G-dim}_R(M)=\sup\{i \mid \mathrm{Ext}^i_R(M,N)\neq 0\}.$$

Proof. See [**1**, Theorem 4.2] and [**21**, Theorem 4.4]. □

THEOREM 2.7. *Let R be a local ring, and M, N two R-modules. If* $\text{CI-dim}_R(M)=0$*, then* $\mathrm{Ext}^i_R(M,N)=0$ *for all* $i>0$ *if and only if* $\mathrm{Tor}_i^R(\mathrm{Tr}\,M,N)=0$ *for all* $i>0$.

Proof. First note that $\text{CI-dim}_R(\mathrm{Tr}\,M)=0$ by [**21**, Lemma 3.3] and $M\approx \mathrm{Tr}\,\mathrm{Tr}\,M$. Now the assertion is clear by [**21**, Proposition 3.4]. □

3. Vanishing of Ext for rigid modules. We start this section by estimating the Gorenstein dimension of the transpose of M in terms of vanishing of cohomology modules, $\mathrm{Ext}^i_R(M,N)$.

LEMMA 3.1. *Let R be a Gorenstein ring, and let M and N be non-zero R-modules. Assume that* $n\geq 0$ *is an integer, and that the following conditions hold.*

(1) $\mathrm{Ext}^i_R(M,N)=0$ *for all* $1\leq i\leq \max\{1,\mathrm{depth}_R(N)-n\}$.
(2) N is rigid.
(3) N has reducible complexity.
Then $\text{G-dim}_R(\mathrm{Tr}\,M)\leq n$ *and* $\mathrm{Tor}_i^R(\mathrm{Tr}\,M,N)=0$ *for all* $i>0$.

Proof. If $\mathrm{Tr}\,M=0$, then $\text{G-dim}_R(\mathrm{Tr}\,M)=0$ and we have nothing to prove, so let $\mathrm{Tr}\,M\neq 0$. As $\mathrm{Ext}^1_R(M,N)=0$, $\mathrm{Tor}_1^R(\mathcal{T}_2M,N)=0$ by the exact sequence (2.3). Since N is rigid, we have $\mathrm{Tor}_i^R(\mathcal{T}_2M,N)=0$ for all $i>0$. It follows from the exact sequence (2.3) again that $\mathrm{Ext}^1_R(M,R)\otimes_R N=0$, and since N is non-zero, $\mathrm{Ext}^1_R(M,R)=0$. Now it is easy to see that $\mathcal{T}_1M\approx\Omega\mathcal{T}_2M$ and so $\mathrm{Tor}_i^R(\mathrm{Tr}\,M,N)=0$ for all $i>0$. Therefore, we have the following equality.

$$\mathrm{depth}_R(\mathrm{Tr}\,M\otimes_R N)+\mathrm{depth}\,R=\mathrm{depth}_R(\mathrm{Tr}\,M)+\mathrm{depth}_R(N) \qquad (3.1)$$

by Theorem 2.3. Set $t=\mathrm{depth}_R(N)-n$. We argue by induction on t. If $t\leq 1$, then $\mathrm{depth}_R(N)\leq n+1$. If $\mathrm{depth}_R(N)=0$, then it is clear that $\mathrm{depth}_R(\mathrm{Tr}\,M)=\mathrm{depth}\,R$ by (3.1) and so $\text{G-dim}_R(\mathrm{Tr}\,M)=0$ by the Auslander–Bridger formula. Now let $0<\mathrm{depth}_R(N)\leq n+1$. As $M\approx\mathrm{Tr}\,\mathrm{Tr}\,M$, we obtain the following exact sequence:

$$0\to\mathrm{Ext}^1_R(M,N)\to\mathrm{Tr}\,M\otimes_R N\to\mathrm{Hom}_R((\mathrm{Tr}\,M)^*,N)\to\mathrm{Ext}^2_R(M,N)$$

from the exact sequence (2.2). As $\mathrm{Ext}^1_R(M,N)=0$, we get the following exact sequence:

$$0\to\mathrm{Tr}\,M\otimes_R N\to\mathrm{Hom}_R((\mathrm{Tr}\,M)^*,N)\to\mathrm{Ext}^2_R(M,N). \qquad (3.2)$$

Therefore, $\mathrm{Ass}_R(\mathrm{Tr}\,M\otimes_R N)\subseteq\mathrm{Ass}_R(\mathrm{Hom}_R((\mathrm{Tr}\,M)^*,N))\subseteq\mathrm{Ass}_R(N)$ by the exact sequence (3.2). Hence, $\mathrm{depth}_R(\mathrm{Tr}\,M\otimes_R N)>0$. Now by (3.1), it is easy to see that $\mathrm{depth}_R(\mathrm{Tr}\,M)\geq\mathrm{depth}\,R-n$ and so $\text{G-dim}_R(\mathrm{Tr}\,M)\leq n$.

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Now suppose that $t > 1$ and consider the following exact sequence,

$$0 \to \Omega M \to F \to M \to 0, \tag{3.3}$$

where F is a free R-module. From the exact sequence (3.3), we obtain the following exact sequence:

$$0 \to M^* \to F^* \to (\Omega M)^* \to \mathbb{D}(M) \to \mathbb{D}(F) \to \mathbb{D}(\Omega M) \to 0,$$

where $\mathbb{D}(X) \approx \operatorname{Tr} X$ for all R-modules X by [**4**, Lemma 3.9]. As $\operatorname{Ext}^1_R(M, R) = 0$, we get the following exact sequence:

$$0 \to \mathbb{D}(M) \to \mathbb{D}(F) \to \mathbb{D}(\Omega M) \to 0. \tag{3.4}$$

Note that $\mathbb{D}(F)$ is free. As $\operatorname{Ext}^i_R(\Omega M, N) \cong \operatorname{Ext}^{i+1}_R(M, N) = 0$ for all $1 \leq i \leq \operatorname{depth}_R(N) - n - 1$, we have $\operatorname{G-dim}_R(\operatorname{Tr} \Omega M) \leq n + 1$ by induction hypothesis. Therefore, $\operatorname{G-dim}_R(\operatorname{Tr} M) \leq n$ by the exact sequence (3.4). □

THEOREM 3.2. *Let R be a Gorenstein ring, and let M and N be non-zero R-modules such that N has reducible complexity. Assume that N is rigid and that $n \geq 0$ is an integer. Then the following statements hold true:*

(i) If $\operatorname{Ext}^i_R(M, N) = 0$ for all $1 \leq i \leq \max\{1, \operatorname{depth}_R(N) - n\}$ and M satisfies (S_n), then $\operatorname{G-dim}_R(M) = 0$.
(ii) If $\operatorname{Ext}^i_R(M, N) = 0$ for all $1 \leq i \leq \max\{1, \operatorname{depth}_R(N) - n\}$, then $\operatorname{G-dim}_R(M^*) \leq n - 2$.

Proof. (i) First note that $\operatorname{G-dim}_R(\operatorname{Tr} M) = \sup\{i \mid \operatorname{Ext}^i_R(\operatorname{Tr} M, R) \neq 0\}$ by [**4**, Theorem 4.13]. As M satisfies (S_n), $\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0$ for all $1 \leq i \leq n$ by [**4**, Theorem 4.25]. On the other hand, $\operatorname{G-dim}_R(\operatorname{Tr} M) \leq n$ by Lemma 3.1. Therefore, $\operatorname{G-dim}_R(\operatorname{Tr} M) = 0$ and so $\operatorname{G-dim}_R(M) = 0$ by [**4**, Lemmm 4.9].

(ii) Note that $M^* \approx \Omega^2 \operatorname{Tr} M$. By Lemma 3.1, $\operatorname{G-dim}_R(\operatorname{Tr} M) \leq n$ and so $\operatorname{G-dim}_R(M^*) \leq n - 2$. □

The following is a generalization of [**16**, Theorem 1].

COROLLARY 3.3. *Let R be a complete intersection, and let M and N be non-zero R-modules. Assume that N is a rigid module of maximal complexity. If $\operatorname{Ext}^i_R(M, N) = 0$ for all i, $1 \leq i \leq \max\{1, \operatorname{depth}_R(N) - 2\}$, then M^* is free.*

Proof. By Lemma 3.1, $\operatorname{Tor}^R_i(\operatorname{Tr} M, N) = 0$ for all $i > 0$. As $M^* \approx \Omega^2 \operatorname{Tr} M$, $\operatorname{Tor}^R_i(M^*, N) = 0$ for all $i > 0$ and so $\operatorname{cx}_R(M^*) + \operatorname{cx}_R(N) \leq \operatorname{codim} R$ by [**5**, Theorem II]. Since N has maximal complexity, it follows that $\operatorname{cx}_R(M^*) = 0$. Therefore, $\operatorname{pd}_R(M^*) = \operatorname{G-dim}_R(M^*) = 0$ by Theorem 3.2(ii). □

It is well known that over a regular local ring, every finite module is rigid. In the following we collect some other examples of rigid modules.

EXAMPLE 3.4.

(i) A class of rigid modules was discovered by Peskine and Szpiro [**19**]. They proved that if R is local, and the minimal free resolution of M over R is of the form

$$0 \to R^m \to R^{k+m} \to R^k \to 0,$$

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for some $m > 0$ and $k > 0$, then M is rigid. In [**22**], Tchernev discovered a new class of rigid modules. He showed that if R is local, and the minimal free resolution of M over R is of the form

$$0 \to R^k \to R^{m+1} \to R^m \to 0,$$

for some $m > 0$ and $k > 0$, then M is rigid [**22**, Theorem 3.6].

(ii) Let R be an admissible hypersurface with isolated singularity, and let N be an R-module. If $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, then N is rigid [**10**, Corollary 4.2].

(iii) Let $(R, \mathfrak{m})$ be a local hypersurface ring such that $\widehat{R} = S/(f)$, where $(S, \mathfrak{n})$ is a complete unramified regular local ring and f is a regular element of S contained in n^2. Let M be an R-module of finite projective dimension. Then M is rigid [**17**, Theorem 3].

In the following, we generalize [**15**, Corollary 1].

THEOREM 3.5. *Let R be a local complete intersection ring, and let M and N be non-zero R-modules. Assume that the following conditions hold:*

(i) N is rigid.
(ii) M satisfies (S_n) for some $n \geq 0$.
(iii) $\mathrm{Ext}^i_R(M, N) = 0$ for some positive integer i such that $i \geq \mathrm{depth}_R(N) - n$.

Then $\mathrm{CI\text{-}dim}_R(M) = \sup\{j \mid \mathrm{Ext}^j_R(M, N) \neq 0\} < i$.

Proof. Set $L = \Omega^{i-1}M$. Note that L satisfies (S_{n+i-1}) and $\mathrm{Ext}^1_R(L, N) = 0$. Now by Theorem 3.2(i), $\mathrm{CI\text{-}dim}_R(L) = \mathrm{G\text{-}dim}_R(L) = 0$. By Lemma 3.1, $\mathrm{Tor}^R_j(\mathrm{Tr}\, L, N) = 0$ for all $j > 0$ and so $\mathrm{Ext}^j_R(L, N) = 0$ for all $j > 0$ by Theorem 2.7. Therefore, $\mathrm{Ext}^j_R(M, N) = 0$ for all $j \geq i$ and so $\mathrm{CI\text{-}dim}_R(M) = \sup\{j \mid \mathrm{Ext}^j_R(M, N) \neq 0\} < i$ by Theorem 2.6. □

The following is a generalization of [**15**, Corollary 2]

THEOREM 3.6. *Let R be a local complete intersection ring, and let M and N be non-zero R-modules. Suppose that N is rigid, and that M satisfies (S_n) for some $n \geq 0$. If* $\mathrm{depth}_R(N) - n \leq \mathrm{CI\text{-}dim}_R(M)$*, then for all $i > 0$ in the range* $\mathrm{depth}_R(N) - n \leq i \leq \mathrm{CI\text{-}dim}_R(M)$*, we have* $\mathrm{Ext}^i_R(M, N) \neq 0$.

Proof. If $\mathrm{Ext}^i_R(M, N) = 0$ for some $\mathrm{depth}_R(N) - n \leq i \leq \mathrm{CI\text{-}dim}_R(M)$, then $\mathrm{Ext}^1_R(\Omega^{i-1}M, N) \cong \mathrm{Ext}^i_R(M, N) = 0$. Note that $\Omega^{i-1}M$ satisfies (S_{n+i-1}). Now by Theorem 3.2(i), we have $\mathrm{CI\text{-}dim}_R(\Omega^{i-1}M) = \mathrm{G\text{-}dim}_R(\Omega^{i-1}M) = 0$. Therefore, $\mathrm{CI\text{-}dim}_R(M) < i$ by [**6**, Lemma 1.9], which is a contradiction. □

Let R be a hypersurface, and let M and N be R-modules such that $\mathrm{length}_R(N) < \infty$. It is well known that if $\mathrm{Ext}^i_R(M, N) = 0$ for some $i > \mathrm{CI\text{-}dim}_R(M)$, then $\mathrm{Ext}^n_R(M, N) = 0$ for all $n > \mathrm{CI\text{-}dim}_R(M)$ (see for example [**8**, Corollary 3.5]). In special cases, we can remove the condition that $i > \mathrm{CI\text{-}dim}_R(M)$.

COROLLARY 3.7. *Let $(R, \mathfrak{m})$ be a local hypersurface ring such that $\widehat{R} = S/(f)$, where $(S, \mathfrak{n})$ is a complete unramified regular local ring and f is a regular element of S contained in $\mathfrak{n}^2$. Let M and N be non-zero R-modules such that* $\mathrm{length}_R(N) < \infty$. *If* $\mathrm{Ext}^n_R(M, N) = 0$ *for some $n \geq 1$, then the following statements hold true:*

(i) $\mathrm{CI\text{-}dim}_R(M) = \sup\{i \mid \mathrm{Ext}^i_R(M, N) \neq 0\} < n$.
(ii) Either $\mathrm{pd}_R(M) < \infty$ *or* $\mathrm{pd}_R(N) < \infty$.

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Proof. First note that N is rigid by [**13**, Theorem 2.4]. It follows from Theorem 3.5 that $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}^i_R(M,N) \neq 0\} < n$. As $\text{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, either $\text{pd}_R(M) < \infty$ or $\text{pd}_R(N) < \infty$ by Theorem 2.5. □

As an application of Theorem 3.2, we have the following result.

COROLLARY 3.8. *Let R be an admissible hypersurface, and let M and N be non-zero R-modules such that $\text{cx}_R(N) = 1$. Assume that the minimal free resolution of N is eventually periodic of period one, and that M satisfies (S_n) for some $n \geq 0$. Then the following statements hold true:*

(i) If $\text{depth}_R(N) - n \leq \text{CI-dim}_R(M)$, then for all $i > 0$ in the range $\text{depth}_R(N) - n \leq i \leq \text{CI-dim}_R(M)$, we have $\text{Ext}^i_R(M,N) \neq 0$.
(ii) If $\text{Ext}^i_R(M,N) = 0$ for some positive integer i such that $i \geq \text{depth}_R(N) - n$, then $\text{pd}_R(M) < i$.
(iii) If $\text{Ext}^i_R(M,N) = 0$ for all i, $1 \leq i \leq \max\{1, \text{depth}_R(N) - 2\}$, then M^* is free.

Proof. Note that N is rigid by [**10**, Corollary 5.6]. Now the first assertion is clear by Theorem 3.6.

(ii) By Theorem 3.5, $\text{Ext}^j_R(M,N) = 0$ for all $j \geq i$. Therefore, $\text{pd}_R(M) < \infty$ by Theorem 2.5 and so $\text{pd}_R(M) < i$.

(iii) Note that N has maximal complexity. Therefore, the assertion is clear by Corollary 3.3. □

Let R be an admissible hypersurface with isolated singularity of dimension $d > 1$. By [**10**, Theorem 3.4], every R-module of dimension less than or equal to one is rigid. As an immediate consequence of Theorem 3.5, we have the following result.

COROLLARY 3.9. *Let R be an admissible hypersurface with isolated singularity of dimension $d > 1$, and let M and N be non-zero R-modules such that $\dim_R(N) \leq 1$. If $\text{Ext}^n_R(M,N) = 0$ for some $n > 0$, then $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}^i_R(M,N) \neq 0\} < n$. Moreover, either $\text{pd}_R(M) < \infty$ or $\text{pd}_R(N) < \infty$.*

In the case of dimension 2, we have the following result.

PROPOSITION 3.10. *Let R be an admissible hypersurface of dimension 2. Assume further that R is normal. Let M and N be non-zero R-modules such that $\text{depth}_R(N) \leq \text{depth}_R(M) + 1$. If $\text{Ext}^1_R(M,N) = 0$, then $\text{CI-dim}_R(M) = 0$ and $\text{Ext}^i_R(M,N) = 0$ for all $i > 0$. Moreover, either M is free or N has finite projective dimension.*

Proof. First note that N is rigid by [**10**, Corollary 3.6]. If $\text{depth}_R(N) \leq 1$, then the assertion is clear by Theorem 3.5. Now let N be maximal Cohen–Macaulay. Then $\text{depth}_R(M) > 0$ and so

$$\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}^i_R(M,R) \neq 0\} = 2 - \text{depth}_R(M) \leq 1. \tag{3.5}$$

By Theorem 2.4, $\text{Tor}^R_1(\mathcal{T}_2 M, N) = 0$. As N is rigid, $\text{Tor}^R_i(\mathcal{T}_2 M, N) = 0$ for all $i > 0$. It follows from Theorem 2.4 again that $\text{Ext}^1_R(M,R) = 0$ and so M is maximal Cohen–Macaulay by (3.5). Now it is easy to see that $\text{Tr}\, M \approx \Omega\mathcal{T}_2 M$ and so $\text{Tor}^R_i(\text{Tr}\, M, N) = 0$ for all $i > 0$. Therefore, $\text{Ext}^i_R(M,N) = 0$ for all $i > 0$ by Theorem 2.7 and so either M is free or N has finite projective dimension by Theorem 2.5. □

4. Vanishing of Ext over complete intersection rings. Let R be a local complete intersection ring of codimension c, and let M and N be R-modules. In [**18**], Murthy proved that if $\mathrm{Tor}_n^R(M,N) = \mathrm{Tor}_{n+1}^R(M,N) = \cdots = \mathrm{Tor}_{n+c}^R(M,N) = 0$ for some $n > 0$, then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq n$. It is easy to see that a similar statement is not true in general, with Tor replaced by Ext. In the following, we prove a similar result for Ext with an extra hypothesis. The following result is a generalization of [**14**, Corollary].

THEOREM 4.1. *Let R be a local complete intersection ring of codimension c, and let M and N be non-zero R-modules. Assume n is a positive integer. If $\mathrm{Ext}_R^i(M,N) = 0$, for all i, $n \leq i \leq n+c$ and $\mathrm{depth}_R(N) \leq n+c$, then $\text{CI-dim}_R(M) = \sup\{i \mid \mathrm{Ext}_R^i(M,N) \neq 0\} < n$.*

Proof. Without loss of generality we may assume that R is complete. We have $R = Q/(x)$ with Q a complete regular local ring and x an Q-sequence of length c contained in the square of the maximal ideal of Q. We argue by induction on c. If $c = 0$, then R is a regular local ring and so $\mathrm{pd}_R(M) = \sup\{i \mid \mathrm{Ext}_R^i(M,N) \neq 0\} < n$ by [**15**, Corollary 1]. For $c > 0$, set $S = Q/(x_1, \ldots, x_{c-1})$. Therefore, $R \cong S/(x_c)$. Note that $\mathrm{depth}_R(N) = \mathrm{depth}_S(N)$.

The change of rings spectral sequence (see [**20**, Theorem 11.66])

$$\mathrm{Ext}_R^p(M, \mathrm{Ext}_S^q(R,N)) \underset{p}{\Rightarrow} \mathrm{Ext}_S^{p+q}(M,N)$$

degenerates into a long exact sequence

$$\cdots \to \mathrm{Ext}_R^i(M,N) \to \mathrm{Ext}_S^i(M,N) \to \mathrm{Ext}_R^{i-1}(M,N) \to \mathrm{Ext}_R^{i+1}(M,N) \to \cdots.$$

It follows that $\mathrm{Ext}_S^i(M,N) = 0$ for all i, $n+1 \leq i \leq n+c$, and so by induction hypothesis we conclude that $\text{CI-dim}_S(M) = \sup\{i \mid \mathrm{Ext}_S^i(M,N) \neq 0\} < n+1$. Therefore, $\mathrm{Ext}_R^{i-1}(M,N) \cong \mathrm{Ext}_R^{i+1}(M,N)$ for all $i > n$. As $c > 0$, it is clear that $\mathrm{Ext}_R^i(M,N) = 0$ for all $i \geq n$ and so $\text{CI-dim}_R(M) = \sup\{i \mid \mathrm{Ext}_R^i(M,N) \neq 0\} < n$ by Theorem 2.6. □

In special cases, one can improve Theorem 4.1 slightly. The following is a generalization of Corollary 3.7.

PROPOSITION 4.2. *Let $(R, \mathfrak{m})$ be a local ring such that $\widehat{R} = S/(f)$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $f = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. Assume that $n \geq 0$ is an integer and that M and N are non-zero finite R-modules such that $\mathrm{length}_R(N) < \infty$. If $\mathrm{Ext}_R^i(M,N) = 0$ for all i, $n+1 \leq i \leq n+c$, then $\text{CI-dim}_R(M) = \sup\{i \mid \mathrm{Ext}_R^i(M,N) \neq 0\} \leq n$.*

Proof. Without loss of generality we may assume that R is complete and $R = S/(f)$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $f = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. We argue by induction on c. If $c = 1$, then the assertion holds by Corollary 3.7. For $c > 1$, set $Q = S/(f_1, \ldots, f_{c-1})$. Therefore, $R \cong Q/(f_c)$. Note that $\mathrm{length}_Q(N) < \infty$. The change of rings spectral sequence

$$\mathrm{Ext}_R^p(M, \mathrm{Ext}_Q^q(R,N)) \underset{p}{\Rightarrow} \mathrm{Ext}_Q^{p+q}(M,N)$$

degenerates into a long exact sequence

$$\cdots \to \mathrm{Ext}_R^i(M,N) \to \mathrm{Ext}_Q^i(M,N) \to \mathrm{Ext}_R^{i-1}(M,N) \to \mathrm{Ext}_R^{i+1}(M,N) \to \cdots.$$

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It follows that $\operatorname{Ext}^i_Q(M,N)=0$ for all i, $n+2 \leq i \leq n+c$, and so by induction hypothesis we conclude that $\text{CI-dim}_Q(M) \leq n+1$ and $\operatorname{Ext}^i_Q(M,N)=0$ for all $i > n+1$. Therefore, $\operatorname{Ext}^{i-1}_R(M,N) \cong \operatorname{Ext}^{i+1}_R(M,N)$ for all $i > n+1$. As $c > 1$, it is clear that $\operatorname{Ext}^i_R(M,N)=0$ for all $i > n$ and so $\text{CI-dim}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(M,N) \neq 0\} \leq n$ by Theorem 2.6. □

As an application of Theorem 4.1, we can generalize [**15**, Corollary 1] as follows.

Corollary 4.3. *Let R be a local complete intersection ring of codimension c, and let M and N be non-zero R-modules. Assume that $n > 0$ and $t \geq 0$ are integers and that the following conditions hold:*

(i) $\operatorname{Ext}^i_R(M,N)=0$ *for all* i, $n \leq i \leq n+c$.
(ii) *M satisfies* (S_t).
(iii) $\operatorname{depth}_R(N) \leq n+c+t$.
Then $\text{CI-dim}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(M,N) \neq 0\} < n$.

Proof. We argue by induction on t. If $t=0$, then the assertion is clear by Theorem 4.1. Now suppose that $t>0$ and consider the universal push forward of M,

$$0 \to M \to F \to M_1 \to 0, \tag{4.1}$$

where F is free. It is easy to see that M_1 satisfies (S_{t-1}). From the exact sequence (4.1), it is clear that

$$\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i+1}_R(M_1,N) \text{ for all } i > 0. \tag{4.2}$$

Therefore, $\operatorname{Ext}^i_R(M_1,N)=0$ for all i, $n+1 \leq i \leq n+c+1$. By induction hypothesis, we conclude that $\operatorname{Ext}^i_R(M_1,N)=0$ for all $i > n$. By (4.2), $\operatorname{Ext}^i_R(M,N)=0$ for all $i \geq n$ and so $\text{CI-dim}_R(M) = \sup\{i \mid \operatorname{Ext}^i_R(M,N) \neq 0\} < n$ by Theorem 2.6. □

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