SOME INEQUALITIES INVOLVING $(r!)^{1/r}$

by HENRYK MINC and LEROY SATHRE
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In a recent investigation of a conjecture on an upper bound for permanents of (0, 1)-matrices (2) we obtained some inequalities involving the function $(r!)^{1/r}$ which are of interest in themselves. Probably the most interesting of them, and certainly the hardest to prove, is the inequality

$$r\phi(r+1)/\phi(r)-(r-1)\phi(r)/\phi(r-1)>1, \quad \ldots\ldots\ldots\ldots(1)$$

where $\phi(r) = (r!)^{1/r}$. In the present paper we prove (1) and other inequalities involving the function $\phi(r)$.

Theorem 1. *If r is a positive integer and $\phi(r) = (r!)^{1/r}$, then*

$$1<\phi(r+1)/\phi(r)<(r+1)/r.$$

Proof. The lower bound is obtained immediately:

$$\phi(r+1)/\phi(r) = ((r+1)^r/r!)^{1/r(r+1)}>1.$$

Since $\log(1+1/r)>1/r-1/2r^2$ and $\log(\sqrt{(2\pi r)})>\frac{1}{2}$, it follows that

$$r\log(1+1/r)+r^{-1}\log(\sqrt{(2\pi r)})-1>0.$$

Therefore

$$r^{-1}\log(\sqrt{(2\pi r)}(r/e)^r)>\log(r)-r\log(1+1/r)$$
$$= (r+1)\log(r)-r\log(r+1),$$

i.e.,

$$(\sqrt{(2\pi r)}(r/e)^r)^{1/r}>r^{r+1}/(r+1)^r.$$

But

$$r!>\sqrt{(2\pi r)}(r/e)^r$$

and thus

$$(r!)^{1/r}>r^{r+1}/(r+1)^r,$$
$$r^{r+1}(r+1)/(r!)^{1/r}<(r+1)^{r+1},$$
$$r^{r+1}(r+1)!/(r!)^{(r+1)/r}<(r+1)^{r+1},$$
$$r^{r+1}(\phi(r+1)/\phi(r))^{r+1}<(r+1)^{r+1},$$

i.e.,

$$\phi(r+1)/\phi(r)<(r+1)/r.$$

Corollary 1. *The functions* $\phi(r)$, $\dfrac{r}{\phi(r)}$ *and* $r\dfrac{\phi(r+1)}{\phi(r)}$ *are strictly increasing.*

Corollary 2.

$$r<r\frac{\phi(r+1)}{\phi(r)}<r+1.$$

We now proceed to prove inequality (1). The method is to prove that the function

$$h(x)=x\frac{(\Gamma(x+2))^{1/(x+1)}}{(\Gamma(x+1))^{1/x}}$$

is strictly concave. The inequality (1) will follow.

Lemma 1. *If $x>1$, then*

$$0<\log(\Gamma(x))-\{(x-\tfrac{1}{2})\log(x)-x+\tfrac{1}{2}\log(2\pi)\}<1/x<1.$$

Proof. We have, by a classical result due to Binet (1) (page 21),

$$\log(\Gamma(x))=(x-\tfrac{1}{2})\log(x)-x+\tfrac{1}{2}\log(2\pi)+\delta(x)$$

where

$$\delta(x)=\int_0^\infty\{\tfrac{1}{2}\cdot t^{-1}+(e^t-1)^{-1}\}e^{-tx}t^{-1}dt.$$

It suffices to prove that, for $t>0$,

$$0<\{\tfrac{1}{2}-t^{-1}+(e^t-1)^{-1}\}t^{-1}<1. \quad\text{.........................(2)}$$

We first show that $f(t)=te^t+t+2-2e^t$ is positive for $t>0$. Now,

$$f'(t)=te^t-e^t+1$$

and $f''(t)=te^t>0$ for $t>0$. Therefore $f'(t)>f'(0)=0$ and thus $f(t)>f(0)=0$ for $t>0$. Hence, for $t>0$,

$$\{\tfrac{1}{2}-t^{-1}+(e^t-1)^{-1}\}t^{-1}=f(t)/2t^2(e^t-1)>0.$$

In order to prove the upper bound of (2) note that for $t>0$

$$\begin{aligned}2t^2+t+2&<t^4+\tfrac{3}{2}t^3+2t^2+t+2,\\&=(2t^2-t+2)(1+t+\tfrac{1}{2}t^2),\\&<(2t^2-t+2)e^t.\end{aligned}$$

Therefore

$$t(e^t-1)+2t-2(e^t-1)<2t^2(e^t-1),$$

i.e.,

$$\{\tfrac{1}{2}+(e^t-1)^{-1}-t^{-1}\}t^{-1}<1.$$

Lemma 2. *If $x>1$, then*

$$-\frac{1}{x}<\frac{\Gamma'(x)}{\Gamma(x)}-\log(x)<-\frac{1}{2x}<0.$$

Proof. We have, by another result due to Binet (1) (page 18),

$$\Gamma'(x)/\Gamma(x)=\log(x)+\varepsilon(x)\ \text{ for }\ x>1,$$

where

$$\varepsilon(x)=\int_0^\infty\{t^{-1}-(1-e^{-t})^{-1}\}e^{-tx}dt.$$

To prove the lemma we show that for positive t

$$-1<t^{-1}-(1-e^{-t})^{-1}<-\tfrac{1}{2}. \quad\text{..............................(3)}$$

Clearly for $t>0$ we have $(t+1)e^{-t}<1$. Therefore

$$t-1+e^{-t}<t-te^{-t}$$

and thus

$$(1-e^{-t})^{-1}-t^{-1}=(t-1+e^{-t})(t-te^{-t})^{-1}<1.$$

To prove the upper bound of (3) note that for $t>0$

$$(t-2)+(t+2)e^{-t}>0;$$

therefore

$$2t-2+2e^{-t}>t-te^{-t}$$

and so

$$(1-e^{-t})^{-1}-t^{-1}=(t-1+e^{-t})(t-te^{-t})^{-1}>\tfrac{1}{2}.$$

Let $\psi(x) = (\log(\Gamma(x)))' = \Gamma'(x)/\Gamma(x)$. Then, since

$$\log(\Gamma(x+1)) = \log(x)+\log(\Gamma(x)), \quad \text{(4)}$$

we have

$$\psi(x+1) = 1/x+\psi(x). \quad \text{(5)}$$

Lemma 3. *If* $x>1$, *then*

$$\psi'(x)>1/x.$$

Proof. It is known (1) (page 22) that

$$\psi'(x) = \sum_{n=0}^{\infty}(x+n)^{-2}.$$

Now,

$$\sum_{n=0}^{\infty}(x+n)^{-2} > \int_0^{\infty}(x+t)^{-2}dt = 1/x.$$

Let $g(x) = \dfrac{(\Gamma(x+2))^{1/(x+1)}}{(\Gamma(x+1))^{1/x}}$ and $h(x) = xg(x)$. We prove now that for $x \geqq 6$ the function $h(x)$ is concave. The result undoubtedly holds also for smaller values of x but the assumption $x \geqq 6$ simplifies our proof and the result is still sufficiently strong to establish our main theorems.

Theorem 2. *The function* $h(x)$ *is strictly concave for* $x \geqq 6$.

Proof. We prove that for $x \geqq 6$ the second derivative of $h(x)$ is negative. A straightforward, though lengthy, computation using (4) and (5) yields

$$g'(x) = g(x)\left\{\frac{1}{(x+1)^2} - \frac{\psi(x+1)}{x(x+1)} + \frac{2x+1}{x^2(x+1)^2}\log(\Gamma(x+1)) - \frac{\log(x+1)}{(x+1)^2}\right\},$$

$$h'(x) = g(x)+xg'(x),$$

$$= g(x)\left\{1+\frac{2x+1}{x(x+1)^2}\log(\Gamma(x+1)) - \frac{\psi(x+1)}{x+1} + \frac{x}{(x+1)^2} - \frac{x\log(x+1)}{(x+1)^2}\right\}.$$

Differentiating again and simplifying we obtain

$$h''(x) = g(x)\{F(x)(1+xF(x))+H(x)\}$$

where

$$F(x) = \frac{2x+1}{x^2(x+1)^2}\log(\Gamma(x+1)) - \frac{\psi(x+1)}{x(x+1)} + \frac{1}{(x+1)^2} - \frac{\log(x+1)}{(x+1)^2}$$

and

$$H(x) = \frac{3x+1}{x(x+1)^2}\psi(x+1) - \frac{4x^2+3x+1}{x^2(x+1)^3}\log(\Gamma(x+1))$$
$$- \frac{\psi'(x+1)}{x+1} - \frac{2x-1}{(x+1)^3} + \frac{x-1}{(x+1)^3}\log(x+1).$$

It remains to prove that $F(x)+x(F(x))^2+H(x)$ is negative. We find suitable upper bounds for $F(x)+H(x)$ and for $x(F(x))^2$. A simple computation gives

$$\begin{aligned}(x+1)^3\{F(x)+H(x)\} &= -2\log(\Gamma(x+1))+2(x+1)\psi(x+1)-(x-2) \\ &\qquad -2\log(x+1)-(x+1)^2\psi'(x+1), \\ &= -2\{(x+\tfrac{1}{2})\log(x+1)-(x+1)+\tfrac{1}{2}\log(2\pi)+\delta(x+1)\} \\ &\quad +2(x+1)\{\log(x+1)+\varepsilon(x+1)\} \\ &\quad -(x-2)-2\log(x+1)-(x+1)^2\psi'(x+1), \\ &= -\log(x+1)+x+4-\log(2\pi)-2\delta(x+1) \\ &\quad +2(x+1)\varepsilon(x+1)-(x+1)^2\psi'(x+1),\end{aligned}$$

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where δ and ε are the functions defined in the proofs of Lemmas 1 and 2. Applying Lemmas 1, 2 and 3 we obtain

$$(x+1)^3\{F(x)+H(x)\} < -\log(x+1)+x+4-\log(2\pi)-1-(x+1).$$

Therefore

$$F(x)+H(x)<(2-\log(2\pi)-\log(x+1))/(x+1)^3. \quad \text{...............(6)}$$

We now show that for $x \geqq 6$ the function $F(x)$ takes negative values and find a lower bound for it. This gives us an upper bound for $x(F(x))^2$.

$$x^2(x+1)^3F(x) = (2x+1)(x+1)\log(\Gamma(x+1))-x(x+1)^2\psi(x+1) -x^2(x+1)\log(x+1)+x^2(x+1).$$

Therefore

$$\begin{aligned} x^2(x+1)^2F(x) &= (2x+1)\{(x+\tfrac{1}{2})\log(x+1)-(x+1)+\tfrac{1}{2}\log(2\pi)+\delta(x+1)\} \\ &\quad -x(x+1)\{\log(x+1)+\varepsilon(x+1)\}-x^2\log(x+1)+x^2, \\ &= (x+\tfrac{1}{2})\log(x+1)-x^2-(3-\log(2\pi))x+\tfrac{1}{2}\log(2\pi) \\ &\quad -1+(2x+1)\delta(x+1)-x(x+1)\varepsilon(x+1). \end{aligned}$$

Now, by Lemmas 1 and 2,

$$\delta(x+1)<1/(x+1) \text{ and } \varepsilon(x+1)> -1/(x+1),$$

and thus

$$\begin{aligned} x^2(x+1)^2F(x) &< (x+\tfrac{1}{2})\log(x+1)-x^2-(3-\log(2\pi))x+\tfrac{1}{2}\log(2\pi) \\ &\quad -1+\frac{2x+1}{x+1}+x, \\ &= (x+\tfrac{1}{2})\log(x+1)-x^2-(2-\log(2\pi))x+\tfrac{1}{2}\log(2\pi)+\frac{x}{x+1}, \\ &< (x+\tfrac{1}{2})\log(x+1)-x^2+2, \end{aligned}$$

which is negative for $x \geqq 3$.

In order to obtain a lower bound for $F(x)$ we use again the two lemmas which state that $\delta>0$ and $\varepsilon<0$ and obtain

$$\begin{aligned} x^2(x+1)^2F(x) &> (x+\tfrac{1}{2})\log(x+1)-x^2-(3-\log(2\pi))x+\tfrac{1}{2}\log(2\pi)-1, \\ &> -x^2-2x-1, \end{aligned}$$

the last inequality holding since $(x+\frac{1}{2})\log(x+1)$ is positive while

$$(3-\log(2\pi))x-2x$$

is negative. We have therefore

$$F(x)> -1/x^2$$

and thus

$$(F(x))^2<1/x^4 \text{ for } x \geqq 3. \quad \text{..............................(7)}$$

It remains to prove that $F(x)+H(x)+x(F(x))^2$ is negative for $x \geqq 6$. Now, we have, from (6) and (7),

$$\begin{aligned} F(x)+H(x)+x(F(x))^2 &< (2-\log(2\pi)-\log(x+1))/(x+1)^3+1/x^3, \\ &= \{x^3(3-\log(2\pi)-\log(x+1))+3x^2+3x+1\}/x^3(x+1)^3, \end{aligned}$$

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and for $x \geqq 6$

$x^3(3-\log(2\pi)-\log(x+1))+3x^2+3x+1 \leqq x^3(3-\log(14\pi))+3x^2+3x+1 < 0.$

Hence $h''(x)<0$ for $x \geqq 6$ and $h(x)$ is strictly concave.

Theorem 3. *If r is an integer greater than* 1 *and* $\phi(m) = (m!)^{1/m}$ *then*

$$r\frac{\phi(r+1)}{\phi(r)} - (r-1)\frac{\phi(r)}{\phi(r-1)} > 1. \qquad (8)$$

Proof. The function $h(x)$ of Theorem 2 is concave for $x \geqq 6$. Therefore $h(x+1)+h(x-1)<2h(x)$ for all $x \geqq 7$. In particular, for an integer $r \geqq 7$,

$$h(r+1)+h(r-1)<2h(r),$$

and so

$$h(r+1)-h(r)<h(r)-h(r-1),$$

i.e.,

$$(r+1)\frac{\phi(r+2)}{\phi(r+1)} - r\frac{\phi(r+1)}{\phi(r)} < r\frac{\phi(r+1)}{\phi(r)} - (r-1)\frac{\phi(r)}{\phi(r-1)}.$$

In other words, the function

$$G(r) = r\frac{\phi(r+1)}{\phi(r)} - (r-1)\frac{\phi(r)}{\phi(r-1)}$$

is strictly decreasing for $r \geqq 7$. But clearly

$$\lim_{r\to\infty} G(r) = 1$$

and therefore

$$G(r) > 1$$

for all $r \geqq 7$.

For $r<7$ we obtain (8) by direct computation. The approximate values of $G(2)$, $G(3)$, $G(4)$, $G(5)$, $G(6)$ are 1·156, 1·084, 1·055, 1·036, 1·028, respectively.

Theorem 4. *If* $r_1, \ldots, r_c$ *are integers greater than* 1, $c \leqq r_t$, $t = 1, \ldots, c$, *and* $\phi(r_t) = (r_t!)^{1/r_t}$, *then*

$$\sum_{t=1}^{c} \frac{1}{\phi(r_t-1)} \leqq \prod_{t=1}^{c} \frac{\phi(r_t)}{\phi(r_t-1)} \qquad (9)$$

with equality if and only if $c = r_1 = \ldots = r_c$.

Proof. We prove that

$$f(r_1, \ldots, r_c) = \sum_{t=1}^{c} \frac{1}{\phi(r_t-1)} \prod_{j=1}^{c} \frac{\phi(r_j-1)}{\phi(r_j)}$$

is a strictly decreasing function of each r_i, i.e. that

$$R = f(r_1, \ldots, r_{c-1}, r_c+1)/f(r_1, \ldots, r_{c-1}, r_c) < 1.$$

For simplicity, let r_c be denoted by r. Then

$$R = \frac{(\phi(r))^2}{\phi(r-1)\phi(r+1)} \frac{K+1/\phi(r)}{K+1/\phi(r-1)}, \qquad (10)$$

where $K = \sum_{t=1}^{c-1} 1/\phi(r_t-1)$.

Since, by Corollary 1 to Theorem 1, $1/\phi(r)$ is a strictly decreasing function, the second fraction in (10) is a proper fraction with a positive numerator and a positive denominator. Thus, for a fixed r, R increases with K. Now, by the same corollary,

$$K \leqq \frac{c-1}{\phi(c-1)} \leqq \frac{r-1}{\phi(r-1)}$$

since $r_t \geqq c$. Therefore

$$\begin{aligned} R &\leqq \frac{(r-1)/\phi(r-1)+1/\phi(r)}{r\phi(r+1)}(\phi(r))^2, \\ &= \frac{\phi(r)}{r\phi(r+1)}\left\{1+(r-1)\frac{\phi(r)}{\phi(r-1)}\right\}, \\ &< \frac{\phi(r)}{r\phi(r+1)}\,\frac{r\phi(r+1)}{\phi(r)}, \text{ by Theorem 3,} \\ &= 1. \end{aligned}$$

It follows immediately that $f(r_1, \ldots, r_c)$ achieves its maximum value when $r_1, \ldots, r_c$ have their minimum permissible value, i.e., for $c \geqq 2$, if and only if $r_1 = \ldots = r_c = c$. Then

$$f(r_1, \ldots, r_c) = \frac{c}{\phi(c-1)}\left(\frac{\phi(c-1)}{\phi(c)}\right)^c = c\,\frac{(\phi(c-1))^{c-1}}{(\phi(c))^c} = c\,\frac{(c-1)!}{c!} = 1,$$

and (9) is an equality. If $c = 1$, then (9) becomes

$$\frac{1}{\phi(r-1)} \leqq \frac{\phi(r)}{\phi(r-1)}$$

which is always strict.

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University of California, Santa Barbara
and
University of Florida, Gainesville