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SOME SMOOTHNESS PROPERTIES OF MEASURES ON TOPOLOGICAL SPACES

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ABSTRACT. V. S. Varadarajan has classified the bounded linear functionals on the algebra $C(X)$ of bounded continuous functions on a topological space X, according to the properties of their smoothness and related this classification to the corresponding natural classification of finitely additive regular measures on the zero sets of X. In this paper, some of these results are extended to the linear functionals on an arbitrary uniformly closed algebra A of bounded functions on a set X.

0. **Introduction.** Starting from the classical theorem of F. Riesz (1909) there have been a number of attempts to give integral representations of bounded linear functionals on the space $C(X)$ of bounded continuous functions on a topological space X under different restrictions on X. Among them are the results of Banach, Kakutani, Alexandroff, Varadarajan, Kirk, Kirk and Crenshaw, and others. These representations have been used to classify the bounded linear functionals in terms of the representing measures. This paper attempts to generalize some of the results of Varadarajan in this direction.

1. **Representation theorems.** Let m be a finitely additive bounded real valued set function on a field Σ of subsets of a set X. For a subfamily ω of Σ, m is said to be *ω-regular* if for every A in Σ and for every positive ε, there exists W in ω such that $W\subset A$ and $|m(B)|<\varepsilon$, for all B in Σ with B contained in $A-W$. A family ω of subsets of a set X is called a *full paving* in X if ω contains ϕ, X and is closed under finite union and finite intersection. For a full paving ω in X, we denote by $\mathscr{F}(\omega)$ the field of subsets of X generated by ω, and by $M(\omega)$ the space of all finitely additive, bounded, real valued, ω-regular set functions defined on $\mathscr{F}(\omega)$. The space $M(\omega)$ is a Banach lattice under the usual pointwise operations and total variation norm. Let A be a uniformly closed algebra of bounded real valued functions on X which contains the constants and separates the points of X, and A^* be its Banach space dual. A^* is a Banach lattice with the usual definition of non-negative linear functionals. We say that $M(\omega)$ *represents* A^* if there exists a Banach lattice isomorphism I of A^* onto $M(\omega)$ such that for every non-negative bounded linear functional ψ

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on A, we have

$$I\psi(W)=\inf\{\psi(f) \mid f\in A, \chi_w \le f\}$$

for all W in ω. Here χ_w denotes the characteristic function on W. Obviously this condition on I makes such an isomorphism unique.

We state two representation theorems. Let X be a topological space. Let $C(X)$ denote the algebra of all bounded real valued continous functions on X with supremum norm, and $C(X)^*$ be its Banach space dual. For an f in $C(X)$, the kernel of f in X is called a *zero set* in X. The full paving of all zero sets in X is denoted by $\mathscr{Z}(X)$. The following theorem is proved by Alexandroff [1]. Also see Varadarajan [5].

THEOREM 1.1. *There exists a Banach lattice isomorphism T of $C(X)^*$ onto $M(\mathscr{Z}(X))$ and the corresponding elements $\Lambda \in C(X)^*$ and $m = T(\Lambda)$ of $M(\mathscr{Z}(X))$ satisfy the identities,*

$$\Lambda(f)=\int f\,dm \quad \text{for every} \quad f\in C(X)$$

and

$$m(Z)=\inf\{\Lambda(g): \chi_Z \le g\} \quad \text{for every} \quad Z\in \mathscr{Z}(X).$$

In other words $M(\mathscr{Z}(X))$ represents $C(X)^*$. Here the integration of f with respect to m is taken in the sense of Dunford and Schwartz [2].

There have been attempts to generalize the Alexandroff's theorem to a wider class of functions. Among them are the works of Kirk [3], Kirk and Crenshaw [4]. Given a set X and a uniformly closed algebra A of bounded real valued functions on X which contains the constants and separates the points of X, there exists, as a consequence of Stone–Weierstrass theorem, a compact Hausdorff space X_A such that X can be embedded into X_A as a dense subspace of X_A and A is isomorphic as a Banach lattice to $C(X_A)$ [2, *p.* 276]. The topology on X inherited by the topology of X_A is denoted by τ_A. For an f in A, let $\bar{f}$ denote the unique continuous extension of f to X_A. For a subset S of X, we denote the closure of S in X_A by $\bar{S}$. The algebra A is said to *separate* a full paving ω in X if whenever W_1, W_2 are disjoint sets in ω, there exists f in A with $f=0$ on W_1 and $f=1$ on W_2. It is obvious that A separates ω if and only if $\bar{W}_1$ and $\bar{W}_2$ are disjoint whenever W_1 and W_2 in ω are disjoint. However for arbitrary W_1, W_2 in ω, $\overline{W_1\cap W_2}$ may fail to be equal to $\overline{W_1}\cap\overline{W_2}$. If $\overline{W_1\cap W_2}=\overline{W_1}\cap\overline{W_2}$ whenever W_1, W_2 are in ω, we say that A *strongly separates* ω. A strongly separates ω implies that A separates ω. A set $Z\subset X$ is called a A-*zero set* if there exists $f\in A$ such that $Z=Z(f)=\{x\in X \mid f(x)=0\}$. We denote the family of A-*zero sets by* $\mathscr{Z}(A)$. $\mathscr{Z}(A)$ is a full paving in X constituting a base for τ_A-closed sets in X. In this case, A separates $\mathscr{Z}(A)$ if and only if A strongly separates $\mathscr{Z}(A)$. Furthermore if ω is any full paving of τ_A-closed sets constituting a base for τ_A-closed sets and contains $\mathscr{Z}(A)$, then

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also A separates ω if and only if A strongly separates ω. Kirk [3] has proved *that* A separates $\mathscr{Z}(A)$ if and only if $M(\mathscr{Z}(A))$ represents A^*. Kirk has also given a class of algebras which do not separate $\mathscr{Z}(A)$.

A more general case is considered by Kirk and Crenshaw. The following theorem is proved in [4].

THEOREM 1.2. *Let A be a uniformly closed algebra of bounded real valued functions on X which contains the constants and separates the points of X and let ω be a full paving of τ_A-closed subsets of X. Then $M(\omega)$ represents A^* if and only if*

1. *A strongly separates ω.*
2. *For any $0 \le \psi \in A^*$ and $Z \in \mathscr{Z}(A)$,*

$$\sup_{W \in \omega(Z)} \inf\{\psi(f) \mid \chi_W \le f\} = \sup\{\psi(g) \mid g \in A, g \le \chi_{X-Z}\}$$

where $\omega(Z) = \{W \in \omega \mid$ there exists $f \in A$ with $f(Z) = 0$ and $f(W) = 1\}$. Furthermore if $M(\omega)$ represents A^* and $m \in M(\omega)$, then each $f \in A$ is m-integrable with $I^{-1}m(f) = \int f\, dm$.

Varadarajan [5] has classified the bounded linear functionals on $C(X)$ in accordance with their properties of smoothness and applied Alexandroff's theorem to relate this classification of functionals to the corresponding natural classification of elements of $M(\mathscr{Z}(X))$. In this paper we apply the theorem of Kirk and Crenshaw to relate the classification of bounded linear functionals on A to the corresponding classification of elements of $M(\omega)$ where $M(\omega)$ represents A^* and thus generalize some results of Varadarajan.

2. **Classification of functionals and measures.** Throughout this section X is an arbitrary set, ω is a full paving in X and A is a uniformly closed algebra of bounded real valued functions on X containing the constants and separating the points of X. The symbols $\mathscr{F}(\omega)$, $M(\omega)$, X_t, $\mathscr{Z}(X)$, $\mathscr{Z}(A)$ all have the same meanings as defined in §1.

For a net $\{f_\alpha\}$ in A we say $\{f_\alpha\}$ decreases to 0 and write $f_\alpha \downarrow 0$ if for each $x \in X$, $0 \le f_\alpha(x) \le f_\beta(x)$ whenever $\alpha \ge \beta$ and $\lim_\alpha f_\alpha(x) = 0$. The symbol $f_n \downarrow 0$ is analogously defined for a sequence $\{f_n\}$ in A.

DEFINITIONS 2.1. Let $\psi \in A^*$. Then ψ is said to be

(i) *τ-smooth* if $\lim_\alpha \psi(f_\alpha) = 0$ whenever $\{f_\alpha\}$ is a net in A with $f_\alpha \downarrow 0$.

(ii) *σ-smooth* if $\lim_n \psi(f_n) = 0$ whenever $\{f_n\}$ is a sequence in A with $f_n \downarrow 0$. The space of all τ-smooth and σ-smooth functionals in A^* are denoted by A^*_τ and A^*_σ respectively. Clearly

$$A^*_\tau \subset A^*_\sigma \subset A^*.$$

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Following the theorems 7 and 8 of part I in [5] we can prove

PROPOSITION 2.2. *$\psi \in A^*$ is τ-smooth (σ-smooth) if and only if its positive and negative variations ψ^+ and ψ^- are τ-smooth (σ-smooth). Consequently ψ is τ-smooth (σ-smooth) if and only if its total variation $|\psi|$ is τ-smooth (σ-smooth).*

DEFINITIONS 2.3. For a net $\{A_\alpha\}$ of subsets of X we say that A_α *decreases to* ϕ and we write $A_\alpha \downarrow \phi$ if $A_\alpha \subset A_\beta$ whenever $\alpha \geq \beta$ and $\bigcap_\alpha A_\alpha = \phi$. We similarly define the symbol $A_n \downarrow \phi$ for a sequence $\{A_n\}$ of subsets of X.

Let $m \in M(\omega)$ and let $|m|$ be the total variation of m. Then m is said to be

(i) *τ-smooth* if $|m|(W_\alpha) \to 0$ for every net $\{W_\alpha\}$ in ω with $W_\alpha \downarrow \phi$.
(ii) *σ-smooth* if $|m|(W_n) \to 0$ for every sequence $\{W_n\}$ in ω with $W_n \downarrow \phi$.

The space of all τ-smooth and σ-smooth elements in $M(\omega)$ are denoted by $M_\tau(\omega)$ and $M_\sigma(\omega)$ respectively. Clearly $M_\tau(\omega) \subset M_\sigma(\omega) \subset M(\omega)$.

It is immediate from the definition that m is τ-smooth (σ-smooth) if and only if its positive and negative variation m^+ and m^- are τ-smooth (σ-smooth). Hence m is τ-smooth (σ-smooth) if and only if $|m|$ is so.

The next result proves that a σ-smooth $m \in M(\omega)$ is σ-additive.

PROPOSITION 2.4. *An $m \in M(\omega)$ is σ-smooth if and only if for every sequence $\{A_n\}$ in $\mathcal{F}(\omega)$ with $A_n \downarrow \phi$, we have $m(A_n) \to 0$.*

Proof. Without loss of generality we assume that m is non-negative so that $|m| = m$. Suppose m is σ-smooth. Suppose $m(A_n) \to \delta > 0$ for some sequence $\{A_n\}$ in $\mathcal{F}(\omega)$ with $A_n \downarrow \phi$. By ω-regularity of m, we can choose $W_n \in \omega$, $W_n \subset A_n$ such that $m(A_n) < m(W_n) + \delta/4^n$. Let $\hat{W}_n = W_1 \cap W_2 \cap \cdots \cap W_n$. Then $\{\hat{W}_n\} \subset \omega$ with $\hat{W}_n \downarrow \phi$. However

$$\begin{aligned} m(\hat{W}_n) &= m(A_n - (A_n - \hat{W}_n)) \\ &\geq m(A_n) - \sum_{i=1}^{n} m(A_i - W_i) \\ &\geq m(A_n) - \sum_{i=1}^{n} \frac{\delta}{4^i} \\ &\to \tfrac{2}{3}\delta > 0. \end{aligned}$$

This contradicts the fact that m is σ-smooth and proves that we must have $m(A_n) \to 0$.

The converse is obvious.

As an immediate consequence we have

COROLLARY 2.5. *An $m \in M(\omega)$ is σ-smooth if and only if for every disjoint sequence $\{A_n\}$ in $\mathcal{F}(\omega)$ for which $\bigcup_{n=1}^\infty A_n \in \mathcal{F}(\omega)$, we have $m(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m(A_n)$.*

We give below two examples one of which is not σ-smooth whereas the other is σ-smooth but not τ-smooth.

EXAMPLE 2.6. Let $X=\mathbb{R}$. Let $\omega=\{(-\infty, a] \mid a\in\mathbb{R}\}\cup\{X, \phi\}$. Then ω is a full paving in X. Define m on $\mathcal{F}(\omega)$ by

$$m(W)=\begin{cases}1 & \text{if either } W=X \text{ or } (-\infty, a]\subset W \text{ for some } a\in\mathbb{R}\\ 0 & \text{otherwise.}\end{cases}$$

Then $m\in M(\omega)$. However m is not σ-smooth. Indeed let $A_n=(-n-1, -n]$ for $n=1,2,\ldots$. Then $\{A_n\}$ is a disjoint sequence in $\mathcal{F}(\omega)$ with $\bigcup_{n=1}^{\infty} A_n=(-\infty, -1]\in\mathcal{F}(\omega)$. Hence $m(\bigcup_{n=1}^{\infty} A_n)=1$ whereas $\sum_{n=1}^{\infty} m(A_n)=0$ since $m(A_n)=0$ for each n.

EXAMPLE 2.7. Let $X=\Omega$ where Ω is the first uncountable ordinal. For each $\alpha\in\Omega$ let $W_\alpha=\{\lambda\in X \mid \alpha<\lambda<\Omega\}$. Let $\omega=\{W_\alpha \mid \alpha\in\Omega\}\cup\{X, \phi\}$. Then ω is a full paving in X. Define m on $\mathcal{F}(\omega)$ by

$$m(W)=\begin{cases}1 & \text{if } W_\alpha\subset W \text{ for some } \alpha\in\Omega \text{ or } W=X\\ 0 & \text{otherwise.}\end{cases}$$

Then $m\in M(\omega)$. Let $\{W_n\}$ be any sequence in ω with $W_n\downarrow\phi$. We can assume that $W_n\neq X$ for any n. Suppose $W_n\neq\phi$ for all n. Let $W_n=W_{\alpha_n}$ where $\alpha_n\in\Omega$. Since the set of positive integers is countable and Ω is the first uncountable ordinal there exists $\alpha\in\Omega$ such that $\alpha_n\leq\alpha$ for all n. Then $W_\alpha\subset W_n$ for each n contradicting the fact that $W_n\downarrow\phi$. Hence $W_n=\phi$ for all $n\geq n_0$ for some n_0 so that $m(W_n)\to 0$. Thus m is σ-smooth. However m is not τ-smooth since $\{W_\alpha \mid \alpha\in\Omega\}$ is a net in ω with $W_\alpha\downarrow\phi$ but $m(W_\alpha)=1$ for each $\alpha\in\Omega$.

Now suppose ω is a full paving of subsets of X such that $M(\omega)$ represents A^*. Then (i) Does $M_\sigma(\omega)$ necessarily correspond to A_σ^*? (ii) Does $M_\tau(\omega)$ necessarily correspond to A_τ^*? With X a topological space, $\omega=\mathcal{Z}(X)$ and $A=C(X)$ both questions (i) and (ii) have been answered affirmatively by Varadarajan [5]. However the Example 2.8 below shows that this is not true in general. We prove in Theorem 2.10 that if $\omega=\mathcal{Z}(A)$ is the full paving of A-zero sets and A is such that $M(\mathcal{Z}(A))$ represents A^* then $M_\sigma(\omega)$ necessarily corresponds to A_σ^*. As is already pointed out, $M(\mathcal{Z}(A))$ represents A^* iff A separates $\mathcal{Z}(A)$. Furthermore if $M(\mathcal{Z}(A))$ represents A^*, in general A need not be equal to $C(X)$; $A=C(X)$ if and only if $\mathcal{Z}(A)=\mathcal{Z}(X)$. (See Example 3.5 and Theorem 3.12 of [3]). Also if $M(\mathcal{Z}(A))$ represents A^* then $\mathcal{Z}(A)$ is a normal base in X and X_A is the Wallman compactification of X relative to the normal base $\mathcal{Z}(A)$. Finally in Theorem 2.11 we prove that if ω is a base of τ_A-closed sets, then $M_\tau(\omega)$ necessarily corresponds to A_τ.

The following example is due to Kirk and Crenshaw [4].

EXAMPLE 2.8. Topologize the closed unit disk S in the plane as follows: For a non-zero $z_0 \in S$ and $\varepsilon > 0$, let $N(z_0, \varepsilon)$ denote the set of all $z \in S$ which lie on the radius of the disk through z_0 and satisfy $|z_0| - \varepsilon < |z| < |z_0| + \varepsilon$. The family of sets $N(z_0, \varepsilon)$, $0 < \varepsilon < |z_0|$ forms a base for the neighbourhood system at z_0. A basic neighbourhood $N(0; z_1, z_2, \ldots, z_n)$ of $0 \in S$ is the whole disk deleting a finite number of closed segments of the radii from z_k to the boundary of S for $k = 1, 2, \ldots, n$. Since every basic open cover of S contains a neighbourhood of 0 which itself covers S except a finite number of compact line segments, S, with this topology, is compact Hausdorff. We take $X = S - \{0\}$.

Let A be the algebra of all restrictions f to X of the elements $\bar{f}$ of $C(S)$. Then $X_A = S$ and τ_A is the relative topology on X induced by the topology of S. Let ω be the full paving on X generated by the A-zero sets together with the closed sets of the form $B_\varepsilon = \{z \in X \mid |z| \le \varepsilon\}$, $0 < \varepsilon \le 1$. For an $f \in A$, if $\bar{f}(0) \ne 0$, then there is a neighbourhood of 0 in S in which f does not vanish. So the zero set of f is compact in X. If $\bar{f}(0) = 0$, then for every positive integer n, there is a neighbourhood of 0 in which $|f(x)| < 1/n$, so that f vanishes in the intersection of these neighbourhoods. Thus, in this case the zero set of f is the whole disk except possibly a countable number of radius-segments deleted. Then a routine verification shows that A separates ω. Also $\omega \supset \mathscr{Z}(A)$ forms a base for closed sets of X. Hence $M(\omega)$ represents A^* (Theorem 5.11 of [3]).

Now define $\psi \in A^*$ by $\psi(f) = \bar{f}(0)$ for every $f \in A$. If $\{f_n\}$ is a sequence in A with $f_n \downarrow 0$ pointwise on X, then $\bar{f}_n(0) \downarrow 0$. For otherwise, by continuity of f_n there is an $x \in X$ such that, $f_n(x) > \alpha > 0$ for infinitely many n which is not possible. Thus ψ is σ-smooth. If $m \in M(\omega)$ is the representing measure of ψ, then $m(B_\varepsilon) = \inf\{\psi(f) \mid \chi_{B_\varepsilon} \le f\} = 1$ for all $\varepsilon > 0$. Thus $B_{1/n} \downarrow \phi$ in ω whereas $m(B_{1/n}) \not\to 0$. Hence m is not σ-smooth.

PROPOSITION 2.9. *Let $M(\omega)$ represent A^* and let $m \in M(\omega)$. If m is σ-smooth then the corresponding linear functional $\psi \in A^*$ is σ-smooth.*

Proof. We may assume $m \ge 0$. Then m is a countably additive measure. The result follows from an application of Lebesgue monotone convergence theorem.

THEOREM 2.10. *Let $M(\mathscr{Z}(A))$ represent A^* and $m \in M(\mathscr{Z}(A))$. Then m is σ-smooth if and only if the corresponding bounded linear functional $\psi \in A^*$ is σ-smooth.*

Proof. If m is σ-smooth then ψ is σ-smooth by Proposition 2.9. Conversely suppose ψ is σ-smooth. Let $\{Z_n\}$ be a sequence in $\mathscr{Z}(A)$ with $Z_n \downarrow \phi$. For each k choose $f_k^* \in A$ with $0 \le f_k^* \le 1$ and $Z_k = Z(f_k^*) = f_k^{*-1}(0)$. For $n = 1, 2, \ldots,$ let $f_n = \max\{f_1^*, \ldots, f_n^*\}$. Then $\{f_n\}$ is an increasing sequence in A.

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Furthermore since $Z_n \downarrow \phi$ we have $Z_n = f_n^{-1}(0)$ for each n. Let

$$U_n = \left\{x \in X_A \mid \bar{f}_n(x) < \frac{1}{n}\right\}$$

and

$$V_n = \left\{x \in X \mid f_n(x) \le \frac{1}{n+1}\right\}.$$

where $\bar{f}_n$ is the unique continuous extension of f_n to X_A. Clearly $\bar{V}_n \subset U_n \subset \bar{V}_{n-1}$ for each n. We claim that $V_n \downarrow \phi$. Let $x \in X$. Since $Z_n \downarrow \phi$, there is some n_0 such that $x \notin Z_n$ for all $n \ge n_0$. Therefore $f_n(x) > 0$ for all $n \ge n_0$. In particular $f_{n_0}(x) > 0$. Choose $n_1 > n_0$ such that $f_{n_0}(x)n_1 > 1$. Then $f_{n_1}(x) > 1/n_1$ and hence $x \notin V_{n_1}$. This proves that $V_n \downarrow \phi$.

Since $\bar{V}_n$ and $X_A - U_n$ are disjoint closed sets in the compact Hausdorff space X_A, there exists by Urysohn's lemma, $\bar{g}_n \in C(X_A)$ such that $\bar{g}_n(\bar{V}_n) = 1$ and $\bar{g}_n(X_A - U_n) = 0$ for each n. Let $\bar{h}_n = \min(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n)$. Then $\bar{h}_n \in C(X_A)$ with $\bar{h}_n(\bar{V}_n) = 1$ and $\bar{h}_n(X_A - U_n) = 0$. Let h_n be the restriction of $\bar{h}_n$ to X. Then $\{h_n\}$ is a decreasing sequence in A with $h_n(V_n) = 1$ and $h_n(X - V_{n-1}) = 0$. Furthermore $h_n \downarrow 0$. In fact, if $x \in X$, since $V_n \downarrow \phi$, there exists n_0 such that $x \notin V_n$ for all $n \ge n_0$. Then

$$h_n(x) = 0 \quad \text{for all} \quad n \ge n_0.$$

Now since $M(\mathscr{Z}(A))$ represents A^*, each h_n is m-integrable and

$$\begin{aligned}\psi(h_n) &= \int h_n\, dm \\ &= \int_{V_n} h_n\, dm + \int_{V_{n-1}-V_n} h_n\, dm + \int_{X-V_{n-1}} h_n\, dm \\ &\ge m(V_n) \ge m(Z_n).\end{aligned}$$

Then $m(Z_n) \le \psi(h_n) \to 0$.

This proves that m is σ-smooth and completes the proof of the theorem.

THEOREM 2.11. *Let ω be a full paving of τ_A-closed sets forming a base for τ_A-closed sets in X. Let $M(\omega)$ represent A^*. Then $m \in M(\omega)$ is τ-smooth if and only if the corresponding $\psi \in A^*$ is τ-smooth.*

Proof. Without loss of generality we may assume that m and ψ are non-negative. Suppose ψ is τ-smooth. Let $\{W_\alpha\} \subset \omega$ with $W_\alpha \downarrow \phi$. Let $D = \{f \in A \mid 0 \le f \le 1\ f = 1 \text{ on some } W_\alpha\} = \{f_\lambda \mid \lambda \in \Lambda\}$. where Λ is so chosen that $\lambda \le \mu$ if and only if $f_\lambda \ge f_\mu$. Since $W_\alpha \downarrow \phi$, it is not hard to check that $\{f_\lambda \mid \lambda \in \Lambda\}$ is a decreasing net. Furthermore $f_\lambda \downarrow 0$. Let $x \in X$. There exists α for which $x \notin W_\alpha$. Since W_α is τ_A-closed, $x \notin \bar{W}_\alpha$. Then by Urysohn's lemma we can choose $\bar{f} \in C(X_A)$ such that $0 \le \bar{f} \le 1$, $\bar{f}(x) = 0$ and $\bar{f} = 1$ on $\bar{W}_\alpha$. If f is the restriction of $\bar{f}$ to X then $f(x) = 0$ and $f \in D$. Let $f = f_{\lambda_0}$. Then for any $\lambda \ge \lambda_0$,

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$f_\lambda(x) \leq f_{\lambda_0}(x) = 0$. Thus $f_\lambda \downarrow 0$. Given $\varepsilon > 0$ choose $\lambda_1 \in \Lambda$ and α_0 such that $\psi(f_{\lambda_1}) < \varepsilon$ and $f_{\lambda_1} = 1$ on W_{α_0}. Then for all $\alpha \geq a_0$ $m(W_\alpha) \leq (m(W_{\alpha_0}) \leq \psi(f_{\lambda_1}) < \varepsilon$. This proves that $\lim m(W_\alpha) = 0$ so that m is τ-smooth.

Conversely suppose m is τ-smooth. Without loss of generality we assume $m(X) = 1$. Let $\{f_\alpha\} \subset A$ with $f_\alpha \downarrow 0$. Assume $\|f_\alpha\| \leq 1$ for each α. Let $\varepsilon > 0$ be arbitrary. For each α let $Z_\alpha = \{x \in X \mid f_\alpha(x) \geq \varepsilon/2\}$. Then Z_α is τ_A-closed with $Z_\alpha \downarrow \phi$. Since ω is a base for τ_A-closed sets for each Z_α there is some $W \in \omega$ with $Z_\alpha \subset W$. Let $D_0 = \{W \in \omega \mid \text{there is some } Z_\alpha \text{ with } Z_\alpha \subset W\} = \{W_\lambda \mid \lambda \in \Lambda_0\}$ where Λ_0 is so chosen that $\lambda \leq \mu$ if and only if $W_\mu \subset W_\lambda$. Since $Z_\alpha \downarrow \phi$ it follows that $\{W_\lambda\}$ is a decreasing net with $W_\lambda \downarrow \phi$. Hence $m(W_\lambda) \to 0$. Choose λ_0 such that $m(W_\lambda) < \varepsilon/2$ for all $\lambda \geq \lambda_0$. Choose Z_{α_0} such that $Z_{\alpha_0} \subset W_{\lambda_0}$. Then for all $\alpha \geq \alpha_0$

$$\begin{aligned}
\psi(f_\alpha) &= \int_X f_\alpha \, dm \\
&= \int_{X - Z_\alpha} f_\alpha \, dm + \int_{Z_\alpha} f_\alpha \, dm \\
&< \frac{\varepsilon}{2} + m(Z_{\alpha_0}) \\
&\leq \frac{\varepsilon}{2} + m(W_{\lambda_0}) < \varepsilon.
\end{aligned}$$

Therefore $\psi(f_\alpha) \to 0$. This completes the proof.

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