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ON TWO LEMMAS OF BROWN AND SHEPP HAVING APPLICATION TO SUM SETS AND FRACTALS, II

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Abstract

Simple proofs are given of improved results of Brown and Shepp which are useful in calculations with fractal sets. A new inequality for convex functions is also obtained.

1. Introduction

Recently there has been a resurgence of interest in sum sets, which have, *inter alia*, application to fractals, iterated function systems and dynamical systems (see the authors [2] for some select references in the area). The calculation of associated Hausdorff dimensions and Hausdorff measures and other properties can be delicate. In [1], Brown and Shepp provided two key lemmas which have proved valuable in making available a number of simple calculations in this area. Improvements of the results of Brown and Shepp were obtained in [2]. Further generalizations of these results are given in [3].

In particular, let E_i be a non-empty set and L_i a class of nonnegative functions $f_i : E_i \to \mathbf{R}$ $(i = 1, 2)$. We consider functionals $A_i : L_i \to \mathbf{R}$ which satisfy the following conditions for $i = 1, 2$.

(a) $f_i \in L_i \Longrightarrow A_i(f_i) \geq 0$.
(b) $f_i \in L_i,\ \lambda_i > 0 \Longrightarrow \lambda_i f_i \in L_i$ and $A_i(\lambda_i f_i) = \lambda_i A_i(f_i)$.
(c) $1 \in L_i$, that is, if $f_i(t) = 1\ \forall t \in E_i$ then $f_i \in L_i$.
(d) $f_i, g_i \in L_i$ with $f_i(t_i) \geq g_i(t_i)$ $(\forall t_i \in E_i)$ $\Longrightarrow$ $A_i(f_i) \geq A_i(g_i)$.
(e) $A_i(f_i + g_i) \leq A_i(f_i) + A_i(g_i)$ $(f_i, g_i \in L_i \Longrightarrow f_i + g_i \in L_i)$.

Then we have the following

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THEOREM A. *Let* $f_i : L_i \to (0, \infty)$ $(i = 1, 2)$ *be real functions and let the functionals* A_i $(i = 1, 2)$ *satisfy the five conditions above. Further, let* s_i, t_i $(i = 0, 1, 2)$ *be positive numbers such that* $as_i^{-1} + bt_i^{-1} = 1$ *for positive constants* a, b *and* $s_1 \le s_0 \le s_2$. *Then*

$$A_1(f_1^{s_0})^{1/s_0} A_2(f_2^{t_0})^{1/t_0} \le \max_{i=1,2} \left\{A_1(f_1^{s_i})^{1/s_i} A_2(f_2^{t_i})^{1/t_i}\right\}.$$

In proving this theorem we used Lemma 1 below from [4] and Theorem B.

LEMMA 1. *If* $f_i^r \in L_i$ $(i = 1, 2)$ *for all* $r \in (0, \infty)$, *then the functions*

$$G_i(r) = A_i(f_i^r) \quad (i = 1, 2)$$

are logarithmically convex on $(0, \infty)$, *that is, the functions* $\log G_i(r)$ *are convex.*

THEOREM B. *Suppose that positive numbers* s_i, t_i *satisfy* $as_i^{-1} + bt_i^{-1} = 1$ $(i = 0, 1, 2)$ *for positive constants* a_i, b_i *and* $s_1 \le s_0 \le s_2$. *If* $f, g : (0, \infty) \to \mathbf{R}$ *are convex functions, then*

$$\frac{f(s_0)}{s_0} + \frac{g(t_0)}{t_0} \le \max_{i=1,2} \left\{\frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i}\right\}. \tag{1}$$

The following generalization of Theorem B is also given in [3].

THEOREM C. *Suppose that positive numbers* $s_{i,j}$ $(i = 0, 1, 2;\ j = 1, \dots, n)$ *satisfy* $s_{1,j} \le s_{0,j} \le s_{2,j}$ $(j = 1, \dots, n)$ *and* $a_j s_{i,1}^{-1} + b_j s_{i,j}^{-1} = 1$ $(i = 0, 1, 2;\ j = 2, \dots, n)$ *for positive constants* a_j, b_j $(j = 2, \dots, n)$. *If* $f_j : (0, \infty) \to \mathbf{R}$, $(j = 1, \dots, n)$ *are convex functions, then*

$$\sum_{j=1}^{n} f_j(s_{0,j})/s_{0,j} \le \max_{i=1,2} \left\{\sum_{j=1}^{n} f_j(s_{i,j})/s_{i,j}\right\}.$$

Here we shall give simpler proofs of Theorems B and C.

2. Results

Our proofs stem from the following lemma, which is of some interest in its own right. For example, it implies from Lemma 1 that the functions

$$H_i(r) = A_i(f_i^{1/r})^r \quad (i = 1, 2)$$

are logarithmically convex or that the means

$$M_i(r) = A_i(f_i^r)^{1/r} \quad (i = 1, 2)$$

are logarithmically convex functions of $1/r$.

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LEMMA 2. *Suppose* $f : (0, \infty) \to \mathbf{R}$. *Then* f *is a convex function if and only if the function* F *given by*

$$F(x) = xf(1/x)$$

is convex.

PROOF. First suppose that f is convex. Then if $x < y < z$, we have

$$(z - x)f(y) \leq (y - x)f(z) + (z - y)f(x). \tag{2}$$

For $b > a > 0$, set $z = 1/a$, $x = 1/b$, $y = 1/[\lambda a + (1 - \lambda)b]$, where $\lambda \in (0, 1)$. Then (2) becomes

$$\left(\frac{1}{a} - \frac{1}{b}\right) f\left(\frac{1}{\lambda a + (1-\lambda)b}\right) \leq \left(\frac{1}{\lambda a + (1-\lambda)b} - \frac{1}{b}\right) f\left(\frac{1}{a}\right) + \left(\frac{1}{a} - \frac{1}{\lambda a + (1-\lambda)b}\right) f\left(\frac{1}{b}\right),$$

that is,

$$[\lambda a + (1-\lambda)b] f\left(\frac{1}{\lambda a + (1-\lambda)b}\right) \leq \lambda a f\left(\frac{1}{a}\right) + (1-\lambda) b f\left(\frac{1}{b}\right),$$

or

$$F(\lambda a + (1-\lambda)b) \leq \lambda F(a) + (1-\lambda)F(b).$$

Therefore F also is convex.

Because $f(x) = xF(1/x)$, the converse follows from the result just shown.

PROOF OF THEOREM B. Let F and G be two convex functions on $(0, \infty)$. Then $F(x) + G(y)$, with $ax + by = 1$ $(a, b > 0)$ is also a convex function of x. Hence if $u_2 \leq u_0 \leq u_1$ and $au_i + bv_i = 1$ for $i = 0, 1, 2$, then

$$F(u_0) + G(v_0) \leq \max_{i=1,2}\{F(u_i) + G(v_i)\}. \tag{3}$$

For the functions f and g of Theorem B we have, by Lemma 2, that the functions given by $F(x) = xf(1/x)$, $G(x) = xg(1/x)$ are convex. Thus (3) becomes

$$u_0 f(1/u_0) + v_0 g(1/v_0) \leq \max_{i=1,2}\{u_i f(1/u_i) + v_i G(1/v_i)\},$$

that is, (1) holds for $u_i = 1/s_i$, $v_i = 1/t_i$ $(i = 0, 1, 2)$.

We now prove a generalization of Theorem C.

THEOREM 1. *Suppose that positive numbers* $u_{i,j}$ $(i = 0, 1, 2;\ j = 1, \dots, n)$ *satisfy*

$$u_{1,j} \geq u_{0,j} \geq u_{2,j}\ (1 \leq j \leq n) \quad \text{and} \quad a_j u_{i,1} + b_j u_{i,j} = 1\ (i = 0, 1, 2;\ 2 \leq j \leq n) \tag{4}$$

for positive constants a_j, b_j $(2 \leq j \leq n)$. *If* $F_j : (0, \infty) \to \mathbf{R}$ $(1 \leq j \leq n)$ *are convex functions, then*

$$\sum_{j=1}^{n} F_j(u_{0,j}) \leq \max_{i=1,2} \left\{ \sum_{j=1}^{n} F_j(u_{i,j}) \right\}.$$

PROOF. From (4) we have for $u_{1,j} > u_{0,j} > u_{2,j}$ that

$$a_j(u_{i,1} - u_{k,1}) + b_j(u_{i,j} - u_{k,j}) = 0$$

for each of the pairs $(i, k) = (1, 0), (2, 1), (2, 0)$. That is, for $\lambda \in (0, 1)$,

$$\begin{aligned} \frac{u_{0,j} - u_{2,j}}{u_{1,j} - u_{2,j}} &= \frac{u_{2,1} - u_{0,1}}{u_{2,1} - u_{1,1}} \quad (:= \lambda), \\ \frac{u_{1,j} - u_{0,j}}{u_{1,j} - u_{2,j}} &= \frac{u_{0,1} - u_{1,1}}{u_{2,1} - u_{1,1}} \quad (:= 1 - \lambda). \end{aligned} \tag{4}$$

On the other hand, the functions F_j are convex, so

$$F(u_{0,j}) \leq \frac{u_{0,j} - u_{2,j}}{u_{1,j} - u_{2,j}} F_j(u_{1,j}) + \frac{u_{1,j} - u_{0,j}}{u_{1,j} - u_{2,j}} F_j(u_{2,j}),$$

that is, $F(u_{0,j}) \leq \lambda F_j(u_{1,j}) + (1 - \lambda) F_j(u_{2,j})$.

Summation gives

$$\begin{aligned} \sum_{j=1}^{n} F_j(u_{0,j}) &\leq \lambda \sum_{j=1}^{n} F_j(u_{1,j}) + (1 - \lambda) \sum_{j=1}^{n} F_j(u_{2,j}) \\ &\leq \max_{i=1,2} \left\{ \sum_{j=1}^{n} F_j(u_{i,j}) \right\}. \end{aligned}$$

Theorem C follows in the particular case $F_j(x) = x f_j(1/x)$ and $u_{i,j} = 1/s_{i,j}$ $(i = 0, 1, 2;\ 1 \leq j \leq n)$.

REMARK 1. Lemma 2 can be generalized as follows.

For an integer $n \geq 1$, the *reciprocal transformation of order n* of a function f whose domain is an interval of positive numbers is the function ϕ_n given by

$$\phi_n(t) = (-1)^n t^{n-1} f(1/t).$$

The reciprocal transformation of order n of ϕ_n is evidently f. We have the following. *The reciprocal transformation of order n preserves n–convexity*, that is, ϕ_n is n–convex if and only if f is n–convex.

Recall that a function f is n–convex if, for $n+1$ distinct points x_n, we have

$$\sum_{k=0}^{n} f(x_k) \Bigg/ \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j) \geq 0 \tag{5}$$

(see [5, pages 14–16]).

To establish the statement enunciated, suppose f is n–convex and set $x_k = 1/t_k$ in (5). Simple manipulations provide

$$\left[\prod_{j=0}^{n} t_j\right] \sum_{k=0}^{n} (-1)^n t_k^{n-1} f(1/t_k) \Bigg/ \prod_{\substack{j=0 \\ j\neq k}}^{n} (t_k - t_j) \geq 0,$$

that is,

$$\sum_{k=0}^{n} \phi_n(t_k) \Bigg/ \prod_{\substack{j=0 \\ j\neq k}}^{n} (t_k - t_j) \geq 0,$$

so ϕ_n is n–convex too.

Since n–convexity coincides with ordinary convexity for $n = 2$, this establishes an alternate proof for Lemma 2.

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