Bull. Aust. Math. Soc. (First published online 2024), page 1 of 8*
doi:10.1017/S0004972724000807
*Provisional—final page numbers to be inserted when paper edition is published

ON A CONJECTURE REGARDING THE SYMMETRIC DIFFERENCE OF CERTAIN SETS

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(Received 22 June 2024; accepted 26 June 2024)

Abstract

Let n be a positive integer and $\underline{n} = \{1, 2, \ldots, n\}$. A conjecture arising from certain polynomial near-ring codes states that if $k \geq 1$ and $a_1, a_2, \ldots, a_k$ are distinct positive integers, then the symmetric difference $a_1\underline{n} \,\Delta\, a_2\underline{n} \,\Delta \cdots \Delta\, a_k\underline{n}$ contains at least n elements. Here, $a_i\underline{n} = \{a_i, 2a_i, \ldots, na_i\}$ for each i. We prove this conjecture for arbitrary n and for $k = 1, 2, 3$.

2020 *Mathematics subject classification*: primary 05A20; secondary 94A60.

Keywords and phrases: symmetric difference, polynomial codes.

1. Introduction

Although the conjecture mentioned in the title originated through its connections with coding theory (see [4] regarding minimum distances of certain linear codes defined *via* polynomial near-rings), we intend to discuss it here from a more informal and general viewpoint, without its connections to coding theory.

Imagine three people with the numbers 1, 2 and 3 on their respective T-shirts entering an empty room. After a minute, three other people with the numbers 2, 4 and 6 on their T-shirts enter the room. There are now two people with the number 2, and these two decide to leave the room, leaving the four people with numbers 1, 3, 4 and 6 behind in the room. After another minute, three further people, with the numbers 3, 6 and 9 on their T-shirts enter the room, and the same procedure is followed: the two with number 3, as well as the two with number 6, leave the room, and the three without matching numbers (1, 4 and 9) stay in the room. Then three more people with the numbers 4, 8 and 12 enter the room, and so on. The conjecture is that there will always be at least three people left in the room. Note that we could reformulate this in terms of symmetric differences of sets, namely, the cardinality of $\{1, 2, 3\} \,\Delta\, \{2, 4, 6\} \,\Delta \cdots \Delta\, \{k, 2k, 3k\}$ is at least 3 for any positive integer k. This result

The second author would like to acknowledge with appreciation the financial support he received from the University of the Free State, South Africa, as well as the National Cheng Kung University, Taiwan.

is fairly easy to prove. A slightly more general form of this conjecture is also not too difficult to establish: the cardinality of

$$\{a_1, 2a_1, 3a_1\}\,\Delta\,\{a_2, 2a_2, 3a_2\}\,\Delta\cdots\Delta\,\{a_k, 2a_k, 3a_k\}$$

is at least three for any sequence $a_1 < a_2 < \cdots < a_k$ of positive integers.

The most general form of the conjecture is the following assertion.

GENERAL CONJECTURE. *For any positive integer n, the cardinality of the set*

$$\{a_1, 2a_1, \dots, na_1\}\,\Delta\,\{a_2, 2a_2, \dots, na_2\}\,\Delta\cdots\Delta\,\{a_k, 2a_k, \dots, na_k\}$$

is at least n for any sequence $a_1 < a_2 < \cdots < a_k$ of positive integers.

Some partial results have been established. For example, the conjecture is known to be true when $\{a_1, a_2, \dots, a_k\} = \{1, 2, \dots, k\}$ (see [2, 3]). We note that the reviewer of the article [3] (Mathematical Reviews, #MR2862558) wrote '... and therefore establishes the value of the distance of the aforementioned code'. This is not true as the minimal distance of the code is only determined when the general conjecture is proved. The general conjecture is also known to be true for all n with $1 \le n \le 6$ (see [4]). There are also combinatorial problems motivated by the conjecture (see [1]).

The aim of this paper is to show that the general conjecture is true for all positive integers n and for all $a_1, a_2, \dots, a_k$, where $1 \le k \le 3$.

Although the general consensus is that the conjecture should be true, no proof is known. One easily senses that the cardinality of such symmetric differences can get as large as possible. In fact, this is true. As shown in [2], when $\{a_1, a_2, \dots, a_k\} = \{1, 2, \dots, k\}$, the resulting cardinality of the symmetric difference is at least $\max\{k, n\}$. As an extreme case, when $\{a_1, a_2, \dots, a_k\} = \{1, n+1, \dots, kn+1\}$, the cardinality of the symmetric difference is kn, since no cancellations occur. However, it is also true that for any fixed $n \ge 2$ and for any $r \ge 0$, one can choose $a_1 < a_2 < \cdots < a_k$, where $k > r$, such that the cardinality of

$$\{a_1, 2a_1, \dots, na_1\}\,\Delta\,\{a_2, 2a_2, \dots, na_2\}\,\Delta\cdots\Delta\,\{a_k, 2a_k, \dots, na_k\}$$

is exactly n. A proof of this fact will be given in the last section. One realises from these facts that there can be no straightforward way to approach the problem. For example, induction may not work on the general situation. Some new ideas are needed.

We hope that the material presented here will spark interest in the problem so that more, if not all, cases will be proved.

2. Terminology

To begin with, we establish some notation and terminology. For any positive integer n, put $\underline{n} = \{1, 2, \dots, n\}$. For positive integers $a_1 < a_2 < \cdots < a_k$, where $k \ge 1$, we consider the symmetric difference of the sets $a_i\underline{n} = \{a_i, 2a_i, \dots, na_i\}$, $i = 1, 2, \dots, k$, that is, $\Delta_{i=1}^{k} a_i\underline{n} = a_1\underline{n}\Delta a_2\underline{n}\Delta\cdots\Delta a_k\underline{n}$. Throughout the article, the k integers $a_1, a_2, \dots, a_k$ will

https://doi.org/10.1017/S0004972724000807 Published online by Cambridge University Press

be referred to as the *multipliers*. By using standard counting techniques based on the inclusion–exclusion principle, it follows that the cardinality of $\Delta_{i=1}^{k} a_i \underline{n}$ is given by

$$\xi_{n,k}(a_1, a_2, \ldots, a_k) = \sum_{r=1}^{k} \sum_{1\leq i_1<i_2<\cdots<i_r\leq k} (-1)^{r-1} 2^{r-1} \left\lfloor \frac{a_{i_1} \cdot n}{[a_{i_1}, a_{i_2}, \ldots, a_{i_r}]} \right\rfloor. \tag{2.1}$$

Here, and throughout the rest of our discussion, we use $[x, y, z, \ldots]$ for the least common multiple of the integers $x, y, z, \ldots$. Likewise, we use $(x, y, z, \ldots)$ for the greatest common divisor of $x, y, z, \ldots$.

The conjecture asserts that $\xi_{n,k}(a_1, a_2, \ldots, a_k) \geq n$ for any $n \geq 1$ and any sequence $a_1 < a_2 < \cdots < a_k$ of k multipliers, for any $k \geq 1$. Our aim is to prove this conjecture for the cases $k = 1, 2, 3$.

3. The cases $k = 1$ and $k = 2$

The conjecture is trivially true when $k = 1$, since $\xi_{n,1}(a_1) = n$, which is the cardinality of $a_1\underline{n} = \{a_1, 2a_1, \ldots, na_1\}$.

For $k = 2$, consider the multipliers $a_1 < a_2$. Since $a_1 < a_2 \leq [a_1, a_2]$, and since $a_1 \mid [a_1, a_2]$, we have $[a_1, a_2] \geq 2a_1$. So, $a_1 n/[a_1, a_2] \leq n/2$, from which it follows that

$$\xi_{n,2}(a_1, a_2) = 2n - 2\left\lfloor \frac{a_1 \cdot n}{[a_1, a_2]} \right\rfloor \geq 2n - 2\left\lfloor \frac{n}{2} \right\rfloor \geq n,$$

since $\lfloor n/2 \rfloor \leq n/2$.

4. The case $k = 3$

The conjecture is known to be true for $1 \leq n \leq 6$ [4, Corollary 2]. Hence, for the remainder of this paper, we will assume that $n \geq 7$. To avoid unnecessary subscripts, we will simply denote the multipliers $a_1 < a_2 < a_3$ by $a < b < c$ in this section. Furthermore, since $a = 1$ does not have any prime divisors, it turns out that we should treat this case separately.

Hence, we will assume first that the multipliers are $1 < b < c$. Here we want to show that

$$\frac{\xi_{n,3}(1, b, c) - n}{2} = n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor + \left\lfloor \frac{bn}{[b, c]} \right\rfloor \right) + 2\left\lfloor \frac{n}{[b, c]} \right\rfloor \geq 0. \tag{4.1}$$

This will be investigated by considering two sub-cases.

(1) Assume $b \mid c$, say $c = tb$, $t \geq 2$. Then,

$$\begin{aligned} n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor + \left\lfloor \frac{bn}{[b, c]} \right\rfloor \right) + 2\left\lfloor \frac{n}{[b, c]} \right\rfloor &= n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor + \left\lfloor \frac{bn}{c} \right\rfloor \right) + 2\left\lfloor \frac{n}{c} \right\rfloor \\ &= n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{tb} \right\rfloor + \left\lfloor \frac{n}{t} \right\rfloor \right) + 2\left\lfloor \frac{n}{tb} \right\rfloor \\ &= n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{t} \right\rfloor \right) + \left\lfloor \frac{n}{tb} \right\rfloor \end{aligned}$$

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$$\geq n - 2\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{tb} \right\rfloor, \text{ since } b \geq 2 \text{ and } t \geq 2$$
$$\geq \left\lfloor \frac{n}{tb} \right\rfloor \geq 0.$$

So (4.1) holds in this case.

(2) Assume $b \nmid c$. Before we proceed with this case, we first mention three results.

(a) For real numbers x and y, it is well known that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.

(b) For a real number x, we have $\lfloor -2x \rfloor = -2\lfloor x \rfloor + \delta$, where $\delta \in \{-2, -1, 0\}$.

PROOF. Consider three cases.

(i) $x = m \in \mathbb{Z}$. Then, $\lfloor -2x \rfloor = -2m = -2\lfloor m \rfloor = -2\lfloor x \rfloor$, which gives $\delta = 0$.

(ii) $x = m + \epsilon$, where $m \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$ with $0 < \epsilon < \frac{1}{2}$. Then, $-2x = -2m - 2\epsilon = -2m - 1 + (1 - 2\epsilon)$ with $0 < 1 - 2\epsilon < 1$. It follows that $\lfloor -2x \rfloor = -2m - 1 = -2\lfloor x \rfloor - 1$, giving $\delta = -1$.

(iii) $x = m + \epsilon$, where $m \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$ with $\frac{1}{2} \leq \epsilon < 1$. As in Case (ii), we see that $\lfloor -2x \rfloor = -2\lfloor x \rfloor - 2$, so that $\delta = -2$. □

(c) Consider the function $f(x, y) = 1/x + 1/y - 2/xy$, where x and y are real variables with $x \geq 2$ and $y \geq 3$. Then the maximum value of $f(x, y)$ is given by $f(2, y) = 1/2$ for any $y \geq 3$.

PROOF. $f(x, y) = (1/x)(1 - 2/y) + 1/y$, and since $1 - 2/y > 0$, $f(x, y)$ achieves its maximum value when x is as small as possible, that is, $x = 2$. However, then $f(2, y) = 1/2$ for any $y \geq 3$. □

We are now ready to proceed with Case (2), where $b \nmid c$. We have $b < c < [b, c]$, and from $b \mid [b, c]$ and $c \mid [b, c]$, we get $3b \leq [b, c]$. Hence, $n/3 \geq bn/[b, c]$, from which it follows that $-\lfloor bn/[b, c] \rfloor \geq -\lfloor n/3 \rfloor$. So we see that

$$n - \left(\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor + \left\lfloor \frac{bn}{[b, c]} \right\rfloor\right) + 2\left\lfloor \frac{n}{[b, c]} \right\rfloor \geq n - \left\lfloor \frac{n}{b} \right\rfloor - \left\lfloor \frac{n}{c} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor + 2\left\lfloor \frac{n}{[b, c]} \right\rfloor.$$

It therefore suffices to show that

$$\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor - 2\left\lfloor \frac{n}{[b, c]} \right\rfloor \leq n - \left\lfloor \frac{n}{3} \right\rfloor. \tag{4.2}$$

From item (c),

$$\frac{n}{2} \geq \frac{n}{b} + \frac{n}{c} - \frac{2n}{bc} \quad \text{so that} \quad \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor + \left\lfloor -\frac{2n}{bc} \right\rfloor,$$

using item (a). It follows that

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor - 2\left\lfloor \frac{n}{bc} \right\rfloor + \delta \quad \text{where } \delta \in \{-2, -1, 0\},$$

by item (b). Since $[b, c] \leq bc$, this gives

$$\left\lfloor \frac{n}{2} \right\rfloor + 2 \geq \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor - 2\left\lfloor \frac{n}{[b, c]} \right\rfloor.$$

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For $n \geq 12$, it easily follows that $n - 2 \geq n/2 + n/3 \geq \lfloor n/2 \rfloor + \lfloor n/3 \rfloor$, that is, $\lfloor n/2 \rfloor + 2 \leq n - \lfloor n/3 \rfloor$. Direct checking shows that this relation is also valid for $7 \leq n \leq 11$. Indeed,

$$\begin{aligned}
n = 7 :& \quad \lfloor \tfrac{7}{2} \rfloor + 2 = 5 \leq 7 - 2 = 7 - \lfloor \tfrac{7}{3} \rfloor,\\
n = 8 :& \quad \lfloor \tfrac{8}{2} \rfloor + 2 = 6 \leq 8 - 2 = 8 - \lfloor \tfrac{8}{3} \rfloor,\\
n = 9 :& \quad \lfloor \tfrac{9}{2} \rfloor + 2 = 6 \leq 9 - 3 = 9 - \lfloor \tfrac{9}{3} \rfloor,\\
n = 10 :& \quad \lfloor \tfrac{10}{2} \rfloor + 2 = 7 \leq 10 - 3 = 10 - \lfloor \tfrac{10}{3} \rfloor,\\
n = 11 :& \quad \lfloor \tfrac{11}{2} \rfloor + 2 = 7 \leq 11 - 3 = 11 - \lfloor \tfrac{11}{3} \rfloor.
\end{aligned}$$

Hence, for all $n \geq 7$,

$$\left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{c} \right\rfloor - 2\left\lfloor \frac{n}{[b,c]} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq n - \left\lfloor \frac{n}{3} \right\rfloor,$$

that is, (4.2) holds. This completes the discussion for $a = 1$.

From here on, we assume that $2 \leq a < b < c$. We may also assume that $(a, b, c) = 1$ (since each fraction $a_{i_1}/[a_{i_1}, a_{i_2}, \ldots, a_{i_r}]$ in (2.1) remains unchanged if we cancel out any common factor between a_{i_1} and $[a_{i_1}, a_{i_2}, \ldots, a_{i_r}]$.) We begin by investigating when

$$g(a, b, c) := \frac{a}{[a, b]} + \frac{a}{[a, c]} + \frac{b}{[b, c]} \leq 1. \tag{4.3}$$

Note that whenever (4.3) holds,

$$\begin{aligned}
\frac{\xi_{n,3}(a, b, c) - n}{2} &= n - \left(\left\lfloor \frac{a \cdot n}{[a, b]} \right\rfloor + \left\lfloor \frac{a \cdot n}{[a, c]} \right\rfloor + \left\lfloor \frac{b \cdot n}{[b, c]} \right\rfloor\right) + 2\left\lfloor \frac{a \cdot n}{[a, b, c]} \right\rfloor\\
&\geq n - \left(\frac{a \cdot n}{[a, b]} + \frac{a \cdot n}{[a, c]} + \frac{b \cdot n}{[b, c]}\right) \quad \text{using Case (2)(a)}\\
&= (1 - g(a, b, c)) \cdot n \geq 0,
\end{aligned}$$

from which it follows that $\xi_{n,3}(a, b, c) \geq n$.

Let $d_1 = (a, b)$, $d_2 = (b, c)$ and $d_3 = (a, c)$. Then, $(d_1, d_2) = (d_1, d_3) = (d_2, d_3) = 1$, because $(a, b, c) = 1$. So $a = d_1 d_3 q_1$, $b = d_1 d_2 q_2$, $c = d_2 d_3 q_3$ for some mutually relatively prime positive integers q_1, q_2 and q_3. It follows that

$$d_1 d_3 q_1 < d_1 d_2 q_2 < d_2 d_3 q_3, \tag{4.4}$$

and

$$g(a, b, c) = \frac{1}{d_2 q_2} + \frac{1}{d_2 q_3} + \frac{1}{d_3 q_3}. \tag{4.5}$$

Furthermore, there exist positive integers s_1, s_2 and s_3 such that $b = a + s_1 d_1$, $c = b + s_2 d_2$ and $c = a + s_3 d_3$.

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We now proceed by partitioning the possible values of the triples (d_1, d_2, d_3) into four different classes.

Class 1: triples (d_1, d_2, d_3) *for which* $d_1 \geq 2$. Here, $b = a + s_1 d_1$ implies that $d_2 q_2 = d_3 q_1 + s_1 \geq 2$. Similarly, $c = b + s_2 d_2$ implies that $d_3 q_3 = d_1 q_2 + s_2 \geq 3$, and also, $c = a + s_3 d_3$ implies that $d_2 q_3 = d_1 q_1 + s_3 \geq 3$. If each of these inequalities happens to be an equality, we obtain $g(a, b, c) = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6}$, which is greater than 1. However, if any one of these inequalities becomes strict, we see that $g(a, b, c) \leq 1$. In the special event of three equalities, we must have $d_2 q_2 = 2$, $d_3 q_3 = 3$ and $d_2 q_3 = 3$. However, this can only happen if $d_2 = d_3 = 1$, $q_2 = 2$ and $q_3 = 3$, giving $c = d_2 d_3 q_3 = 3$, which is not possible, since $a \geq 2$. Therefore, $g(a, b, c) \leq 1$ for all triples (a, b, c) for which $d_1 = (a, b) \geq 2$.

Class 2: triples (d_1, d_2, d_3) *for which* $d_1 = 1$ *and* $d_2 \geq 2$. Here, $b = d_2 q_2 = d_3 q_1 + s_1 \geq 2$, $d_3 q_3 = q_2 + s_2 \geq 2$ and $d_2 q_3 = q_1 + s_3 \geq 2$.

If $d_2 = 2$, then, since $a = d_3 q_1 \geq 2$ and $(a, b, c) = 1$, we must have either $d_3 \geq 3$ or $q_1 \geq 3$. If $d_3 \geq 3$, then, from the inequalities above, $b = d_2 q_2 = d_3 q_1 + s_1 \geq 4$, $d_3 q_3 = q_2 + s_2 \geq 3$ and $d_2 q_3 = q_1 + s_3 \geq 2$, so that $g(a, b, c) \leq \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = 13/12$. By checking the small cases, there is only one instance where $1 < g(a, b, c) \leq 13/12$, namely $g(3, 4, 6) = 13/12$. In this case, referring to (4.1),

$$\frac{\xi_{n,3}(3, 4, 6) - n}{2} = n + \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \geq 0 \quad \text{for all } n \geq 1.$$

However, if $q_1 \geq 3$, then, as above, $b = d_2 q_2 = d_3 q_1 + s_1 \geq 4$, $d_3 q_3 = q_2 + s_2 \geq 2$ and $d_2 q_3 = q_1 + s_3 \geq 4$, implying that $g(a, b, c) \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$.

Next, consider $d_2 \geq 3$. Using the same inequalities as above, we see that now $g(a, b, c) \leq \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6}$. Again, checking small cases, there is only one instance here where $1 < g(a, b, c) \leq \frac{7}{6}$, namely $g(2, 3, 6) = \frac{7}{6}$. In this case,

$$\frac{\xi_{n,3}(2, 3, 6) - n}{2} = n - \left\lfloor \frac{n}{2} \right\rfloor \geq 0 \quad \text{for all } n \geq 1.$$

We see that, apart from these two exceptional cases (which satisfy $\xi_{n,3}(a, b, c) \geq n$ by direct checking), all the other triples (a, b, c) in this class have $g(a, b, c) \leq 1$, so that $\frac{1}{2}(\xi_{n,3}(a, b, c) - n) \geq (1 - g(a, b, c))n \geq 0$.

Class 3: triples (d_1, d_2, d_3) *for which* $d_1 = d_2 = 1$ *and* $d_3 \geq 2$. Now we have $b = q_2 = d_3 q_1 + s_1 \geq 3$, $c = d_3 q_3 = q_2 + s_2 \geq 2$ and $c = d_3 q_3 = d_3 q_1 + s_3 d_3$, giving $q_3 = q_1 + s_3 \geq 2$. From $q_3 \geq 2$ and $d_3 \geq 2$, it follows that $c = d_3 q_3$ is actually greater than or equal to 4. For $c = 4$, there is only one possible triple (a, b, c), namely $a = 2$, $b = 3$ and $c = 4$, and we already know that $\xi_{n,3}(2, 3, 4) \geq n$. So we may assume that $q_3 \geq 3$ or $d_3 \geq 3$. In the former case, $g(a, b, c) = g(d_3 q_1, q_2, d_3 q_3) = 1/q_2 + 1/q_3 + 1/d_3 q_3 \leq 1/3 + 1/3 + 1/6 = 5/6 \leq 1$, and in the latter case, $g(a, b, c) =$

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$1/q_2 + 1/q_3 + 1/d_3q_3 \leq 1/3 + 1/2 + 1/6 = 1$. As in the previous paragraphs, we conclude that $\xi_{n,3}(a,b,c) \geq n$ for all triples (a,b,c) that belong to this class.

Class 4: $d_1 = d_2 = d_3 = 1$. From (4.4), $2 \leq q_1 < q_2 < q_3$. Then, $g(a,b,c) = g(q_1,q_2,q_3) \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{5} = 11/15 < 1$, and we again have $\xi_{n,3}(a,b,c) \geq n$.

5. Infinitely many cases where $\xi_{n,k}(a_1, a_2, \ldots, a_k) = n$.

One should not be misled by thinking that $\xi_{n,k}(a_1, a_2, \ldots, a_k)$ would grow without bound as k gets bigger. In this section, we conclude our discussion by proving that $\xi_{n,k}(a_1, a_2, \ldots, a_k) = n$ is possible for arbitrarily large k.

THEOREM 5.1. *Let $n \in \mathbb{N}$, $n \geq 2$. For each $r \geq 0$, we can find multipliers $a_1 < a_2 < \cdots < a_k$, where $k > r$, such that $\Delta_{i=1}^{k} a_i \underline{n} = \{a_1,\ 2^{2^r} \cdot a_1,\ 3^{2^r} \cdot a_1,\ \ldots,\ n^{2^r} \cdot a_1\}$.*

PROOF. For the sake of this theorem, we denote

$$D_{n,k}^{r}(a_1, a_2, \ldots, a_k) = \Delta_{i=1}^{k} a_i \underline{n} = \{a_1,\ 2^{2^r} \cdot a_1,\ 3^{2^r} \cdot a_1,\ \ldots,\ n^{2^r} \cdot a_1\}.$$

The proof is by induction on r. For $r = 0$, take $k = 1$ and $a_1 = 1$ so that

$$D_{n,1}^{0}(1) = \{1, 2, 3, \ldots, n\} = \{1,\ 2^{2^0} \cdot 1,\ 3^{2^0} \cdot 1, \ldots,\ n^{2^0} \cdot 1\}.$$

Assume that the statement is true for some $r \geq 0$, and $a_1 < a_2 < \cdots < a_k$, where $k > r$, are the multipliers used to produce the symmetric difference

$$D_{n,k}^{r}(a_1, a_2, \ldots, a_k) = \{a_1,\ 2^{2^r} \cdot a_1,\ 3^{2^r} \cdot a_1,\ \ldots,\ n^{2^r} \cdot a_1\}.$$

Then, for each m, $1 \leq m \leq n$,

$$\begin{aligned}
\Delta_{j=1}^{k}(m^{2^r} a_j \cdot \underline{n}) &= m^{2^r} \Delta_{j=1}^{k}(a_j \underline{n}) \\
&= \{m^{2^r} \cdot a_1,\ m^{2^r}(2^{2^r} \cdot a_1),\ m^{2^r}(3^{2^r} \cdot a_1),\ \ldots,\ m^{2^r}(n^{2^r} \cdot a_1)\} \\
&= m^{2^r} D_{n,k}^{r}(a_1, a_2, \ldots, a_k).
\end{aligned}$$

Now take the symmetric difference between the n sets:

$$\begin{array}{ccccc}
\{\ \ a_1, & 2^{2^r} a_1\ , & 3^{2^r} a_1\ , & \ldots, & n^{2^r} a_1\ \}, \\
\{2^{2^r} a_1, & 2^{2^r}(2^{2^r} a_1), & 2^{2^r}(3^{2^r} a_1), & \ldots, & 2^{2^r}(n^{2^r} a_1)\}, \\
\{3^{2^r} a_1, & 3^{2^r}(2^{2^r} a_1), & 3^{2^r}(3^{2^r} a_1), & \ldots, & 3^{2^r}(n^{2^r} a_1)\}, \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\{n^{2^r} a_1, & n^{2^r}(2^{2^r} a_1), & n^{2^r}(3^{2^r} a_1), & \ldots, & n^{2^r}(n^{2^r} a_1)\}.
\end{array}$$

Due to the symmetry of this 'matrix', the symmetric difference is the 'diagonal',

$$D = \{a_1,\ 2^{2^{r+1}} \cdot a_1,\ 3^{2^{r+1}} \cdot a_1, \ldots,\ n^{2^{r+1}} \cdot a_1\}.$$

https://doi.org/10.1017/S0004972724000807 Published online by Cambridge University Press

This symmetric difference uses the list of multipliers

$$a_1, a_2, \ldots, a_k;\ 2^{2^r} \cdot a_1, 2^{2^r} \cdot a_2, \ldots, 2^{2^r} \cdot a_k;\ \ \ldots;\ n^{2^r} \cdot a_1, n^{2^r} \cdot a_2, \ldots, n^{2^r} \cdot a_k.$$

There may be duplicates in this list. Since two identical entries will not have any effect on D, our final list of multipliers $a'_1 < a'_2 < \cdots < a'_{k'}$ is given by

$$A' = \Delta_{m=1}^{n}(m^{2^r} A) = \{a'_1, a'_2, \ldots, a'_{k'}\}$$

and

$$D_{n,k'}^{r+1}(a'_1, a'_2, \ldots, a'_{k'}) = D.$$

Since $n > 1$, there is a prime p with $n/2 < p \leq n$. Then, $\{a_1, p^{2^r} a_1, p^{2^r} a_2, \ldots, p^{2^r} a_k\} \subseteq A'$, and so $r + 1 < k + 1 \leq k'$. The induction is complete. □

Acknowledgement

The second author would like to express his gratitude to Professor Ke and his family for their hospitality during his visit to Tainan, Taiwan.

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