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ON THE CONVERGENCE OF NON-INTEGER LINEAR HOPF FLOW

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Abstract The evolution of a rotationally symmetric surface by a linear combination of its radii of curvature is considered. It is known that if the coefficients form certain integer ratios the flow is smooth and can be integrated explicitly. In this paper the non-integer case is considered for certain values of the coefficients and with mild analytic restrictions on the initial surface.

We prove that if the focal points at the north and south poles on the initial surface coincide, the flow converges to a round sphere. Otherwise the flow converges to a non-round Hopf sphere. Conditions on the fall-off of the astigmatism at the poles of the initial surface are also given that ensure the convergence of the flow.

The proof uses the spectral theory of singular Sturm-Liouville operators to construct an eigenbasis for an appropriate space in which the evolution is shown to converge.

Keywords: parabolic equations; singular Sturm-Liouville operator; curvature flow; extrinsic curvature flow

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1. Introduction and results

The literature on extrinsic curvature flows in $\mathbb{R}^3$ primarily concerns flows whose normal speed is a symmetric function in the radii of curvature [1], for which flows by mean curvature, inverse mean curvature and powers of Gauss curvature are examples [4, 5, 13]. Symmetry is necessary for the normal speed to be well defined on general surfaces [2] as the radii of curvature may be exchanged via re-parameterisation. However when stationary solutions of such flows exist, they must necessarily be Weingarten surfaces satisfying a symmetric Weingarten relationship – in which case if they are homeomorphic to S^2 (we will call such surfaces simply *spheres*), they are in fact isometric to S^2 [7, 12], in which case we will call them *round spheres*. The current work differs from the literature in that the curvature flow considered, the *linear Hopf flow*, is not a symmetric curvature flow, hence offers novelty in that stationary spheres of the (unscaled) flow may be non-round. Rotationally symmetric surfaces support a canonical labelling of the radii of curvature associated to the meridian and profile principal foliations, allowing asymmetric curvature flows such as the linear Hopf flow to be sensibly defined on them. Furthermore, rotational symmetry reduces the problem to one spatial variable, affording extra tractability and allowing for a wider class of curvature flows to be considered on such surfaces, e.g. [17, 18]. Explicitly, we consider in this paper solving for the curvature flow $\vec{X} : S^2 \times [0, \infty) \to \mathbb{R}^3$ satisfying

$$\left(\frac{\partial \vec{X}}{t}\right)^{\perp} = (ar_1 + br_2 + c)\widehat{n}, \qquad \text{and} \qquad \vec{X}(S^2, 0) = \mathcal{S}_0, \tag{1.1}$$

with the initial data $\mathcal{S}_0$ being a C^2-smooth, rotationally symmetric topological 2-sphere. Here $a, b, c \in \mathbb{R}$ and $\vec{X}$, $\hat{n}$ and r_1, r_2 are respectively the position vector, normal vector and radii of curvature of the evolving surface. A flow linear in the radii of curvature is called a *linear Hopf flow* [11]. We will further assume that $b > 0$ which implies the parabolicy of a related flow (cf. equation 3.4) and also that a is constrained by

$$-a/b = 2n + 3, \tag{1.2}$$

with $n \in (-1, 1)$ – note this implies $a < 0$. The quantity $-a/b$ is called the *flow slope*. The flow slope places both topological and regularity restrictions on stationary solutions to the flow: Stationary solutions of the flow can only be non-round, strictly convex and topological 2-spheres when $-a/b > 1$, i.e. $n > -1$ ([12], Corollary 3.5.). In addition the radii of curvature of such stationary solutions can only be smooth functions if $n \in \mathbb{N}_0$ (cf. Theorem 2.6). Convergence of the linear Hopf flow was shown in [11] when $n \in \mathbb{N}_0$, wherein r_1 and r_2 are shown to converge as $t \to \infty$ through families of smooth functions to limits that are also smooth – namely the radii of curvature of a stationary solution. However in the non-integer case $n \in (-1, 1)$ considered here, stationary solutions with smooth radii of curvature do not exist and r_1, r_2 converge to functions which are not smooth, complicating the analysis (cf. the discussion in S 3) To state the main results, let θ be the angle between the normal of $\mathcal{S}$ at a point and the axis of rotational symmetry and denote the second order Legendre differential operator for $\mu, \nu \in \mathbb{R}$ by $\mathscr{L}_\nu^\mu$ (cf. equation (3.6)). Let $L^2_{\sin\theta}(0, \pi)$ be the space of square integrable functions with weight $\sin\theta$ on the interval $(0, \pi)$.

Theorem 1.1 *Consider the linear Hopf flow (1.1) and (1.2) for $n \in (-1, 1)$. Let $\mathcal{S}_0$ be a C^4-smooth strictly convex rotationally symmetric initial sphere. Assume that $\mathcal{S}_0$ satisfies the following conditions*

(1) $\mathscr{L}_n^n \left(\dfrac{s_0(\theta)}{\sin^{n+2}\theta}\right) \in L^2_{\sin\theta}(0, \pi)$,

(2) $n \cdot \lim\limits_{\theta \to 0} \left(\dfrac{s_0(\theta)}{\sin^2\theta}\right) = n \cdot \lim\limits_{\theta \to \pi} \left(\dfrac{s_0(\theta)}{\sin^2\theta}\right) = 0.$

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If the focal points of $\mathcal{S}_0$ at the north and south poles coincide, the flow converges to a round sphere of radius $-\frac{c}{a+b}$. Otherwise the flow converges to a non-round Hopf sphere with astigmatism at the equator given by the signed distance between the focal points at the poles of S_0.

If $\mathcal{S}_0$ has isolated umbilic points at the north or south pole, a sufficient condition to imply (2) can be given in terms of the surfaces *umbilic slopes*, which quantify the rate at which $\mathcal{S}_0$ becomes umbilic (see S 2 for details).

1. (2*) At each pole of $\mathcal{S}_0$, the umbilic slope is greater than 3.

It is not immediately obvious which classes of surfaces satisfy condition (1). However a sufficient condition can be given in terms of the asymptotic fall-off of s and its derivatives as $\theta \to 0, \pi$.

1. (3) $s_0^{(i)} \sim c_i \sin^{m-i}\theta$ at $\theta = 0, \pi$ for $m > n+3$, $i = 0, 1, 2$ with possibly different constants c_i at $\theta = 0$ and $\theta = \pi$.

Condition (3) is in fact enough for convergence as it implies both conditions (1) and (2). Theorem 1.1 will be a consequence of the following more technical theorem.

Theorem 1.2 *Let* S_0 *be a* C^2*-smooth strictly convex rotationally symmetric 2-sphere. If the astigmatism* s *of* $\mathcal{S}_0$ *satisfies both of the following conditions*

(i) $\dfrac{s}{\sin^{n+2}\theta} \in L^2_{\sin\theta}(0,\pi)$, $\dfrac{\mathrm{d}s}{\mathrm{d}\theta} \in AC_{loc}(0,\pi)$, $\mathscr{L}^n_n\left(\dfrac{s}{\sin^{n+2}\theta}\right) \in L^2_{\sin\theta}(0,\pi)$,

(ii) $\displaystyle\lim_{\theta\to 0,\pi}\left(\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{s}{\sin^{2n+2}\theta}\right)\right) = 0$,

then the conclusion of Theorem 1.1 holds.

Theorem 1.2 has a weaker differentiability requirement of the initial surface than Theorem 1.1. Indeed, (I) implies $\mathcal{S}_0$ is C^3-smooth everywhere apart from possibly at the north and south umbilic points, and C^4-smooth almost everywhere. Under the additional assumption that $\mathcal{S}_0$ is C^4-smooth, conditions (I) and (II) are implied by (1) and (2) respectively and Theorem 1.1 follows.

Our method of proof involves a spectral expansion in an eigenbasis of the non-integer Legendre operator, defined on the weighted Hilbert space $L^2_{\sin\theta}(0,\pi)$. The eigenbasis is adapted to contain the stationary solutions of the linear Hopf flow. The restriction $n \in (-1, 1)$, is needed to guarantee such an eigenbasis expansion is possible. In Theorem 1.1 Condition (1) is a technical assumption needed to apply $L^2_{\sin\theta}(0,\pi)$ spectral theory. If this assumption were dropped one may still expect convergence of the flow in a suitably weak sense, just with geometric quantities becoming more singular at the poles $\theta = 0, \pi$. Condition (2) is controlled by the umbilic slope of the initial surface. In [11] it was shown that if the initial surface umbilic slope is less than the flow slope, divergence of the linear Hopf flow can occur. Hence one might expect such behaviour if condition (2) is removed.

Section 2 fixes the notation used to describe geometrical quantities and derives some consequences of rotationally symmetry. Section 3 discusses the geometrical properties of possible stationary solutions of the linear Hopf flow and describes the evolution of important geometric quantities under the flow. Critical to this description is the second order Legendre differential operator $\mathscr{L}_n^n$.

Section 4 reviews the theory of singular Sturm-Liouville operators, which we require to prove the existence of the adapted eigenbasis. In particular the LC property of a singular Sturm-Liouville operator and boundary conditions required to generate possible self-adjoint domains are discussed. An application of the Spectral Theorem 4.4 for LC operators then guarantees the existence of a complete orthonormal basis of eigenfunctions of such operators.

This is then applied in S 5 where the main results are proven. This done by showing that $\mathscr{L}_n^n$ is LC for $n \in (-1, 1)$ and by finding boundary conditions (namely conditions (I) and (II) of Theorem 1.2) to define a self-adjoint domain for $\mathscr{L}_n^n$, which is appropriate for the flow.

The eigenbasis is then given explicitly in terms of Legendre functions and the various geometric quantities are similarly expressed. Finally the evolution is solved in terms of the eigenbasis as an expansion which decays exponentially in time to the stationary solution of the flow.

2. Rotationally symmetric surfaces

The class of surfaces we concern ourselves within this work are elements of the set

$$\mathscr{W} = \Big\{ S \subset \mathbb{R}^3 : S \text{ is an embedded } C^2\text{smooth topological 2-sphere which is}$$
$$\text{rotationally symmetric and strictly convex.} \Big\}.$$

The terminology "topological sphere" which we will abbreviate as just "sphere" is taken to mean a closed surface of genus 0. A given surface $\mathcal{S} \in \mathscr{W}$ will be orientated with outwards pointing normal $\hat{n}$ in a right handed coordinate system $\vec{X} = (x^1, x^2, x^3)$. Align the x^3-axis with the axis of rotational symmetry of $\mathcal{S}$. $\mathcal{S}$ is parametrized by pushing forward the standard polar coordinates (θ, ϕ) of S^2 onto $\mathcal{S}$ by the inverse of the Guass map $\mathcal{N}^{-1} : S^2 \to \mathcal{S}$. Hence $\theta \in [0, \pi]$ measures the angle made between the normal vector $\hat{n}$ of $\mathcal{S}$, and the axis of rotational symmetry whereas $\phi \in [0, 2\pi]$ measures the angle made by a clockwise rotation from the x^2x^3-plane. As a consequence of rotational symmetry many quantities $f : \mathcal{S} \to \mathbb{R}$ are independent of ϕ, in which case we write for short-hand $f(\theta)$ instead of $f(\vec{X}(\theta, \phi))$. In particular the radii of curvature of $\mathcal{S}$, r_1 and r_2, associated to the meridian and profile principal foliations respectively, are functions of θ only. If $\mathcal{S}$ is assumed strictly convex then away from the umbilic points of S, $r_1(\theta)$ and $r_2(\theta)$ are C^{m-2}-smooth functions whenever $\mathcal{S}$ is C^m-smooth. In the case of rotational symmetry the radii of curvature inherit the regularity of S even at the umbilic points of S (i.e. at $\theta = 0$ and $\theta = \pi$). If S is C^3, then the radii are C^1 even at the umbilics. If S is C^4 they are C^2. See [12, pg 6] for further details. Of critical importance to our study is the

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astigmatism of $\mathcal{S}$ given by

$$s = r_2 - r_1.$$

It is noted that s vanishes at, and only at, the umbilic points of $\mathcal{S}$. In particular $s(0) = s(\pi) = 0$ and $s \equiv 0$ if and only if $\mathcal{S}$ is a round sphere.

Our description of $\mathcal{S}$ in $\mathbb{R}^3$ will be facilitated by the support function $r : \mathcal{S} \to \mathbb{R}$ defined by $r = \vec{X} \cdot \hat{n}$. In the present setting r is a function of θ only and $\vec{X}(\theta, \phi)$ can be recovered from r and its derivatives via:

$$x^1 + ix^2 = (\sin\phi + i\cos\phi)\left(r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\right), \qquad x^3 = r\cos\theta - \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta.$$

Remark 2.1. For later convenience, note that the focal points [10] of $\mathcal{S}$ at $\theta = 0$ and π lie on the axis of rotational symmetry with x^3 coordinates

$$f_0 = r(0) - r_1(0), \qquad f_\pi = -r(\pi) + r_1(\pi).$$

This can be deduced by the above equation for $\vec{X}$ and the equations of the focal sheets $\vec{F}_i = \vec{X} - r_i\widehat{n}$, $i = 1, 2$.

We collect together some useful relationships between the above quantities on rotationally symmetric surfaces.

Proposition 2.2. *The following relationships hold*

$$r_1 = \frac{\cos^2\theta}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{r}{\cos\theta}\right), \qquad r_2 = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + r, \tag{2.1}$$

$$r = C_2\cos\theta + C_1 + \int \sin\theta\left[\int \frac{s}{\sin\theta}d\theta\right]d\theta, \tag{2.2}$$

for constants C_1,C_2. *If in addition surface is* C^3, *the derived Codazzi-Mainardi equation holds:*

$$\frac{\mathrm{d}r_1}{\mathrm{d}\theta} = (r_2 - r_1)\cot\theta. \tag{2.3}$$

Proof. The derivation of equation (2.1) can be found in [8] or derived from the definition of the support function. Equations (2.2) and (2.3) are derived by integrating or differentiating equation (2.1) respectively, with respect to θ. $\square$

Remark 2.3. Given $\mathcal{S} \in \mathscr{W}$ with support function r, the transformation $r \mapsto r + C_1$ translates $\mathcal{S}$ at each point along its normal line by a distance C_1, i.e. $\mathcal{S}$ moves to a parallel surface. The transformation $r \mapsto r + C_2\cos\theta$ translates the entire surface a distance of C_2 along the x^3-axis. Therefore s determines the oriented, affine normal lines of the surface. See [9] for further details.

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2.1. The slope at an isolated umbilic

If $\mathcal{S}$ possesses isolated umbilic points at its north and south poles, i.e. at $\theta = 0, \pi$, then we define the *umbilic slopes* of $\mathcal{S}$ as

$$\mu_0 = \lim_{\theta \to 0} \left(\frac{r_2(\theta) - r_2(0)}{r_1(\theta) - r_1(0)} \right), \qquad \mu_\pi = \lim_{\theta \to \pi} \left(\frac{r_2(\theta) - r_2(\pi)}{r_1(\theta) - r_1(\pi)} \right).$$

If $\mathcal{S}$ is C^3-smooth, the umbilic slopes are just $\frac{\mathrm{d}r_2}{\mathrm{d}r_1}$ evaluated at $\theta = 0$ and π respectively. We remark that coordinate substitution $\theta \mapsto \pi - \theta$ corresponds to reversing the direction of the x^3 axis, transforming μ_0 into μ_π and vice-versa. Hence arguments that are made concerning the umbilic at one pole will often hold at the other pole also. When this is the case we will simply say the argument follows by reflection. The question of what values of umbilic slope are possible on various spheres is an area of active research [7, 12, 14]. The derived Codazzi-Mainardi equation (2.3) is a necessary integrability condition for a C^3-smooth surface to be rotationally symmetric and has some striking consequences in this direction, two of which are the following proposition and theorem.

Proposition 2.4. *Let $\mathcal{S} \in \mathscr{W}$ be a* C^3 *sphere with an isolated umbilic at $\theta = 0$. For any $\alpha \in \mathbb{R}$*

(1) $\mu_0 > \alpha + 1 \implies \lim\limits_{\theta \to 0} \left(\frac{s}{\sin^\alpha \theta} \right) = 0,$

(2) $\mu_0 < \alpha + 1 \implies \lim\limits_{\theta \to 0} \left(\frac{s}{\sin^\alpha \theta} \right) = \pm\infty.$

If $\mathcal{S}$ has an isolated umbilic at $\theta = \pi$ then the value of μ_π dictates the behaviour of s *as $\theta \to \pi$ in the same way.*

Proof. We prove the result for $\theta = 0$, the $\theta = \pi$ case follows by reflection. Since the umbilic at $\theta = 0$ is isolated, there exists $\delta \in (0, \frac{\pi}{2})$ such that $s \neq 0$ for $\theta \in (0, \delta]$. Therefore by the Codazzi-Mainardi equation (2.3), $\frac{\mathrm{d}r_1}{\mathrm{d}\theta} \neq 0$ on this interval also. Furthermore since the surface is assumed strictly convex and C^3-smooth, $\frac{\mathrm{d}r_2}{\mathrm{d}\theta}$ is continuous and bounded. Therefore $\frac{\mathrm{d}r_2}{\mathrm{d}r_1}$ is continuous on $(0, \delta]$.

From the Codazzi-Mainardi equation and the definition of s one can derive the separable ODE:

$$\frac{\mathrm{d}r_2}{\mathrm{d}r_1} = 1 + \frac{\tan\theta}{s} \frac{\mathrm{d}s}{\mathrm{d}\theta}(\theta).$$

Integrating from θ to δ gives

$$\begin{aligned} |s| &= |s(\delta)| \exp\left\{ \int_\theta^\delta \cot\theta \left(1 - \frac{\mathrm{d}r_2}{\mathrm{d}r_1} \right) d\theta \right\} \\ &= \left| \frac{s(\delta)}{\sin^\alpha \delta} \right| \sin^\alpha \theta \cdot \exp\left\{ \int_\theta^\delta \cot\theta \left(\alpha + 1 - \frac{\mathrm{d}r_2}{\mathrm{d}r_1} \right) d\theta \right\}, \end{aligned}$$

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for all $\theta \in (0, \delta]$. Dividing both sides by $\sin^\alpha \theta$ we derive the relationship

$$\left|\frac{s(\theta)}{\sin^\alpha \theta}\right| = \left|\frac{s(\delta)}{\sin^\alpha \delta}\right| \exp\left\{\int_\theta^\delta \cot\theta \left(\alpha + 1 - \frac{\mathrm{d}r_2}{\mathrm{d}r_1}\right) d\theta\right\} \qquad \forall \theta \in (0, \delta].$$

Now let $\theta \to 0$. If $\mu_0 > \alpha + 1$ the quantity in parenthesis diverges to $-\infty$, implying $s/\sin^\alpha \theta \to 0$. On the other hand if $\mu_0 < \alpha + 1$ the quantity in parenthesis diverges to $+\infty$ which implies $|s/\sin^\alpha \theta| \to \infty$. □

Remark 2.5. Proposition 2.4 shows how the umbilic slopes dictate the rate of vanishing of s at the north and south poles. We remark that if it is the case that μ_0 or μ_π is equal to $\alpha + 1$, then $s/\sin^\alpha \theta$ can exhibit either of the two behaviours in Proposition 2.4 or tend to a non-zero constant, as illustrated by the three examples $s = \sin^\alpha \theta \cdot \ln(2\csc\theta)^\epsilon$ for $\epsilon = -1, 0, 1$.

Theorem 2.6 *If the radii of curvature of $\mathcal{S} \in \mathscr{W}$ are C^∞ and* S *has an isolated umbilic at the north or south pole, then the umbilic slope at that pole when it exists, takes a value of an odd integer greater than or equal to 3.*

If furthermore $\alpha + 1$ is the value of the umbilic slope at a given pole, the limit of $s/\sin^\alpha \theta$ as we approach the pole is finite, non-zero.

Proof. Let μ be the umbilic slope at $\theta = 0$ and assume it is finite. Since $\mathcal{S}$ is strictly convex, rotationally symmetric and smooth, s and all odd derivatives of s vanish at $\theta = 0$, i.e. $s(0) = s^{(2m+1)}(0) = 0$ for all $m \in \mathbb{N}$.

Now assume for contradiction that all even derivatives also vanish. Then $s^{(m)}(0) = 0$ for all $m \in \mathbb{N}_0$. In particular for any $\beta \in \mathbb{N}$, by L'Hôpitals rule

$$\lim_{\theta \to 0} \left(\frac{s}{\sin^\beta \theta}\right) = \ldots = \frac{1}{\beta!} s^{(\beta)}(0) = 0.$$

Using the contrapositive of (2) in Proposition 2.4, it follows that $\mu \geq \beta + 1$ which contradicts the assumption of μ being finite. Therefore there exists some $k \in \mathbb{N}$ such that $s^{(2k)}(0) \neq 0$, without loss of generality take k to be the smallest natural number such that this holds, so $s^{(m)}(0) = 0$ for all $m < 2k$. The smoothness assumption on $\mathcal{S}$ grantees the existence of $s^{(2k)}(0)$ and therefore we have the following limit

$$\lim_{\theta \to 0} \left(\frac{s}{\sin^{2k} \theta}\right) = \ldots = \frac{1}{(2k)!} s^{(2k)}(0) \neq 0, \pm\infty.$$

This time using the contrapositive of both (1) and (2) in Proposition 2.4 we have $\mu \geq 2k + 1$ and $\mu \leq 2k + 1$ which proves the first claim. The second claim follows from the above limit. If the isolated umbilic is at $\theta = \pi$ we argue by reflection. □

Although the umbilic slopes are not generally quantised for non-smooth spheres with isolated umbilic points, the regularity of a sphere still places restrictions on the possible values of the umbilic slope.

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Proposition 2.7. *If $\mathcal{S} \in \mathscr{W}$ is of regularity* C^4 *and has isolated umbilic points, then the umbilic slopes of $\mathcal{S}$ are greater or equal to 3.*

Proof. First argue at the $\theta = 0$ umbilic. By L'Hôpital and the assumed regularity of $\mathcal{S}$, we have the existence of the following limit:

$$\lim_{\theta\to 0}\left(\frac{s}{\sin^2\theta}\right) = \lim_{\theta\to 0}\left(\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{2\cos\theta\sin\theta}\right) = \frac{1}{2}\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}\bigg|_{\theta=0} \tag{2.4}$$

Hence by the converse of (2) in Proposition (2.4), we have that $\mu_0 \geq 3$. The case at $\theta = \pi$ follows by reflection. □

3. Linear Hopf flow

3.1. Stationary solutions

The stationary solutions of the linear Hopf flow (1.1) are surfaces satisfying the curvature relationship

$$0 = ar_1 + br_2 + c. \tag{3.1}$$

The surfaces in $\mathscr{W}$ for which a general linear curvature relationship such as equation (3.1) holds are called *linear Hopf spheres.*

Proposition 3.1. *The linear Hopf spheres which solve equation (3.1) for given parameter values* a, b *and* c *have astigmatisms of the form*

$$s_{Hopf} = C_0 \sin^{2n+2}\theta, \tag{3.2}$$

and support functions of the form

$$r_{Hopf} = -\frac{c}{a+b} + C_1\cos\theta + C_0\left[\frac{\sin^{2n+2}\theta}{2n+2} - \cos\theta\int_0^\theta \sin^{2n+1}\theta d\theta\right], \tag{3.3}$$

where C_0 *and* C_1 *are constants and* $-a/b = 2n+3$.

Proof. Combing the derived Codazzi-Mainardi equation (2.3) with the curvature relationship (3.1) results in a separable ODE which is solved to give equation (3.2). The support function (3.3) follows by integrating equation (3.2) by quadrature as in equation (2.2), except with an extra constant other than C_0 and C_1. This additional constant is then determined by requiring that the radii of curvature derived from the support function satisfies equation (3.1). Equation (3.3) then follows. □

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Any surface with isolated umbilic points satisfying equation (3.1) necessarily has umbilic slopes taking the common value

$$\mu_0 = \mu_\pi = -a/b.$$

Furthermore, if this surface is in $\mathscr{W}$ and smooth, Theorem 2.6 implies $-a/b = 2n+3$ for some $n \in \mathbb{N}_0$. Therefore the only non-round strictly convex linear Hopf spheres with smooth radii of curvature are ones with odd umbilic slope greater than or equal to 3. As to allow for convergence of the linear Hopf flow to non-round, smooth and convex spheres, the work undertaken in [11] investigated the linear Hopf flow with $-a/b$ restricted to an odd integer ≥ 3 with initial data in $\mathscr{W}$. The central result is then:

Theorem 3.2 [***11***] *Let $\mathcal{S}_0 \in \mathscr{W}$ be a smooth initial surface with equal umbilic slopes $\mu = \mu_0 = \mu_\pi$. The linear Hopf flow (1.1) and (1.2) behaves in the following manner:*

(1) *if $2n+3 < \mu$, the evolving sphere converges exponentially through smooth convex spheres to the round sphere of radius $\frac{c}{2(n+1)}$,*
(2) *if $2n+3 = \mu$, an initial non-round sphere converges exponentially thorough smooth convex spheres to a non-round linear Hopf sphere,*
(3) *if $2n+3 > \mu$, the sphere diverges exponentially.*

Since in the current paper we consider n which is generically non-integer, we cannot expect non-round stationary solutions with smooth radii of curvature as in Theorem 3.2. We also remark that the flow in Theorem 3.2 fixes $b=1$, whereas in this work we allow $b>0$. These conventions are equivalent up to a parabolic scaling $t \mapsto bt$ of equation (1.1).

3.2. Time evolution of geometric quantities

The curvature flow equation (1.1) is equivalent to the following evolution equation for the support function

$$\frac{\partial r}{\partial t} = b\frac{\partial^2 r}{\partial \theta^2} + a\cot\theta\frac{\partial r}{\partial \theta} + (a+b)r + c, \tag{3.4}$$

as can be seen from the definition of r and equations (2.1). We remark that the coordinate singularities in equation (3.4) prevent us from using regular Sturm-Liouville theory to derive an associated eigenbasis, which motivates the singular theory discussed in $S\,4$.

The solution $r(t,\theta)$ to equation (3.4) may be found by first considering the behaviour of the astigmatism $s(t,\theta)$. The support function $r(t,\theta)$ may then be recovered via quadrature by equation (2.2) up to two time-dependent constants determined by an initial condition and equation (3.4).

Proposition 3.3. *Under the linear Hopf flow the astigmatism evolves as*

$$\frac{\partial}{\partial t}\left(\frac{s}{\sin^{n+2}\theta}\right) = b \cdot \mathscr{L}_n^n\left(\frac{s}{\sin^{n+2}\theta}\right), \tag{3.5}$$

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where $\theta \in (0, \pi)$, $-a/b = 2n+3$ *and*

$$\mathscr{L}_\nu^\mu = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + (\nu+1)\nu - \frac{\mu^2}{\sin^2\theta}, \tag{3.6}$$

is the Legendre operator.

Proof. Differentiating equation (2.2) twice gives a relationship between s and the derivatives of r. The claim follows after inserting this relationship into equation (3.4) and performing some algebraic manipulation. □

Remark 3.4. As expected the astigmatism of the appropriate linear Hopf sphere is stationary under the flow since

$$\frac{s_{\text{Hopf}}}{\sin^{n+2}\theta} = \sin^n\theta \in \text{Ker } \mathscr{L}_n^n.$$

A crucial ingredient for solving equation (3.5) in the case $n \in \mathbb{N}_0$, is that the eigenfunctions of $\mathscr{L}_n^n$ are the associated Legendre polynomials. The associated Legendre polynomials, $\mathrm{P}_m^n(\cos\theta)$, are polynomials in sine and cosine and form an orthogonal basis of $C^0[0, \pi]$. In particular they span the higher index terms ($l \geq n$) of the following astigmatism decomposition for smooth surfaces:

$$s = \sum_{l=0}^{\infty}(a_l + b_l\cos\theta)\sin^{2l+2}\theta. \tag{3.7}$$

In [11] this decomposition enabled the flow to be solved explicitly when $n \in \mathbb{N}_0$ by projecting the flow equation (3.5) into each eigenbasis of $\mathscr{L}_n^n$ and solving for the time dependency of the coefficients $a_l(t)$,$b_l(t)$. The term with coefficient a_n is the astigmatism of the linear Hopf sphere with umbilic slope $2n+3$, therefore in the case of convergence of s to a linear Hopf sphere, $b_l \to 0$ for all $l \in \mathbb{N}_0$ and $a_l \to 0$ for all $l \in \mathbb{N}_0\backslash\{n\}$ as $t \to \infty$, leaving only the linear Hopf term. In the case of convergence to a round sphere, $a_n = 0$.

We try to emulate the above argument in this paper for $n \notin \mathbb{N}$. In the non-integer case there are some significant differences; the eigenfunctions of $\mathscr{L}_n^n$ are no longer trigonometric polynomials and are not terms in the series expansion (3.7). Furthermore it is no longer clear if the non-integer Legendre functions are orthogonal or form a basis. Finally, when $n \notin \mathbb{N}$ the astigmatism of the stationary solution s_{Hopf} is non-smooth, hence we expect the eigen basis to be non-smooth. To address these points we will write the Legendre operator in its Sturm-Liouville form

$$\mathscr{L}_\nu^\mu = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \nu(\nu+1)\sin\theta - \frac{\mu^2}{\sin\theta}\right], \tag{3.8}$$

and show using Singular Sturm-Liouville theory that we can find an orthogonal basis in which the surfaces astigmatism can be decomposed.

4. Singular Sturm-Liouville operators

In this section, we summarize the theory of singular Sturm-Liouville problems following [21]. In $S\,5$, the theory will be used to deduce the existence of an orthogonal eigenbasis for the non-integer Legendre operator associated with the linear Hopf flow.

Consider the general Sturm-Liouville operator

$$T = -\frac{1}{w(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right], \tag{4.1}$$

where $1/p, q, w \in L_{loc}\big((a,b);\mathbb{R}\big)$ are locally Lebesgue integrable real-valued functions on the interval (a,b) and $w > 0$. Operators of this form are called *singular Sturm-Liouville operators*. The Legendre operator discussed in $S\,3$ is an example of such.

View T as a linear operator on $L^2\big((a,b), w(x)dx; \mathbb{C}\big)$, the Hilbert space of complex-valued square integrable functions with weight w, denoted simply by $L^2_w(a,b)$. Under the standard inner-product of the $L^2_w(a,b)$ spaces

$$\langle f, g\rangle_w = \int_a^b f(x)\cdot\overline{g(x)}\cdot w(x)dx,$$

the operator T satisfies the so-called Greens formula

$$\langle Tf, g\rangle_w = p(x)\left(f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right)\Big|_{x=a^+}^{x=b^-} + \langle f, Tg\rangle_w, \tag{4.2}$$

where the evaluation of the boundary term is to be understood as a limit. Greens formula allows us to investigate the symmetry of T in $L^2_w(a,b)$ so long as the functions f and g are chosen to be such that the terms in (4.2) are well defined. For this purpose the *maximal domain* is introduced:

$$D_{\max} = \big\{f \in L^2_w(a,b) : f, pf' \in \mathrm{AC}_{\mathrm{loc}}(a,b), Tf \in L^2_w(a,b)\big\},$$

where $\mathrm{AC}_{\mathrm{loc}}(a,b)$ is the space of functions which are absolutely continuous on all compact intervals of (a,b).

The requirement that $f, pf' \in \mathrm{AC}_{\mathrm{loc}}(a,b)$, is enough to ensure that f, pf' are differentiable almost everywhere and their derivatives are Lebesgue integrable, which gives meaning to equation (4.2).

In the case that the coefficient p satisfies $1/p \in \mathrm{AC}_{\mathrm{loc}}(a,b)$, the description of $D_{\max}$ simplifies to

$$D_{\max} = \big\{f \in L^2_w(a,b) : f' \in \mathrm{AC}_{\mathrm{loc}}(a,b), Tf \in L^2_w(a,b)\big\}, \tag{4.3}$$

in particular the elements of $D_{\max}$ must be $C^1(a,b)$ and have second derivative a.e. Such is the case with the Legendre operator (3.8).

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Sturm-Liouville operators often come supplied with boundary conditions as to make the boundary term in equation (4.2) vanish, i.e.

$$\langle Tf, g\rangle_w = \langle f, Tg\rangle_w. \tag{4.4}$$

These boundary conditions constitute part of T's domain of definition, which naturally must be a subset of $D_{\max}$.

In non-singular Sturm-Liouville theory[1] the well-known boundary conditions

$$\alpha_1 f(a) + \alpha_2 p(a) f'(a) = 0, \qquad \beta_1 f(b) + \beta_2 p(b) f'(b) = 0, \qquad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}, \tag{4.5}$$

define a domain $D_{\text{S.A.}}$ for T which make $T|_{D_{\text{S.A.}}}$ self adjoint [16]. It is for this reason $D_{\text{S.A.}}$ will be referred to as a self adjoint domain for T.

In singular problems however, the quantities $f(x)$ and $p(x)f'(x)$ may not exist as $x \to a$ or b, even if $f \in D_{\max}$. Therefore boundary conditions such as (4.5) are not appropriate. To facilitate the description of boundary conditions for singular problems the *Lagrange bracket*

$$[f, g]_p(x) = p(x)\left(f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right), \tag{4.6}$$

is introduced. Unlike the terms in boundary condition (4.5), the Lagrange bracket $[f, g]_p(x)$ *is* finite in the limits $x \to a, b$ so long as $f, g \in D_{\max}$.

In order to give appropriate boundary conditions in the singular case, we first give a definition.

Definition 4.1. *Given a singular Sturm-Liouville operator* T*, we say* T *is* limit-circle *(LC) at* x = a *if for a given* $\chi \in \mathbb{C}$ *all solutions of the eigenvalue equation*

$$Ty = \chi y,$$

are in $L^2_w(a, c)$ *for some* $c \in (a, b)$*. Otherwise we say* T *is* limit-point *(LP) at* a.

Similarly we say T *is LC at* x = b *if correspondingly* $y \in L^2_w(c, b)$ *and LP at* b *otherwise.*

T *is said to be LC(LP) if it is LC(LP) at both* a *and* b.

Remark 2. T being LC or LP is independent of $\chi \in \mathbb{C}$ [21].

The next theorem states the parallel of boundary condition (4.5) for LC Sturm-Liouville operators.

Theorem 4.3 [*21*] *Let* T *be a LC Sturm-Liouville operator and* η, ψ *be real valued functions in* D_{max} *such that* $[\eta, \psi]_p(a) = 1$ *and* $[\eta, \psi]_p(b) = 1$*. Consider the separated*

[1] which requires the stronger condition $1/p, q, w$ are Lebesgue integrable over (a, b)

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boundary condition

$$\alpha_1[u,\eta]_p(a) - \alpha_2[u,\psi]_p(a) = 0 \qquad (\alpha_1,\alpha_2) \in \mathbb{C}^2\backslash\{0\} \tag{4.7}$$

$$\beta_1[u,\eta]_p(b) - \beta_2[u,\psi]_p(b) = 0 \qquad (\beta_1,\beta_2) \in \mathbb{C}^2\backslash\{0\}, \tag{4.8}$$

where

$$[u,f]_p(a) = \lim_{x\to a^+}[u,f]_p(x) \qquad [u,f]_p(b) = \lim_{x\to b^-}[u,f]_p(x).$$

Then given α_{i}*,* β_{i}*,* $i = 1, 2$ *as above, the domain*

$$D_{S.A.} = \{y \in D_{max} : equations (4.7)\text{-}(4.8)\ hold\},$$

is a self adjoint-domain for T*, i.e.* $T|_{D_{S.A.}}$ *is a self-adjoint operator.*

In addition if T is a LC Sturm-Liouville operator then T has a compact resolvent [20]. Together Theorem 4.3 and LC operators having compact resolvents allow us to apply the following spectral theorem once an appropriate $D_{\mathrm{S.A.}}$ has been found.

Theorem 4.4 Spectral Theorem [*19*] *Let* $\mathcal{H}$ *be a separable complex Hilbert space with inner product* $\langle\cdot,\cdot\rangle$ *and let* $T : Dom(T) \subseteq \mathcal{H} \to \mathcal{H}$ *be a linear, self-adjoint operator on* $\mathcal{H}$ *with compact resolvent. Then, there exists a sequence* $(\lambda_n)_n \subset \mathbb{R}$ *and a complete orthonormal basis* $(e_n)_n$ *of* $\mathcal{H}$ *with* $e_n \in Dom(T)$ *for all* $n \in \mathbb{N}$ *such that*

(1) $Te_n = \lambda_n e_n$,
(2) $Dom(T) = \{x \in \mathcal{H} | (\lambda_n\langle x, e_n\rangle)_n \in \ell^2\}$,
(3) $Tx = \sum_{n=1}^{\infty} \lambda_n\langle x, e_n\rangle e_n$ *for all* $x \in Dom(T)$.

5. Proof of Theorem 1.2

We will solve equation (3.5) in terms of an eigenbasis expansion of the operator $\mathscr{L}_n^n$. Theorem 1.2 will then follow by the asymptotic behaviour of the solution as $t \to \infty$. The proof is organised into three parts: Firstly we show the existence of the appropriate eigenbasis using the theory in S 4. Secondly we determine the basis explicitly as the Legendre functions $\{\mathrm{P}_{n+m}^{-n}(\cos\theta)\}_{m=0}^{\infty}$ and derive the corresponding expansions for s, r_1 and r. Finally, we use this basis to solve the time evolution problem

$$\begin{cases} \partial_t u = b \cdot \mathscr{L}_n^n u, & [0,\infty) \times [0,\pi] \\ u(t,\cdot) \in D_{\mathrm{S.A.}}, & t \in [0,\infty) \\ u = u_0, & \{t = 0\} \times [0,\pi]. \end{cases} \tag{5.1}$$

The function space $D_{\mathrm{S.A.}}$ is a self adjoint domain for $\mathscr{L}_n^n$ and plays the role of an effective a parabolic boundary condition in (5.1). If s is the astigmatism of a rotationally symmetric

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surface, making the substitution $u = s/\sin^{n+2}\theta$ turns the time evolution problem (5.1) into one describing the evolution of a surfaces astigmatism under the linear Hopf flow, i.e. equation (3.5). The solution $s(t,\theta)$ is then integrated for the support function $r(t,\theta)$ and both are shown to exhibit the asymptotic behaviour as $t \to \infty$

$$s(t,\theta) \sim \widetilde{\gamma} \cdot s_{\mathrm{Hopf}}, \qquad r(t,\theta) \sim r_{\mathrm{Sphere}} + \widetilde{\gamma} \cdot r_{\mathrm{Hopf}}, \tag{5.2}$$

where r_{Sphere} is the support function of a sphere with radius $-\frac{c}{a+b}$ and $\widetilde{\gamma}$ is the signed distance between the focal points of the initial surface at $\theta = 0$ and π.

5.1. Existence of the eigenbasis and the self-adjoint domain

First we remark for which values of n that $\mathscr{L}_n^n$ is LC.

Proposition 5.1. *$\mathscr{L}_n^n$ is LC if and only if $n \in (-1, 1)$.*

Proof. To check if $\mathscr{L}_n^n$ is LC, solve the eigenvalue problem $\mathscr{L}_n^n u = \chi u$. Recall when checking LC/LP, we are free to choose χ as we please. If we set $\chi := -n(n+1)$ we must solve the problem

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) - \frac{n^2}{\sin^2\theta}u = 0.$$

If we can show that any two linearly independent solutions are square integrable (with weight $w = \sin\theta$), it follows that every solution is square integrable by the triangle inequality. First assume $n = 0$. Then the eigenvalue problem is simply

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = 0,$$

with two linearly independent solutions $\ln\cot\left(\frac{\theta}{2}\right)$ and a constant function. These are square integrable. Now assume $n \neq 0$, then two linearly independent solutions are

$$u_\pm = \cot^{\pm n}\left(\frac{\theta}{2}\right).$$

We have then,

$$||u_\pm||^2_{L^2_{\sin\theta}(0,\pi)} = \int_0^\pi \cot^{\pm 2n}\left(\frac{\theta}{2}\right) \cdot \sin\theta d\theta = 2\int_0^\pi \left[\cos\left(\frac{\theta}{2}\right)\right]^{1\pm 2n} \cdot \left[\sin\left(\frac{\theta}{2}\right)\right]^{1\mp 2n} d\theta,$$

which is convergent if and only if $-1 < n < 1$ and therefore $\mathscr{L}_n^n$ is LC if and only if $n \in (-1, 1)$. □

For such values of n we may now use Theorem 4.3 to find self-adjoint domains for $\mathscr{L}_n^n$. Furthermore, as to make the convergence obvious, we'd like the eigenbasis of $\mathscr{L}_n^n$

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to contain explicitly the stationary solution to equation (3.5):

$$\frac{s_{\text{Hopf}}}{\sin^{n+2}\theta},$$

it is therefore necessary that $\frac{s_{\text{Hopf}}}{\sin^{n+2}\theta} \in D_{\text{S.A.}}$. Finding such a self-adjoint domain will be the content of the next proposition.

Proposition 5.2. *The separated boundary condition*

$$\lim_{\theta\to 0}\left[\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\sin^n\theta}\right)\right]=0, \qquad \lim_{\theta\to \pi}\left[\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\sin^n\theta}\right)\right]=0, \tag{5.3}$$

generates a self-adjoint domain for the Legendre operator

$$D_{S.A.}=D_{max}\cap\left\{u\in L^2_{\sin\theta}(0,\pi):\textit{boundary condition } (5.3) \textit{holds.}\right\}, \tag{5.4}$$

where D_{max} *is given by equation (4.3) for* $(a,b)=(0,\pi)$ *and* $T=\mathscr{L}_n^n$. *Furthermore* $s_{Hopf}/\sin^{n+2}\theta\in D_{S.A.}$.

Proof. The expression for $D_{\max}$ for the Legendre operator takes the form

$$D_{\max}=\left\{u\in L^2_{\sin\theta}(0,\pi):\tfrac{\mathrm{d}u}{\mathrm{d}\theta}\in \mathrm{AC}_{\text{loc}}(0,\pi),\mathscr{L}_n^n u\in L^2_{\sin\theta}(0,\pi)\right\}.$$

Theorem 4.3 tells us that all self adjoint domains of the Legendre operator generated by separated boundary conditions are given by functions $u\in D_{\max}$ satisfying

$$\alpha_1[u,\eta]_{\sin\theta}(0)+\alpha_2[u,\psi]_{\sin\theta}(\pi)=0\&\beta_1[u,\eta]_{\sin\theta}(0)+\beta_2[u,\psi]_{\sin\theta}(\pi)=0,$$

where $\{\eta,\psi\}\subset D_{\max}$ and $[\eta,\psi]_{\sin\theta}(0)=[\eta,\psi]_{\sin\theta}(\pi)=1$, with $(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in\mathbb{C}^2\backslash\{0\}$.

Take $(\eta,\psi)=\frac{1}{\sqrt{n}}\left(\sin^n\theta,\sin^{-n}\theta\right)$. It is straightforward to check that this choice of η and ψ satisfy the above requirements. From Theorem 4.3, the boundary conditions with this choice of η and ψ are

$$\lim_{\theta\to 0}\left[\alpha_1\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\sin^n\theta}\right)+\frac{\alpha_2}{\sin^{2n-1}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(u\cdot\sin^n\theta\right)\right]=0, \tag{5.5}$$

$$\lim_{\theta\to \pi}\left[\beta_1\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\sin^n\theta}\right)+\frac{\beta_2}{\sin^{2n-1}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(u\cdot\sin^n\theta\right)\right]=0. \tag{5.6}$$

If we let $u_{\text{Hopf}}=s_{\text{Hopf}}/\sin^{n+2}\theta$ then

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u_{\text{Hopf}}}{\sin^n\theta}\right)\equiv 0, \qquad \frac{1}{\sin^{2n-1}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(u_{\text{Hopf}}\cdot\sin^n\theta\right)=2nC\cos\theta,$$

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and so boundary conditions (5.5)-(5.6) are satisfied by u_{Hopf} only when $\alpha_2 = \beta_2 = 0$. Therefore setting $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, gives the self adjoint domain

$$D_{\mathrm{S.A.}} = D_{\max} \cap \left\{ u \in L^2_{\sin\theta}(0,\pi) : \lim_{\theta\to 0,\pi} \left[\sin^{2n+1}\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\sin^n\theta}\right)\right] = 0 \right\}.$$

It is quick to check that $u_{\mathrm{Hopf}} \in D_{\max}$, completing the proof. □

Proposition 5.3. *If* s *is the astigmatism of* $\mathcal{S} \in \mathscr{W}$, *then* $s/\sin^{n+2}\theta \in D_{S.A.}$ *if and only if (I) and (II) of Theorem 1.2 hold.*

Proof. Making the substitution $u = s/\sin^{n+2}\theta$ turns the maximal domain conditions (5.1) and boundary condition (5.3) into conditions (I) and (II) respectively. □

Under stronger assumptions on the surfaces regularity, conditions (I) and (II) can be replaced by less technical, sufficient conditions, i.e. those which appear in Theorem 1.1. First an elementary lemma is given concerning the asymptotic behaviour of functions in $L^2_{\sin\theta}(0,\pi)$ near the boundary points $\theta = 0, \pi$.

Lemma 5.4. *If* $f \in C^0((0,\pi);\mathbb{R})$ *satisfies asymptotic conditions*

$$f \sim \frac{k_0}{\sin^p\theta}\, as\ \theta\to 0 \quad and \quad f \sim \frac{k_\pi}{\sin^p\theta}\, as\ \theta \to \pi, \qquad p<1,$$

for constants k_0 *and* k_π *then* $f \in L^2_{\sin\theta}(0,\pi)$.

Proposition 5.5. *If* $\mathcal{S} \in \mathscr{W}$ *is in addition* C^4*-smooth, it is sufficient for* $s/\sin^{n+2}\theta \in D_{S.A.}$, *that both (1) and (2) of Theorem 1.1 hold.*

Proof. Under the assumption $\mathcal{S}$ is C^4 it will be shown that (1) implies (I) and (2) implies (II).

Assume the astigmatism s of $\mathcal{S}$ satisfies (1). There are two sub-conditions of (I) left to demonstrate. It immediately follows that $\frac{\mathrm{d}s}{\mathrm{d}\theta} \in \mathrm{AC}_{\mathrm{loc}}(0,\pi)$ since $\mathcal{S} \in \mathscr{W}$ and is C^4. The last subcondition of (I) follows by first applying L'Hôpital's rule to show s has the following asymptotic behaviours near the boundary of $(0,\pi)$;

$$\frac{s}{\sin^{n+2}\theta} \sim \frac{1}{2\sin^n\theta}\left.\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}\right|_{\theta=0} \text{ as } \theta\to 0 \quad \text{and} \quad \frac{s}{\sin^{n+2}\theta} \sim \frac{1}{2\sin^n\theta}\left.\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}\right|_{\theta=\pi} \text{ as } \theta\to \pi.$$

Therefore since $n<1$, Lemma 5.4 gives $s/\sin^{n+2}\theta \in L^2_{\sin\theta}(0,\pi)$ and (I) is satisfied.

To show (II) holds, re-write (II) as

$$\lim_{\theta\to 0,\pi}\left(\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{\sin\theta} - 2(n+1)\cos\theta\frac{s}{\sin^2\theta}\right) = 0.$$

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Again using L'Hôpital's rule (remarking that condition (2) implies the existence of the relevant limits) one can write condition (II) as the pair of equations

$$n \cdot \lim_{\theta\to 0}\left(\frac{s}{\sin^2\theta}\right) = 0, \qquad\qquad n \cdot \lim_{\theta\to \pi}\left(\frac{s}{\sin^2\theta}\right) = 0,$$

which is (2). □

Proposition 5.6. *Let $\mathcal{S} \in \mathscr{W}$ be C^4 and satisfy (3). Then both (1) and (2) of Theorem 1.1 hold.*

Proof. First we show that (3) implies (1). By lemma 5.4, it is enough to show that (3) implies

$$\mathscr{L}_n^n\left(\frac{s}{\sin^{n+2}\theta}\right) \sim \frac{k}{\sin^p\theta},$$

around $\theta = 0$ for some $p < 1$ and some k, and likewise at $\theta = \pi$. We have

$$\mathscr{L}_n^n\left(\frac{s}{\sin^{n+2}\theta}\right) = \frac{1}{\sin^{n+2}\theta}\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2} - \frac{(2n+3)\cos\theta}{\sin^{n+3}\theta}\frac{\mathrm{d}s}{\mathrm{d}\theta} + \frac{2(n+1)(1+\cos^2\theta)s}{\sin^{n+4}\theta}.$$

It is easily shown that if s and its derivatives have the asymptotic behaviour given by (3), then by Lemma 5.4 the asymptotic fall off of $\mathscr{L}_n^n\left(s/\sin^{n+2}\theta\right)$ is sufficient for $\mathscr{L}_n^n\left(s/\sin^{n+2}\theta\right) \in L^2_{\sin\theta}(0,\pi)$. To prove (2) we have

$$\frac{s}{\sin^2\theta} = \frac{s}{\sin^m\theta}\cdot\sin^{m-2}\theta \to 0 \text{as } \theta \to 0, \pi,$$

since $m > n+3 > 2$ for $n \in (-1, 1)$. □

5.2. The eigenbasis expansion

Now the domain $D_{\text{S.A.}}$ has been determined, the next proposition will determine explicitly the eigenfunctions of $\mathscr{L}_n^n$ when $\mathrm{Dom}(\mathscr{L}_n^n) = D_{\text{S.A.}}$.

Proposition 5.7. *If $u \in L^2_{\sin\theta}(0,\pi)$ and $n \in (-1,1)$, then* u *can be decomposed in $L^2_{\sin\theta}(0,\pi)$ as*

$$u = \gamma_{0,n}\sin^n\theta + \sum_{m=1}^{\infty}\gamma_{m,n}P^{-n}_{n+m}(\cos\theta). \tag{5.7}$$

where P^μ_ν is the Legendre function of order μ and degree ν.

Proof. Take $\mathscr{L}_n^n$ to be the Legendre operator with the self-adjoint domain $D_{\text{S.A.}}$ given by Proposition 5.2. Since by Proposition 5.1 $\mathscr{L}_n^n$ is LC, it is a compact operator and therefore by the Spectral Theorem 4.4, $L^2_{\sin\theta}(0,\pi)$ has a complete orthonormal basis consisting of $\mathscr{L}_n^n$'s eigenfunctions and kernel. The kernel and eigenspaces of $\mathscr{L}_n^n$ are spanned by the following linearly independent sets of functions depending on if $n=0$ or $n\neq 0$ [3, pp. 352-353]:

	$n=0$	$0<\lvert n\rvert<1$
Kernel	$\{\mathrm{Q}_0(\cos\theta),1\}$	$\{\mathrm{P}_n^n(\cos\theta),\sin^n\theta\}$
Eigenspaces	$\{\mathrm{Q}_\nu(\cos\theta),\mathrm{P}_\nu(\cos\theta)\}$	$\{\mathrm{P}_\nu^n(\cos\theta),\mathrm{P}_\nu^{-n}(\cos\theta)\}$

The functions $\{\mathrm{P}_{n+m}^{-n}(\cos\theta)\}_{m=0}^{\infty}$ will be shown to be the only functions from the above list belonging to $D_{\text{S.A.}}$, i.e. the only functions that satisfy the boundary conditions (5.3). It is first remarked that the Legendre operator $\mathscr{L}_\nu^\mu$ satisfies

$$\mathscr{L}_\nu^\mu=\mathscr{L}_{-\nu-1}^\mu,\qquad \forall\mu,\nu\in\mathbb{R}.$$

This is easily checked by noting the quantity $\nu(\nu+1)$ is preserved under the transformation $\nu\mapsto-\nu-1$, corresponding to a reflection of the ν axis over the point $\nu=-1/2$. Since the set Eigenfunctions of $\mathscr{L}_\nu^\mu$ are clearly invariant under such a transformation of ν, it may be assumed without loss of generality that $\nu\geq-1/2$.

The function $\sin^n\theta$ and the constant function 1 are easily seen to satisfy the boundary conditions (5.3). For the other functions, the derivative formula [3, p362]

$$\sin\theta\frac{\mathrm{d}R_\nu^\mu(\cos\theta)}{\mathrm{d}\theta}=(1-\mu+\nu)R_{\nu+1}^\mu(\cos\theta)-(\nu+1)\cos\theta R_\nu^\mu(\cos\theta)\tag{5.8}$$

for $R_\nu^\mu(\cos\theta)$ being either $\mathrm{P}_\nu^\mu(\cos\theta)$ or $\mathrm{Q}_\nu^\mu(\cos\theta)$, can be used to write the boundary conditions (5.3) (with $u=R_\nu^\mu(\cos\theta)$) as

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_\nu^\mu(\cos\theta)}{\sin^n\theta}\right)=\sin^n\theta\left\{(1-\mu+\nu)R_{\nu+1}^\mu(\cos\theta)-(1+n+\nu)\cos\theta R_\nu^\mu(\cos\theta)\right\}.\tag{5.9}$$

Boundary term asymptotics at $\theta=0$.

At $\theta=0$, the asymptotic behaviour of $\mathrm{P}_\nu^\mu(\cos\theta)$ is given by [3, p361]

$$\mathrm{P}_\nu^\mu(\cos\theta)\sim\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{\sin\theta}\right)^\mu,\mu\neq1,2,3,\ldots.\tag{5.10}$$

It follows from equation (5.9) with $R_\nu^\mu(\cos\theta)=\mathrm{P}_\nu^\mu(\cos\theta)$ that near $\theta=0$ the asymptotic behaviour of the boundary term is

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{P}_\nu^\mu(\cos\theta)}{\sin^n\theta}\right)\sim-\frac{2^\mu(n+\mu)}{\Gamma(1-\mu)}\sin^{n-\mu}\theta,$$

which vanishes in the limit $\theta\to0$ if and only if either $\mu=-n$ or $n-\mu>0$. Recall that in the above eigenfunctions $\mu=\pm n$ and so the boundary condition at $\theta=0$ is only

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met when $\mu = -n$. Hence $\mathrm{P}_\nu^n(\cos\theta) \notin D_{\text{S.A.}}$. To show that $\mathrm{Q}_\nu(\cos\theta)$ doesn't satisfy the boundary conditions, use the following asymptotic formula as $\theta \to 0$ [3, p361] :

$$\mathrm{Q}_\nu(\cos\theta) = \frac{1}{2}\ln\left(\frac{2}{1-\cos\theta}\right) - \gamma - \psi(\nu+1) + O(1-\cos\theta), \nu \neq -1, -2, -3, \dots \quad (5.11)$$

where γ is Euler's constant and ψ is the logarithmic derivative of the Gamma function: $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$. Hence inserting equation (5.11) into equation (5.9) yields the asymptotic behaviour of the boundary term for $n=0$:

$$\begin{aligned}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{Q}_\nu(\cos\theta)\right) =& (1+\nu)\left\{\mathrm{Q}_{\nu+1}(\cos\theta) - \cos\theta\mathrm{Q}_\nu(\cos\theta)\right\} \\ &= (1+\nu)\left\{\cos\theta\psi(\nu+1) - \psi(\nu+2) + \frac{1}{2}(1-\cos\theta)\ln\left(\frac{2}{1-\cos\theta}\right)\right\} \\ &+ O(\sin^2\theta).\end{aligned}$$

Letting $\theta \to 0$ gives the limit

$$\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{Q}_\nu(\cos\theta)\right) \to (1+\nu)(\psi(\nu+1) - \psi(\nu+2)) \neq 0.$$

showing $\mathrm{Q}_\nu(\cos\theta)$ does not satisfy the required boundary condition at $\theta = 0$. The remaining eigenfunctions are therefore

	$n=0$	$0 < \lvert n\rvert < 1$
Kernel	1	$\sin^n\theta$
Eigenspaces	$\mathrm{P}_\nu(\cos\theta)$	$\mathrm{P}_\nu^{-n}(\cos\theta)$

Boundary term asymptotics at $\theta = \pi$.
The boundary term now takes the simpler form

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{P}_\nu^{-n}(\cos\theta)}{\sin^n\theta}\right) = (1+n+\nu)\sin^n\theta\left\{\mathrm{P}_{\nu+1}^{-n}(\cos\theta) - \cos\theta\mathrm{P}_\nu^{-n}(\cos\theta)\right\}, \quad (5.12)$$

and the possible values of ν such that the boundary term vanishes at $\theta = \pi$ will be determined.

Case 1: $n=0$
Suppose that $\mathrm{P}_\nu(\cos\theta)$ satisfies the boundary condition at $\theta = \pi$. From the *connection formula* [3, p.362]:

$$\frac{2}{\pi}\sin((\nu-\mu)\pi)\mathrm{Q}_\nu^{-\mu}(\cos\theta) = \cos((\nu-\mu)\pi)\mathrm{P}_\nu^{-\mu}(\cos\theta) - \mathrm{P}_\nu^{-\mu}(-\cos\theta),$$

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it follows that

$$\begin{aligned}\mathrm{P}_\nu(\cos\theta) &= \cos(\nu\pi)\mathrm{P}_\nu(-\cos\theta) - \frac{2}{\pi}\sin(\nu\pi)\mathrm{Q}_\nu(-\cos\theta)\\ &= \cos(\nu\pi)\mathrm{P}_\nu(-\cos\theta) - \frac{2\sin(\nu\pi)}{\pi}\left\{\frac{1}{2}\ln\left(\frac{2}{1+\cos\theta}\right) - \gamma - \psi(\nu+1)\right\}\\ &\quad + O(1+\cos\theta)\end{aligned} \tag{5.13}$$

as $\theta \to \pi$ with the second equality following from the asymptotic behaviour of Q_ν given by formula (5.11). Inserting this into the boundary term (5.12) with $n = 0$ gives

$$\begin{aligned}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{P}_\nu(\cos\theta)\right) =&(1+\nu)\Big[-\cos(\nu\pi)(\mathrm{P}_{\nu+1}(-\cos\theta) + \cos\theta\mathrm{P}_\nu(-\cos\theta))\\ &+ \frac{\sin(\nu\pi)(1+\cos\theta)}{\pi}\left\{\ln\left(\frac{2}{1+\cos\theta}\right) - 2\gamma\right\}\\ &- \frac{2\sin(\nu\pi)}{\pi}\big(\psi(\nu+2) + \cos\theta\psi(\nu+1)\big)\Big] + O(1+\cos\theta)\\ &\to \frac{2\sin(\nu\pi)}{\pi}(\psi(\nu+1) - \psi(\nu+2)) \neq 0,\end{aligned}$$

as $\theta \to \pi$, where the limit of $\mathrm{P}_\nu(-\cos\theta)$ and $\mathrm{P}_{\nu+1}(-\cos\theta)$ have been calculated by formula (5.10). Hence the boundary condition is not satisfied unless $\nu \neq 0, 1, 2, \ldots$.

Case 2: $n \neq 0$

First the following representation for $\mathrm{P}_\nu^{-n}(\cos\theta)$ is introduced [3, p353]:

$$\mathrm{P}_\nu^{-n}(\cos\theta) = \tfrac{1}{\Gamma(1+n)}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{-n/2} {}_2F_1\left(\nu+1, -\nu; 1+n; \frac{1-\cos\theta}{2}\right), \tag{5.14}$$

where the the function ${}_2F_1(a, b; c, z)$ is the *hyper-geometric series* defined as

$${}_2F_1(a, b; c, z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k, \qquad z \in \{z \in \mathbb{C} : |z| < 1\}, \tag{5.15}$$

with

$$(x)_k = \begin{cases} x(x+1)\ldots(x+k-1), & k = 1, 2, 3, \ldots,\\ 1, & k = 0,\end{cases}$$

being the *rising factorial*. It is remarked that ${}_2F_1(a, b; c, z)$ is not defined when c is a negative integer. Furthermore ${}_2F_1(a, b; c, 0) = 1$ and the convergence of the series on the circle $|z| = 1$ depends on the sign of $c - a - b$. To investigate the behaviour of the

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$\mathrm{P}_\nu^{-n}(\cos\theta)$ at $\theta=\pi$ the following transformation is used [3, p390]:

$$\begin{aligned}{}_2F_1(a,b;c,z) =& \tfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b+1-c;1-z)\\ &+\tfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z).\end{aligned}$$

The above transformation is valid whenever $c-a-b\neq 0,-1,-2,\ldots$ as $\Gamma(z)$ is singular whenever z is a non-positive integer. For the hyper-geometric series in equation (5.14) for any ν, $c-a-b=n$ and since it is assumed that $n\in(-1,1)\backslash\{0\}$, the above transformation is valid for all ν. The series representation for $\mathrm{P}_\nu^{-n}(\cos\theta)$ then takes the form

$$\begin{aligned}\mathrm{P}_\nu^{-n}(\cos\theta) =& \frac{\Gamma(n)}{\Gamma(n-\nu)\Gamma(1+n+\nu)}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{-n/2}{}_2F_1\left(\nu+1,-\nu;1+n,\frac{1+\cos\theta}{2}\right)\\ &+\frac{2^{-n}\Gamma(-n)\sin^n\theta}{\Gamma(1+\nu)\Gamma(-\nu)}{}_2F_1\left(n-\nu,1+n+\nu;1+n,\frac{1+\cos\theta}{2}\right).\end{aligned}\tag{5.16}$$

Rewriting the boundary term (5.12) with the above representation of $\mathrm{P}_\nu^{-n}(\cos\theta)$ leads to

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{P}_\nu^{-n}(\cos\theta)}{\sin^n\theta}\right) = (1+n+\nu)\left\{\Gamma(n)(1-\cos\theta)^n\chi(\theta)+2^{-n}\Gamma(-n)\sin^{2n}\theta\Upsilon(\theta)\right\},$$

where

$$\chi(\theta) = \frac{{}_2F_1\left(\nu+2,-\nu-1;1+n,\frac{1+\cos\theta}{2}\right)}{\Gamma(n-\nu-1)\Gamma(2+n+\nu)} - \cos\theta\frac{{}_2F_1\left(\nu+1,-\nu;1+n,\frac{1+\cos\theta}{2}\right)}{\Gamma(n-\nu)\Gamma(1+n+\nu)},$$

and

$$\Upsilon(\theta) = \frac{{}_2F_1\left(n-\nu-1,2+n+\nu;1+n,\frac{1+\cos\theta}{2}\right)}{\Gamma(2+\nu)\Gamma(-\nu-1)} - \cos\theta\frac{{}_2F_1\left(n-\nu,1+n+\nu;1+n,\frac{1+\cos\theta}{2}\right)}{\Gamma(1+\nu)\Gamma(-\nu)}.$$

After first using the *reflection formula* [3, p138]:

$$\Gamma(z+1)=z\Gamma(z),\qquad z\in\mathbb{C}.$$

and then the series representation (5.15) for ${}_2F_1(a,b;c,z)$, one can write $\Upsilon(\theta)$ as

$$\begin{aligned}\Upsilon(\theta) =& \frac{{}_2F_1\left(n-\nu-1,2+n+\nu;1+n,\frac{1+\cos\theta}{2}\right)+\cos\theta{}_2F_1\left(n-\nu,1+n+\nu;1+n,\frac{1+\cos\theta}{2}\right)}{\Gamma(2+\nu)\Gamma(-\nu-1)}\\ =& \frac{1}{\Gamma(\nu+2)\Gamma(-\nu-1)}\Bigg[1+\cos\theta+\sum_{l=1}^{\infty}\\ &\left(\frac{(n-\nu-1)_l(2+n+\nu)_l+\cos\theta(n-\nu)_l(1+n+\nu)_l}{l!(1+n)_l}\right)\left(\frac{1+\cos\theta}{2}\right)^l\Bigg].\end{aligned}$$

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It follows that as $\theta \to \pi$ the behaviour of $\chi(\theta)$ and $\sin^{2n}\theta\Upsilon(\theta)$ are

$$\chi(\theta) \to \frac{1}{\Gamma(n-\nu-1)\Gamma(2+n+\nu)} + \frac{1}{\Gamma(n-\nu)\Gamma(1+n+\nu)},$$
$$\sin^{2n}\theta\Upsilon(\theta) \to 0$$

leading to the asymptotic behaviour of the boundary term

$$\sin^{2n+1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{P}_\nu^{-n}(\cos\theta)}{\sin^n\theta}\right) \sim \frac{2^n(1+\nu+n)\Gamma(n)}{\Gamma(\nu+n+2)\Gamma(n-\nu-1)} + \frac{2^n(1+\nu+n)\Gamma(n)}{\Gamma(\nu+n+1)\Gamma(n-\nu)}.$$

To satisfy the boundary condition, we require the above terms to be 0 which is only the case when $\nu = n+m$ for $m \in \mathbb{N}\cup\{0\}$, so that $1/\Gamma(n-\nu-1) = 1/\Gamma(n-\nu) = 0$.

The family $\{\mathrm{P}_{n+m}^{-n}(\cos\theta)\}_{m=0}^\infty$ are thus the only family of eigenfunctions in $D_{\mathrm{S.A.}}$ and form an orthogonal basis of $L^2_{\sin\theta}(0,\pi)$, therefore if $u \in L^2_{\sin\theta}(0,\pi)$

$$u = \gamma_{0,n}\sin^n\theta + \sum_{m=1}^\infty \gamma_{m,n}\mathrm{P}_{n+m}^{-n}(\cos\theta),$$

for expansion coefficients $\{\gamma_{m,n}\}_{m=0}^\infty$. □

For well-behaved surfaces, Proposition 5.7 gives an expansion of the surfaces astigmatism in terms of the Legendre functions.

Theorem 5.8 *Let $\mathcal{S}$ be a C^2-smooth rotationally symmetric, strictly convex sphere with astigmatism satisfying $s/\sin^{n+2}\theta \in L^2_{\sin\theta}(0,\pi)$. If $n \in (-1,1)$, the following geometric quantities associated to $\mathcal{S}$ decompose as:*

$$s = \gamma_{0,n}\sin^{2n+2}\theta + \sin^{n+2}\theta\sum_{m=1}^\infty \gamma_{n,m}P_{n+m}^{-n}(\cos\theta),$$

$$r_1 = C_1 + \frac{\gamma_{0,n}}{2(n+1)}\sin^{2n+2}\theta + \sin^{n+2}\theta\sum_{m=1}^\infty \gamma_{m,n}\left\{P_{n+m}^{-(n+2)}(\cos\theta) + \cot\theta P_{n+m}^{-(n+1)}(\cos\theta)\right\},$$

$$r = C_2\cos\theta + C_1 + \gamma_{0,n}\left[\frac{\sin^{2n+2}\theta}{2(n+1)} - \cos\theta\int_0^\theta \sin^{2n+1}\theta d\theta\right] + \sin^{n+2}\theta\sum_{m=1}^\infty \gamma_{m,n}P_{n+m}^{-(n+2)}(\cos\theta),$$

where $C_1 = r_1(0)$, $C_2 = r(0) - r_1(0)$ and the first equality is to be understood as an equality in $L^2_{\sin\theta}(0,\pi)$ where as the last two are point-wise.

Proof. The expansion for s follows from Proposition 5.7 by letting $s = \sin^{n+2}\theta \cdot u$. The expansion for r_1 can be derived by inserting the expansion of s into the integrated

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derived Codazzi-Mainardi equation (2.3):

$$\begin{aligned} r_1(\theta) &= r_1(0) + \int_0^\theta \frac{s\cos\theta}{\sin\theta} d\theta \\ &= r_1(0) + \frac{\gamma_{0,n}}{2(n+1)} \sin^{2n+2}\theta + \sum_{m=1}^{\infty} \gamma_{m,n} \int_0^\theta \cos\theta \sin^{n+1}\theta \mathrm{P}_{n+m}^{-n}(\cos\theta) d\theta. \end{aligned}$$

We then use the standard integral [3, p368)]

$$\int \sin^{\mu+1}\theta \cdot \mathrm{P}_\nu^{-\mu}(\cos\theta)d\theta = \sin^{\mu+1}\theta \cdot \mathrm{P}_\nu^{-(\mu+1)}(\cos\theta) \forall \mu, \nu \in \mathbb{R},$$

to evaluate the following by parts:

$$\int_0^\theta \cos\theta \sin^{n+1}\theta \mathrm{P}_{n+m}^{-n}(\cos\theta)d\theta = \sin^{n+2}\theta \mathrm{P}_{n+m}^{-(n+2)}(\cos\theta) + \cos\theta \sin^{n+1}\theta \mathrm{P}_{n+m}^{-(n+1)}(\cos\theta),$$

where the boundary term at $\theta = 0$ vanishes because of the asymptotic behaviour of $\mathrm{P}_{n+m}^{-(n+1)}$, see equation (5.10). This completes the derivation for r_1, to derive the expansion of r one may either insert the astigmatism decomposition into equation (2.2) and proceed as above, or we can integrate the decomposition of r_1 by virtue of equation (2.1). The calculation is analogous. □

For a given rotationally symmetric surface $\mathcal{S}$ with astigmatism s, it is intriguing to ask in what way does the expansion of s given in Theorem 5.8 convey geometric information about $\mathcal{S}$?

Corollary 5.9. *Let $\gamma_{m,n}$ be the expansion coefficients as given in Theorem 5.8 for $n \in (-1, 1)$. We have the following equalities*

$$\gamma_{0,n} = \frac{\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+1)} \cdot (f_0 - f_\pi), \qquad \gamma_{1,n} = \frac{\Gamma(2n+4)}{2^{n+1}\Gamma(n+1)} \cdot (r_1(\pi) - r_1(0)).$$

hence $\gamma_{0,n}$ characterises the distance between the focal points of $\mathcal{S}$ while $\gamma_{1,n}$ characterises the differences between the radii of curvature at each pole.

Proof. Starting with the expansion formula for s in Theorem 5.8, we divide by $\sin\theta$ and integrate to find the following expression satisfied by $\gamma_{0,n}$:

$$\int_0^\pi \frac{s}{\sin\theta} d\theta = \gamma_{0,n} \int_0^\pi \sin^{2n+1}\theta d\theta,$$

where the higher order terms in the expansion vanish via orthogonality. The integral on the left hand side can be evaluated by using sequentially, equation (2.3), integration by parts, equation (2.1) and Remark 2.1:

$$\int_0^\pi \frac{s}{\sin\theta}d\theta = f_0 - f_\pi,$$

which proves the claim for $\gamma_{0,n}$ once the remaining integral on the right hand side is calculated. The claim concerning $\gamma_{1,n}$ follows by letting $\theta = \pi$ in the expansion of r_1. All higher order terms tend to zero as $\theta \to 0$ apart from the $m = 1$ which tends to a constant. This follows by taking the limit of equation (5.16) as $\theta \to \pi$. Therefore

$$r_1(\pi) = r_1(0) + \gamma_{1,n}\left\{\sin^{n+2}\theta \mathrm{P}_{n+1}^{-(n+2)}(\cos\theta) + \sin^{n+1}\theta\cos\theta \mathrm{P}_{n+1}^{-(n+1)}(\cos\theta)\right\}\Big|_{\theta=\pi}.$$

The $m = 1$ term satisfies

$$\sin^{n+2}\theta \mathrm{P}_{n+1}^{-(n+2)}(\cos\theta) + \sin^{n+1}\theta\cos\theta \mathrm{P}_{n+1}^{-(n+1)}(\cos\theta) \to \frac{\Gamma(n+1)2^{n+2}}{\Gamma(2n+4)},$$

as $\theta \to \pi$, again following from equation (5.16), which completes the proof. □

5.3. Time evolution of the eigenbasis expansion

In this section, we give a solution for each $n \in (-1, 1)$ to the time evolution problem (5.1) for all $t \geq 0$ and for almost every $\theta \in [0, \pi]$. Furthermore this solution is unique in the sense that it is equal point-wise to a strong solution of (5.1) whenever a strong solution exists, for all most all $\theta \in [0, \pi]$, for every $t \geq 0$.

Proposition 5.10. *Fix $n \in (-1, 1)$ and let $\{e_m\}_{m=1}^\infty$ be the Legendre functions $\{P_{n+m}^{-n}(\cos\theta)\}_{m=0}^\infty$, normalised with respect to the $L^2_{\sin\theta}(0, \pi)$ inner product, i.e.*

$$e_m = \frac{P_{n+m}^{-n}(\cos\theta)}{||P_{n+m}^{-n}(\cos\theta)||}.$$

Let $u_0 \in D_{S.A.}$ have the eigenbasis decomposition

$$u_0 = \sum_{m=0}^\infty a_m e_m.$$

and $u : [0, \infty) \to L^2_{\sin\theta}(0, \pi)$ be the mapping defined by the series

$$u(t) = \sum_{m=0}^\infty A_m(t) e_m, \qquad A_m(t) = a_m \exp\{\lambda_m bt\}$$

where $\lambda_m = -(2n + 1 + m)m$. Then

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(a) *For each* $t \geq 0$, $\mathrm{u}(\mathrm{t})$ *is an element of* $D_{S.A.}$ *and*

$$[u(t)](\theta) = \sum_{m=0}^{\infty} A_m(t) e_m(\theta), \tag{5.17}$$

point-wise for almost all $\theta \in (0, \pi)$ *(with respect to the Lebesgue measure).*

(b) *Furthermore,* $\mathrm{u}(\mathrm{t})$ *is the unique solution to equation (5.1) with initial data* u_0.

Proof. First we prove (a). Let $t \geq 0$. Since $n \in (-1, 1)$ and $b > 0$, we have that $\lambda_m \leq 0$ for all $m \in \mathbb{N}$ and so $A_m(t)$ is decreasing in t, in particular $|A_m(t)| \leq |a_m|$. It follows that since $u_0 \in L^2_{\sin\theta}(0, \pi)$, $u(t) \in L^2_{\sin\theta}(0, \pi)$ also by the comparison test for series in $L^2_{\sin\theta}(0, \pi)$. To show membership of $D_{\mathrm{S.A.}}$ note that by the Spectral Theorem 4.4, $D_{\mathrm{S.A.}}$ is characterised by those elements $x \in L^2_{\sin\theta}(0, \pi)$ such that $(\lambda_n \langle x, e_n \rangle)_n \in \ell^2$, therefore since $u_0 \in D_{\mathrm{S.A.}}$,

$$\sum_{m=0}^{\infty} |\lambda_m A_m(t)|^2 \leq \sum_{m=0}^{\infty} |\lambda_m a_m|^2 < \infty,$$

and $u(t) \in D_{\mathrm{S.A.}}$ for all $t \geq 0$. To show (5.17) holds we invoke the following theorem.

Theorem 5.11 [*15*] p267 *Let* $\{e_n\}_n$ *be an orthonormal series with respect to* $L^2_{\sin\theta}(0, \pi)$. *The series*

$$\sum_{n=0}^{\infty} b_n e_n(\theta),$$

converges absolutely for almost all $\theta \in (0, \pi)$ *if the sequence* $(b_n)_n$ *satisfies*

$$\sum_{n=2}^{\infty} |b_n|^2 (\log_2(n))^2 (\log_2(\log_2 n))^{1+\varepsilon} < \infty$$

for some $\varepsilon > 0$.

To see that the sequence $(a_m)_m$ satisfies the above requirement it is enough to notice that the sequence $(\lambda_m)_m$ grows quadratically with m, hence taking $\varepsilon = 1$;

$$\sum_{m=M}^{\infty} |a_m|^2 (\log_2(m))^2 (\log_2(\log_2 m))^2 < \sum_{m=M}^{\infty} |a_m|^2 \lambda_m^2 < \infty$$

for some $M \in \mathbb{N}$. Therefore $\sum_{m=0}^{\infty} a_m e_m(\theta)$ converges absolutely a.e., and so must $\sum_{m=0}^{\infty} A_m(t) e_m(\theta)$ for all $t \geq 0$, by a comparison of series. Now we have proved a.e. convergence of the RHS of (5.17), the stated equality follows by uniqueness of limits, since the partial sums of $u(t)$ converge a.e. along some sub-sequence to $u(t)$.

To show (b) note that $u(t)$ satisfies the boundary conditions in (5.1) since $u(0) = u_0$ and $u(t) \in D_{\mathrm{S.A.}}$. All that is left to is to show $u(t)$ solves the time evolution equation

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in (5.1). Fix $\theta \in [0,\pi]$ such that we have the absolute pointwise convergence of $u_0(\theta) = \sum_{m=0}^{\infty} a_m e_m(\theta)$ and therefore of $[u(t)](\theta) = \sum_{m=0}^{\infty} A_m(t) e_m(\theta)$ for all $t \geq 0$. Let us justify the term-by-term differentiation of this function w.r.t. t. Fix $T > 0$ and consider the functions $f : \mathbb{N}_0 \times [T,\infty) \to \mathbb{R}$ and $g : \mathbb{N}_0 \to \mathbb{R}$ given by

$$f(m,t) = A_m(t) e_m(\theta), \qquad \text{and} \qquad g(m) = (eT)^{-1} |a_m e_m(\theta)|.$$

For any $x > 0$ and $r < 0$ the inequality $|r| e^{rx} < (ex)^{-1}$ gives

$$\left| \frac{\partial f(m,t)}{\partial t} \right| = |a_m \cdot \lambda_m b e^{\lambda_m b t} \cdot e_m(\theta)| \leq (et)^{-1} \cdot |a_m e_m(\theta)| \leq (eT)^{-1} \cdot |a_m e_m(\theta)| = g(m)$$

for all $(m,t) \in \mathbb{N}_0 \times [T,\infty)$. Hence $|\frac{\partial}{\partial t} f(m,t)|$ is dominated by a function whose sum is convergent. Hence by a corrollary of the dominated convergence theorem [6, pg 56], f is differentiable with respect to t for all $t > T$ and

$$[\partial_t u(t)](\theta) = \partial_t \sum_{m=0}^{\infty} f(m,t) = b \sum_{m=0}^{\infty} \lambda_m A_m(t) e_m(\theta) = b \cdot [\mathscr{L}_n^n u(t)](\theta),$$

with the last equality following from the fact that λ_m is the eigen value of the operator $\mathscr{L}_n^n$ associated to e_m and part (3) of the Spectral Theorem 4.4. Finally since $T > 0$ was arbitrary we have

$$[\partial_t u(t)](\theta) = b \cdot [\mathscr{L}_n^n u(t)]\,(\theta) \qquad \forall t \in (0,\infty), \text{a.e. } \theta \in (0,\pi).$$

Now projecting the time evolution equation into each eigen space derives an ODE for the basis components of $u(t)$. Uniqueness now follows from ODE theorem. □

Making the substitution $u = s/\sin^{n+2}\theta$ turns the time evolution problem (5.1) into

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{s}{\sin^{n+2}\theta}\right) = b \cdot \mathscr{L}_n^n \left(\dfrac{s}{\sin^{n+2}\theta}\right), & [0,\infty) \times [0,\pi] \\ \dfrac{s(t,\cdot)}{\sin^{n+2}(\cdot)} \in D_{\text{S.A.}} & t \in [0,\infty) \\ s = s_0 & \{t = 0\} \times [0,\pi], \end{cases} \tag{5.18}$$

which describes the evolution of a surfaces astigmatism under the linear Hopf flow (3.5).

Corollary 5.12. *Let $\mathcal{S}_0 \in \mathscr{W}$ be an initial surface with an astigmatism s_0 satisfying $s_0/\sin^{n+2}\theta \in D_{S.A.}$. If $\mathcal{S}_t$ is a strong solution of the linear Hopf flow (1.1) and (1.2) with $n \in (-1,1)$ then the astigmatism s, radius of curvature r_1 and the support function*

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r *of* $\mathcal{S}_t$ *evolve as*

$$s(t,\theta) = \gamma_{0,n}\sin^{2n+2}\theta + \sum_{m=1}^{\infty}\Gamma_{n,m}(t)\sin^{n+2}\theta P_{n+m}^{-n}(\cos\theta), \qquad \big(a.e.\ \theta\in[0,\pi]\big),$$

for almost all $\theta\in[0,\pi]$ *and all* $t\geq 0$*, and*

$$r_1(t,\theta) = C_1(t) + \frac{\gamma_{0,n}\sin^{2n+2}\theta}{2n+2} + \sin^{n+2}\theta\sum_{m=1}^{\infty}\Gamma_{n,m}(t)\left[P_{n+m}^{-(n+2)}(\cos\theta) + \cot\theta P_{n+m}^{-(n+1)}(\cos\theta)\right],$$

$$\begin{aligned} r(t,\theta) &= C_2(t)\cos\theta + C_1(t) + \gamma_{0,n}\left[\frac{\sin^{2n+2}\theta}{2n+2} - \cos\theta\int_0^{\theta}\sin^{2n+1}\theta d\theta\right] \\ &+ \sin^{n+2}\theta\sum_{m=1}^{\infty}\Gamma_{n,m}(t)P_{n+m}^{-(n+2)}(\cos\theta), \end{aligned}$$

for all $\theta\in[0,\pi]$*,* $t\geq 0$*, where* $\Gamma_{m,n}(t) = \gamma_{m,n}\exp\big\{-(2n+1+m)mb\cdot t\big\}$ *and* $\gamma_{m,n}$ *are the decomposition coefficients of the initial surface* $\mathcal{S}_0$ *in the basis* $\{P_{n+m}^{-n}(\cos\theta)\}_{m=0}^{\infty}$*. Furthermore the constants* $C_1(t)$ *and* $C_2(t)$ *evolve as*

$$C_1(t) = C_1(0)e^{-2(n+1)bt} + \frac{c\left(1-e^{-2(n+1)bt}\right)}{2(n+1)b}, \qquad C_2 = constant\ .$$

Proof. Fix $n\in(-1,1)$. Since $u(t,\cdot) := s(t,\cdot)/\sin^{n+2}(\cdot)$ solves Equation (5.1), we may expand it as in Proposition 5.10, giving a series which convergence point-wise almost everywhere. After noting the coefficients $\Gamma_{m,n}(t)$ are related to $A_m(t)$ as in Proposition 5.10 by $A_m(t) = \Gamma_{n,m}(t)||\mathrm{P}_{n+m}^{-n}(\cos\theta)||$ this derives the series expansion for $s(t,\theta)$. Since $\mathcal{S}_t$ is a strong solution, it follows that the support function $r(t,\theta)$ must be C^2-smooth in the θ variable for all t. This implies via Equations (2.1) that $r_1(t,\theta)$ and $r_2(t,\theta)$ must be finite for $\theta\in(0,\pi)$. The expressions for $r_1(t,\theta)$ and $r(t,\theta)$ are then derived as in Theorem 5.8 by integration. To derive the behaviour of $C_1(t)$ and $C_2(t)$, we ensure they satisfy equation (3.4) for the linear Hopf flow, i.e.

$$\begin{aligned} \frac{\partial r(t,\theta)}{\partial t} &= ar_1(t,\theta) + br_2(t,\theta) + c \\ &= c + b\left[s(t,\theta) - 2(n+1)r_1(t,\theta)\right]. \end{aligned}$$

where we have first written the right hand side of (3.4) in terms of r_1 & r_2, and then used the identity $-a/b = 2n+3$. Substituting the expansions of $r(t,\theta)$, $r_1(t,\theta)$ and $s(t,\theta)$ into this equation and collecting together terms gives us the relationship

$$\partial_t C_2(t)\cos\theta + \partial_t C_1(t) = c - 2(n+1)bC_1(t) + b\sin^{n+2}\theta\sum_{m=1}^{\infty}\Gamma_{n,m}(t)\Delta_{m,n},$$

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where

$$\Delta_{m,n} = \mathrm{P}_{n+m}^{-n}(\cos\theta) - 2(n+1)\cot\theta \mathrm{P}_{n+m}^{-(n+1)}(\cos\theta) - (2n+2+m)(m-1)\mathrm{P}_{n+m}^{-(n+2)}(\cos\theta).$$

However, by letting $\mu = -(n+2)$ and $v = n+m$ in the following recurrence relation between the Legendre functions [3, p362]

$$\mathrm{P}_{\nu}^{\mu+2}(x) + 2(\mu+1)x\left(1-x^2\right)^{-1/2}\mathrm{P}_{\nu}^{\mu+1}(x) + (\nu-\mu)(\nu+\mu+1)\mathrm{P}_{\nu}^{\mu}(x) = 0,$$

we can see that $\Delta_{m,n}$ is identically 0, hence we have the following evolution equation

$$\partial_t C_2(t)\cos\theta + \partial_t C_1(t) = c - 2(n+1)bC_1(t).$$

It is easy to see that $\partial_t C_2(t) = 0$. Solving the remaining ODE gives the stated time evolution of $C_1(t)$. □

Remark 5.13. We have the following asymptotic behaviour as $t \to \infty$,

$$r(t,\theta) \sim C_2\cos\theta + \frac{c}{2(n+1)b} + \gamma_{0,n}\left[\frac{\sin^{2n+2}\theta}{2n+2} - \cos\theta\int_0^\theta \sin^{2n+1}\theta d\theta\right].$$

This is the support function of a linear Hopf sphere (cf. Equation (3.3)). If $\gamma_{0,n} = 0$, i.e. if the focal points of the initial surface coincide, then the support function is that of a sphere with radius

$$\frac{c}{2(n+1)b} = -\frac{c}{a+b},$$

as claimed.

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