UNITARY REPRESENTATIONS OF GENERALIZED SYMMETRIC GROUPS

B. M. PUTTASWAMAIAH

In this paper all representations are over the complex field K. The generalized symmetric group $S(n, m)$ of order $n!m^n$ is isomorphic to the semi-direct product of the group of $n \times n$ diagonal matrices whose mth powers are the unit matrix by the group of all $n \times n$ permutation matrices over K. As a permutation group, $S(n, m)$ consists of all permutations of the mn symbols $\{1, 2, \ldots, mn\}$ which commute with

$$(1, n+1, \ldots, (m-1)n+1)(2, n+2, \ldots, (m-1)n+2) \ldots (n, 2n, \ldots, mn).$$

Obviously, $S(1, m)$ is a cyclic group of order m, while $S(n, 1)$ is the symmetric group of order $n!$. If $c_i = (i, n+i, \ldots, (m-1)n+i)$ and

$$s_j = (j, j+1)(n+j, n+j+1) \ldots ((m-1)n+j, (m-1)n+j+1),$$

then $\{c_1, c_2, \ldots, c_n\}$ generate a normal subgroup $Q(n)$ of order m^n and $\{s_1, s_2, \ldots, s_{n-1}\}$ generate a subgroup $S(n)$ isomorphic to $S(n, 1)$. The group rings $KS(n, 2)$ and $KS(n, m)$ were studied by Young (**7**) and Osima (**4**), respectively; however, they did not give the construction theory of their irreducible representations. The first part of this paper (which is based on the work done in (**5**)) fills this gap and also includes a result of Frame (**2**) as a special case of Theorem 2. In the second part of the paper we are concerned with the development of an operator theory for $S(n, m)$, similar to that of the symmetric group (**6**).

1. If G is any finite group, let KG be the group ring of G over the field K. If M is a finite-dimensional vector space, we denote by $\operatorname{aut}(M)$ the group of all automorphisms of M. If there is a homomorphism of G into $\operatorname{aut}(M)$, then M can be regarded as a (right) KG-module. Conversely, any KG-module M defines a homomorphism of G into $\operatorname{aut}(M)$. Let B be a normal subgroup of index m in G. Then $G = \bigcup_{i=1}^{m} Ba_i$. Let M and N be KB-modules which afford the representations U and V, respectively. For any $a \in G$, U_a defined by $U_a(x) = U(a^{-1}xa)$, $x \in B$, is also a representation of KB. Let $U \oplus V$ and $U \otimes V$ be the representations of KB afforded by the direct sum $M \oplus N$ and tensor product $M \otimes N$, respectively. Let U^m denote the representation of KB defined by $U^m(x) = \sum_{i=1}^{m} \oplus\, U(a_i^{-1}xa_i)$. If W is a representation of a subgroup of G, then W^G denotes the representation of G induced by W.

Received January 13, 1967 and in revised form, August 14, 1968.

https://doi.org/10.4153/CJM-1969-003-3 Published online by Cambridge University Press

THEOREM 1. *Let G be a semi-direct product of B by A and let D be a subgroup of A of index n.*

(i) *If λ and μ are homomorphisms of A and B into* $\operatorname{aut}(M)$, *respectively, then $\lambda\mu$ (defined by $\lambda\mu(ab) = \lambda(a)\mu(b)$) is also a homomorphism if and only if $\lambda(A) \cap \mu(B) = \lambda(1)$ and $\lambda(a^{-1})\mu(b)\lambda(a) = \mu(a^{-1}ba)$ for all a in A and b in B.*

(ii) *If α and β are homomorphisms of D and B into* $\operatorname{aut}(N)$ *such that $\alpha\beta$ is also a homomorphism, then $(\alpha\beta)^G = \alpha^A\beta^n$.*

Proof. (i) Let $\lambda(A) \cap \mu(B) = \lambda(1)$ and $\lambda(a^{-1})\mu(b)\lambda(a) = \mu(a^{-1}ba)$ for all a in A and b in B. If $ab, cd \in AB$, then

$$\lambda\mu(abcd) = \lambda\mu(acc^{-1}bcd) = \lambda(ac)\mu(c^{-1}bcd) = \lambda(a)\lambda(c)\mu(c^{-1}bc)\mu(d) =$$
$$\lambda(a)\lambda(c)\lambda(c^{-1})\mu(b)\lambda(c)\mu(d) = \lambda(a)\mu(b)\lambda(c)\mu(d) = \lambda\mu(ab)\lambda\mu(cd)$$

so that $\lambda\mu$ is a homomorphism.

Conversely, let $\lambda\mu$ be a homomorphism. Then $\lambda(A) \cap \mu(B) = \lambda(1)$ follows from the fact that $A \cap B = 1$. Also, each element $a \in A$ defines an automorphism $\mu(b) \to \lambda(a^{-1})\mu(b)\lambda(a)$ of $\mu(B)$ if and only if $\lambda(a^{-1})\mu(b)\lambda(a) = \mu(a^{-1}ba)$ for all $b \in B$.

(ii) Let $A = \bigcup_{i=1}^{n} Da_i$. Since B is normal in G, we have that

$$G = BA = B \bigcup_{i=1}^{n} Da_i = \bigcup_{i=1}^{n} (BD)a_i.$$

Assume that α and β are matrix representations. If we let $\alpha(x) = 0$ for $x \in A - D$, we obtain

$$(\beta\alpha)^G(ba) = (\beta\alpha(a_i^{-1}baa_j)) = (\beta\alpha(a_i^{-1}ba_ia_i^{-1}aa_j)) =$$
$$(\beta(a_i^{-1}ba_i)\alpha(a_i^{-1}aa_j)) = (\beta(a_i^{-1}ba_j)\delta_{ij})(\alpha(a_i^{-1}aa_j)) = \beta^n(b)\alpha^A(a)$$

for all $b \in B$ and $a \in A$. Clearly, β^n is a representation of B. Furthermore,

$$(\beta\alpha)^G(a^{-1}ba) = (\beta\alpha)^G(a^{-1}ba \cdot 1) = \beta^n(a^{-1}ba)\alpha^A(1) =$$
$$\beta^n(a^{-1}ba) = (\beta\alpha)^G(1a^{-1}ba) = (\beta\alpha)^G(1a^{-1})(\beta\alpha)^G(ba) =$$
$$\beta^n(1)\alpha^A(a^{-1})\beta^n(b)\alpha^A(a) = \alpha^A(a^{-1})\beta^n(b)\alpha^A(a)$$

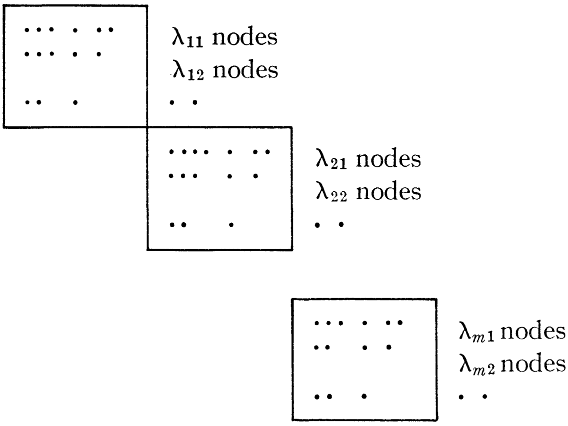
for all $a \in A$ and $b \in B$. Therefore, $\beta^n\alpha^A$ is a representation of G and $(\beta\alpha)^G = \beta^n\alpha^A$, which proves the result.

The theory of characters and the associated properties of $S(n, m)$ were studied by Osima (**4**), who showed that there is a one-to-one correspondence between the distinct irreducible representations of $KS(n, m)$ and the distinct m-quotient diagrams (for definition see (**6**)) of n nodes. In the following, the irreducible representation of $KS(n, m)$ corresponding to the m-quotient diagram $[\lambda]_m{}^*$ of n nodes will be denoted by $\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle = \langle[\lambda]\rangle$ where $[\lambda_i]$ means $[\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \ldots]$ with $\lambda_{i1} \geqq \lambda_{i2} \geqq \lambda_{i3} \geqq \ldots \geqq 0$ and $\lambda_{i1} + \lambda_{i2} + \lambda_{i3} + \ldots = n_i$, subject to the condition: $n_1 + n_2 + \ldots + n_m = n$. If $n_i = 0$, then it is still necessary to write $[\lambda_i] = [\emptyset]$ in $\langle[\lambda]\rangle$.

Generalized Young diagram and tableau. Let $n = n_1 + n_2 + \ldots + n_m$, where $0 \leqq n_i \leqq n$. Corresponding to each partition (λ_i) of

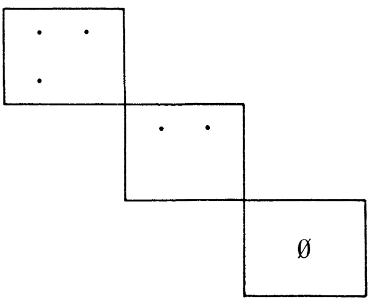
$$n_i = \lambda_{i1} + \lambda_{i2} + \lambda_{i3} + \ldots,$$

we define a unique generalized Young diagram denoted by the same symbol $\langle[\lambda]\rangle$ in the form:

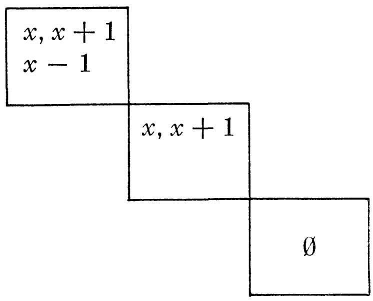


In the diagram $\langle[\lambda]\rangle$, the part corresponding to $[\lambda_i]$ will be called the ith constituent of the diagram. The graph $\langle[\lambda]\rangle_x$ of $\langle[\lambda]\rangle$ is obtained by replacing the (u, v)th node by the quantity $x - u + v$, where x is an arbitrary parameter.

Example 1. Let $n = 3 + 2 + 0$ and $[\lambda_1] = [21]$, $[\lambda_2] = [2]$, $[\lambda_3] = [\emptyset]$. Then the diagram $\langle[21]; [2]; [\emptyset]\rangle$ and its graph are



and



If (u, v) and (u', v') are positions of two nodes in $\langle[\lambda]\rangle$ with $u < u'$ and $v' < v$, then the axial distance $1/\rho$ between these two nodes is defined to be (i) $(u' - u) - (v' - v)$ if both nodes are in a constituent or (ii) arbitrarily large if the two nodes are in different constituents. Thus,

$$\rho = ((u' - u) - (v' - v))^{-1}$$

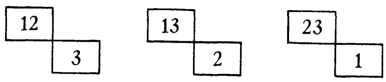
if both the nodes are in the same constituent, otherwise $\rho = 0$.

If we now arrange the n symbols $1, 2, \ldots, n$ in all possible ways in places of the nodes of the diagram $\langle[\lambda]\rangle$, we obtain a generalized Young tableau. Of these generalized tableaux, there are a certain number $f = f\langle[\lambda]\rangle$ of tableaux, in which the symbols in each row and column appear in their natural order, which may be called the generalized standard Young tableaux. If f_i is the number of standard tableaux of n_i nodes of the ith constituent, then f is given (**3**) by

$$f = \frac{n!}{n_1!\, n_2! \ldots n_m!} f_1 f_2 \ldots f_m,$$

where we set $f_i = 1$ if $n_i = 0$. Then f is the K-dimension of the minimal right ideal of $KS(n, m)$ which affords the irreducible representation $\langle[\lambda]\rangle$.

Example 2. The generalized standard Young tableaux associated with the irreducible representation $\langle[2]; [1]\rangle$ of $KS(3, 2)$ are



Furthermore, the K-dimension f of the minimal right ideal of $KS(3, 2)$ which affords the representation $\langle[2]; [1]\rangle$ of $KS(3, 2)$ is 3.

The generalized standard Young tableaux associated with $\langle[\lambda]\rangle$ will be denoted by $t_1, t_2, \ldots, t_f$. A tableau t_v (with $u < v$) will be called the j-conjugate of t_u if t_v is obtained from t_u by interchanging j and $j + 1$. In the following theorem, t_v is the j-conjugate of t_u.

THEOREM 2. *Let $\{e_1, e_2, \ldots, e_f\}$ be a K-basis of a vector space M. Then M is an irreducible $KS(n, m)$-module with respect to the action of $S(n, m)$ defined as follows:*

(i) *$e_u s_j = e_u$ or $-e_u$ if j and $j + 1$ lie in the same row or column of t_u, respectively;*

(ii) *$e_u s_j = -\rho e_u + (1 - \rho^2)^{\frac{1}{2}} e_v$ and $e_v s_j = (1 - \rho^2)^{\frac{1}{2}} e_u + \rho e_v$ if j and $j + 1$ are not in the same row or column;*

(iii) *$e_u c_j = \omega^{r-1} e_u$, where r is the cardinality of the constituent of t_u in which j appears and ω is a primitive mth root of unity.*

Proof. By (i) and (ii), it follows that M is a $KS(n)$-module and by (iii) it follows that M is a $KQ(n)$-module. Let U and V be the corresponding representations of $KS(n)$ and $KQ(n)$, respectively. If a constituent $[\lambda_i]$ of $\langle[\lambda]\rangle$ has n nodes, then U is an irreducible representation of $KS(n)$ (see **6**) and each $V(c_j)$ is a scalar multiple of the identity transformation. Therefore, we have that $U(a^{-1}) V(c_j) U(a) = V(a^{-1} c_j a)$ for all $a \in S(n)$. It follows from Theorem 1 that M is an irreducible $KS(n, m)$-module.

Next, assume that not all the n nodes belong to a single constituent. Since the argument is similar in the general case, it is not a restriction to assume that the n nodes are distributed between two constituents $[\lambda_r]$ and $[\lambda_s]$ with $r \neq s$.

Let p nodes belong to $[\lambda_r]$ and q nodes belong to $[\lambda_s]$. Let $S(p)$ and $S(q)$ be the corresponding symmetric subgroups of $S(n, m)$.

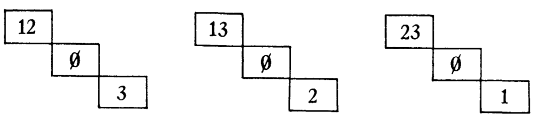
The tableaux can be divided into $l = n!/p!q!$ packs $T_1, T_2, \ldots, T_l$, each of $f_r f_s$ tableaux, in the following manner. First, let $t_1, t_2, t_3, \ldots$ be the standard tableaux obtained by the arrangement of the symbols $1, 2, \ldots, p$ while $p + 1, p + 2, \ldots, n$ of $[\lambda_s]$ are invariant. Then the pack T_1 is obtained by arranging $p + 1, p + 2, \ldots, n$ of $[\lambda_s]$ in $t_1, t_2, t_3, \ldots$, while $1, 2, \ldots, p$ are invariant. Then, clearly, T_1 has $f_r f_s$ tableaux. The pack T_2 is obtained by arranging $1, 2, \ldots, p - 1, p + 1$ in place of $1, 2, \ldots, p$ and $p, p + 2, \ldots, n$ in place of $p + 1, p + 2, \ldots, n$. Similarly, the other packs are obtained. By restricting the construction to T_1, we obtain a representation U_1 of $KS(p) \times S(q)$ and a representation V_1 of $KQ(n)$. Again, each element $V_1(b)$ is a multiple of identity transformation, so that $U_1(a^{-1}) V_1(b) U_1(a) = V_1(a^{-1}ba)$ for all $a \in S(p) \times S(q)$ and $b \in Q(n)$. By Theorem 1, it follows that $U_1 V_1$ is an irreducible representation of $KS(p, m) \times S(q, m)$. Therefore, we have that

$$(U_1 V_1)^{S(n,m)} = U_1^{S(n)} V_1^{l}.$$

Then, clearly, $U = U_1^{S(n)}$ and $V = V_1^{l}$. Thus, M is a $KS(n, m)$-module. Since $S(p, m) \times S(q, m)$ is the stability subgroup of the representation V_1 of the normal subgroup $Q(n)$, Clifford's result (**1**) implies that M is an irreducible $KS(n, m)$-module.

COROLLARY. *The representation of $S(n, m)$ afforded by the module M with respect to $\{e_1, e_2, \ldots, e_f\}$ is an irreducible unitary representation.*

Example 3. The representation $\langle[2]; [\emptyset]; [1]\rangle$ of $S(3, 3)$ is of degree 3. The standard tableaux associated with $\langle[2]; [\emptyset]; [1]\rangle$ are



The irreducible unitary matrix representation V of $S(3, 3)$ based on this arrangement is given by

$$V(s_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V(s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V(c_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix},$$

where $s_1 = (12)(45)(78)$, $s_2 = (23)(56)(89)$, $c_1 = (147)$, and w is a primitive cube root of unity.

2. Young's raising operators. In the ordinary representation theory of $S(n, 1)$, an interesting role is played by Young's raising operators; see (**6**). If $(\lambda) = (\lambda_1 \lambda_2 \ldots \lambda_r)$; $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_r = n$ is any partition of n, let $S(\lambda_1, 1)$ be the symmetric subgroup on $1, 2, \ldots, \lambda_1$,

https://doi.org/10.4153/CJM-1969-003-3 Published online by Cambridge University Press

$S(\lambda_2, 1)$ the symmetric subgroup on $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_2$, and so on. Then the representation of $S(n, 1)$ induced by the identity representation of Young's subgroup $S(\lambda_1, 1) \times S(\lambda_2, 1) \times \ldots \times S(\lambda_r, 1)$ is denoted by $[\lambda_1] \cdot [\lambda_2] \cdot \ldots \cdot [\lambda_r]$, which is in general reducible. Then Young's reduction formula (**6**) states that

$$[\lambda_1] \cdot [\lambda_2] \cdot \ldots \cdot [\lambda_r] = \sum \prod R_{s,t}[\lambda_1\lambda_2 \ldots \lambda_r],$$

where $R_{s,t}$ is a Young's operator which, for $s < t$, represents the raising of a node from the tth row to the sth row of $[\lambda_1\lambda_2 \ldots \lambda_r]$ to obtain a new diagram and $\prod R_{s,t}$ represents successive raisings of nodes in $[\lambda] = [\lambda_1\lambda_2 \ldots \lambda_r]$, where $s = 1, 2, \ldots, r - 1$ and $t = 2, 3, \ldots, r$. In applying Young's raising operator, the resulting diagram is to be disregarded (i) if any row contains more symbols than a previous row, or (ii) if two symbols from the same row appear in the same column. In the following we give a formula for the reduction of the representations of $S(n, m)$ induced by linear representations of certain subgroups. This will generalize Young's reduction formula given above.

Let $n = n_1 + n_2 + \ldots + n_m$, $0 \leqq n_i \leqq n$, and $(\lambda_i) = (\lambda_{i1}\lambda_{i2} \ldots)$, $\lambda_{i1} \geqq \lambda_{i2} \geqq \lambda_{i3} \geqq \ldots > 0$, be a partition of n_i for $i = 1, 2, \ldots, m$. Let $S(\lambda_{ij}, m)$ be the corresponding generalized symmetric subgroups. Then the subgroup $S(\lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots) = S(\lambda_{11}, m) \times S(\lambda_{12}, m) \times S(\lambda_{13}, m) \times \ldots$ may be called a (generalized) Young subgroup of $S(n, m)$. A skew diagram is formed of separated diagrams of one line each, containing λ_{ij} nodes for $j = 1, 2, \ldots$ and $i = 1, 2, \ldots, m$, such that no two nodes are in the same column. If we now replace the nodes by the symbols $1, 2, \ldots, n$, then the number of standard skew tableaux is seen to be

$$\partial(\lambda) = \frac{n!}{\lambda_{11}!\, \lambda_{12}!\, \lambda_{13}! \ldots}.$$

The $\partial(\lambda)$-dimensional $KS(n, m)$-module M constructed according to Theorem 2 is no longer irreducible. The representation of $KS(n, m)$ afforded by M will be denoted by the symbol $\langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle$.

THEOREM 3. *The representation* $\langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle$ *of* $S(n, m)$ *is induced by a linear representation of the Young subgroup* $S(\lambda_{11}, \lambda_{12}, \ldots)$. *The reduction of the representation into the direct sum of the irreducible representations is given by the formula*

$$\langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle = \prod_{i=1}^{m} \sum^{i} \prod^{i} R_{st}{}^{i}\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle,$$

where $R_{st}{}^{i}$ *operates only on the ith constituent* $[\lambda_i]$ *for* $i = 1, 2, \ldots, m$.

Proof. It is no restriction to assume that $n = 0 + \ldots + 0 + \lambda_r + 0 + \ldots + \lambda_s + \ldots + 0$. As in the proof of Theorem 2, divide the standard skew tableaux into $l = n!/p!q!$ packs $T_1, T_2, \ldots, T_l$ each of

$$\frac{p!\, q!}{\lambda_{r1}!\, \lambda_{r2}! \ldots \lambda_{s1}!\, \lambda_{s2}! \ldots}$$

tableaux. The restriction of the construction to pack T_1 gives the representation

$$U_1 = ([\lambda_{r1}] \cdot [\lambda_{r2}] \cdot \ldots) \otimes ([\lambda_{s1}] \cdot [\lambda_{s2}] \cdot \ldots)$$

of the subgroup $S(p) \times S(q)$ and the representation V_1 of $Q(n)$ in which each element is mapped on a scalar multiple of the identity transformation, so that $U_1(a^{-1})V_1(b)U_1(a) = V_1(a^{-1}ba)$ for all $a \in S(p) \times S(q)$. By Theorem 1, it follows that U_1V_1 is a representation of $S(p, m) \times S(q, m)$ such that

$$(U_1V_1)^{S(n,m)} = U_1{}^{S(n)}V_1{}^l = \langle[\lambda_r]\rangle \cdot \langle[\lambda_s]\rangle.$$

However, $[\lambda_{r1}] \cdot [\lambda_{r2}] \cdot \ldots \cdot [\lambda_{rk}] = I^{S(p)}$, where I is the identity representation of $S(\lambda_{r1}, 1) \times S(\lambda_{r2}, 1) \times \ldots \times S(\lambda_{rk}, 1)$. Also, V_1 is a direct sum of a linear representation of $Q(n)$. This proves the first assertion.

Next we have that

$$\begin{aligned}
U_1 &= ([\lambda_{r1}] \cdot [\lambda_{r2}] \cdot \ldots \cdot [\lambda_{rk}]) \otimes ([\lambda_{s1}] \cdot [\lambda_{s2}] \cdot \ldots \cdot [\lambda_{st}]) \\
&= \left(\sum^{r} \prod^{r} R_{uv}{}^{r}[\lambda_{r1}\lambda_{r2} \ldots \lambda_{rk}]\right) \otimes \left(\sum^{s} \prod^{s} R_{vu}{}^{s}[\lambda_{s1}\lambda_{s2} \ldots \lambda_{st}]\right) \\
&= \prod_{i=r,s} \sum^{i} R_{uv}{}^{i}[\lambda_r] \otimes [\lambda_s],
\end{aligned}$$

where $R_{uv}{}^{i}$ operates only on the ith constituent $[\lambda_i]$, for $i = r, s$. Since each element $V_1(b)$ is a scalar multiple of identity, we have that

$$U_1V_1 = \left(\prod_{i=r,s} \sum^{i} R_{uv}{}^{i}[\lambda_r] \otimes [\lambda_s]\right)V_1 = \prod_{i=r,s} \sum^{i} R_{uv}{}^{i}([\lambda_r] \otimes [\lambda_s]V_1'),$$

where $V_1 = \sum V_1'$, the summands depending on the operators. This in turn implies that

$$\begin{aligned}
\langle[\lambda_r]\rangle \cdot \langle[\lambda_s]\rangle &= (U_1V_1)^{S(n,m)} \\
&= \left(\prod_{i=r,s} \sum^{i} R_{uv}{}^{i}([\lambda_r] \otimes [\lambda_s]V_1')\right)^{S(n,m)} \\
&= \prod_{i=r,s} \sum^{i} R_{uv}{}^{i}([\lambda_r] \otimes [\lambda_s]V_1')^{S(n,m)} \\
&= \prod_{i=r,s} \sum^{i} R_{uv}{}^{i}\langle[\emptyset]; \ldots ; [\lambda_r]; \ldots ; [\lambda_s]; \ldots [\emptyset]\rangle.
\end{aligned}$$

Thus, we have proved the reduction formula

$$\langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle = \prod_{i=1}^{m} \sum^{i} R_{uv}{}^{i}\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle,$$

where $R_{uv}{}^{i}$ operates only on the ith constituent $[\lambda_i]$.

COROLLARY. *In the representation* $\langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle$ *of* $S(n, m)$, *each of the irreducible representations*

$$\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle \textit{ and } \langle[n_1]; [n_2]; \ldots ; [n_m]\rangle$$

appears exactly once.

Example 4. As an application of Theorem 3, let us determine the irreducible representations of $S(6, 3)$ in $\langle[21]\rangle \cdot \langle[1^2]\rangle \cdot \langle[1]\rangle$. In this case we have that

$$\Pi^1 R_{uv}{}^1 = R_{11}{}^1 R_{12}{}^1, \quad \Pi^2 R_{uv}{}^2 = R_{11}{}^2 R_{12}{}^2, \quad \text{and } \Pi^3 R_{uv}{}^3 = R_{11}{}^3$$

while the other factors do not appear. Therefore,

$$\begin{aligned}\langle[21]\rangle \cdot \langle[1^2]\rangle \cdot \langle[1]\rangle &= \prod_{i=1}^{3} \sum^{i} R_{ut}{}^i \langle[21]; [1^2]; [1]\rangle \\ &= \sum^{1} \langle R_{11}{}^1 R_{12}{}^1 [21]; [2]; [1]\rangle + \langle R_{11}{}^1 R_{12}{}^1 [21]; [1^2]; [1]\rangle \\ &= \langle[3]; [2]; [1]\rangle + \langle[21]; [2]; [1]\rangle + \langle[3]; [1^2]; [1]\rangle \\ &\qquad + \langle[21]; [1^2]; [1]\rangle.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\langle[21]\rangle \cdot \langle[1^2]\rangle \cdot \langle[1]\rangle = \langle[3]; [2]; [1]\rangle + \langle[21]; [2]; [1]\rangle + \langle[3]; [1^2]; [1]\rangle \\ + \langle[21]; [1^2]; [1]\rangle.\end{aligned}$$

The converse of Theorem 3 can be stated as follows.

THEOREM 4. *The character of the irreducible representation*

$$\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle$$

of $S(n, m)$ *is a linear combination with rational integral coefficients of the characters of* $S(n, m)$ *induced by linear characters of generalized Young subgroups and the coefficients are given by the formula*

$$\langle[\lambda_1]; [\lambda_2]; \ldots ; [\lambda_m]\rangle = \prod_{i=1}^{m} \prod^{i} (1 - R_{st}{}^i) \langle[\lambda_1]\rangle \cdot \langle[\lambda_2]\rangle \cdot \ldots \cdot \langle[\lambda_m]\rangle,$$

where $\Pi^i(1 - R_{st}{}^i)$ *operates only on* $[\lambda_{i1}] \cdot [\lambda_{i2}] \cdot [\lambda_{i3}] \cdot \ldots$ *for* $i = 1, 2, \ldots, m$.

Proof. With the same notation as in Theorem 3, we have that

$$([\lambda_{r1}] \cdot [\lambda_{r2}] \cdot \ldots) \otimes ([\lambda_{s1}] \cdot [\lambda_{s2}] \cdot \ldots) V_1 = \prod_{i=1}^{m} \sum^{i} R_{uv}{}^i ([\lambda_r] \otimes [\lambda_s]) V_1'$$

which can be written (see **6**) in the form

$$([\lambda_r] \otimes [\lambda_s]) V_1' = \prod_{i=1}^{m} \prod^{i} (1 - R_{uv}{}^i)([\lambda_{r1}] \cdot [\lambda_{r2}] \cdot \ldots) \otimes ([\lambda_{s1}] \cdot [\lambda_{s2}] \cdot \ldots) V_1.$$

This implies that

$$\begin{aligned}\langle[\emptyset]; \ldots ; [\lambda_r]; \ldots ; [\lambda_s]; \ldots\rangle &= ([\lambda_r] \otimes [\lambda_s] V_1')^{S(n,m)} = \\ &\prod_{i=1}^{m} \prod^{i} (1 - R_{uv}{}^i) \langle[\lambda_1]\rangle \ldots \langle[\lambda_r]\rangle \ldots \langle[\lambda_s]\rangle \ldots \langle[\lambda_m]\rangle,\end{aligned}$$

where no conditions on $R_{uv}{}^i$ are required.

Example 5. As an illustration, let us express the character $\phi(21)(1^2)(1)$ of the irreducible representation $\langle[21];[1^2];[1]\rangle$ of $S(6,3)$ in terms of the characters $\Phi(\lambda_1)(\lambda_2)(\lambda_3)$ of $\langle[\lambda_1]\rangle\cdot\langle[\lambda_2]\rangle\cdot\langle[\lambda_3]\rangle$ of $S(6,3)$. Here, the relevant operators are

$$\prod^1(1-R_{st}^1) = 1-R_{12}^1,$$
$$\prod^2(1-R_{st}^2) = 1-R_{12}^2,$$
$$\prod^3(1-R_{st}^3) = 1,$$

so that we obtain

$$\begin{aligned}\langle[21];[1^2];[1]\rangle &= (1-R_{12}^1)(1-R_{12}^2)\langle[21]\rangle\cdot\langle[1^2]\rangle\cdot\langle[1]\rangle\\ &= (1-R_{12}^1-R_{12}^2+R_{12}^1R_{12}^2)\langle[21]\rangle\cdot\langle[1^2]\rangle\cdot\langle[1]\rangle\\ &= \langle[21]\rangle\cdot\langle[1^2]\rangle\cdot\langle[1]\rangle-\langle[3]\rangle\cdot\langle[1^2]\rangle\cdot\langle[1]\rangle\\ &\qquad -\langle[21]\rangle\cdot\langle[2]\rangle\cdot\langle[1]\rangle+\langle[3]\rangle\cdot\langle[2]\rangle\cdot\langle[1]\rangle.\end{aligned}$$

Thus, we have that

$$\phi(21)(1^2)(1) = \Phi(21)(1^2)(1)-\Phi(3)(1^2)(1)-\Phi(21)(2)(1)+\Phi(3)(2)(1).$$

Let $A\times B$ be a subgroup of a group G. If U and V are representations of A and B, respectively, then $U\cdot V=(U\otimes V)^G$ is the outer tensor product representation of G. But, in practice, the process is somewhat reversed. For arbitrary groups A and B, one can always form the direct product $A\times B$ but the group G is not well-defined in general. If $A=S(n,1)$ and $B=S(n',1)$, then $G=S(n+n',1)$ is well-defined. The representation $[\lambda]\cdot[\mu]$ of $S(n+n',1)$ induced by $[\lambda]\otimes[\mu]$ of $S(n,1)\times S(n',1)$ is the subject of an extensive study by several authors; see (**6**). In the following, we show that the situation is similar for a generalized symmetric group and then we give a generalization of the Littlewood-Richardson rule (**6**) for the reduction of such a representation.

If $\langle[\lambda]\rangle$ and $\langle[\mu]\rangle$ are irreducible representations of $S(n,m)$ and $S(n',m)$, respectively, then $\langle[\lambda]\rangle\cdot\langle[\mu]\rangle$ is the outer tensor product representation of $S(n+n',m)$. In general, $\langle[\lambda]\rangle\cdot\langle[\mu]\rangle$ is reducible. The irreducible components of the representation $\langle[\lambda]\rangle\cdot\langle[\mu]\rangle$ can be determined by the application of the following generalization of the Littlewood-Richardson rule.

THEOREM 5. *Each diagram constructed according to* (a) *and* (b) *below defines an irreducible component of* $\langle[\lambda]\rangle\cdot\langle[\mu]\rangle$ *and all components are obtained in this manner.*

(a) *To each tableau of* $[\lambda_i]$, *add the symbols of the first row of a tableau of* $[\mu_i]$ *for* $i=1,2,\ldots,m$. *These may be added to one row or divided into any number of sets, preserving their order, the first set being added to one row of* $[\lambda_i]$, *the second set to a subsequent row, the third to a row subsequent to this, and so on. After the addition, no row of the compound tableau may contain more symbols than a preceding row, and no two added symbols may appear in the same column. Next,* *add the second row of* $[\mu_i]$, *according to the same rules, followed by the remaining rows in succession until all the symbols of* $[\mu_i]$ *have been used.*

(b) *These additions must be such that each symbol from* $[\mu_i]$ *shall appear in a later row of the compound tableau than that occupied by the symbol immediately above it in* $[\mu_i]$, *for* $i = 1, 2, \ldots, m$.

Proof. In order to prove this result, we employ the corresponding result for the symmetric group; see (**6**). In order to limit the size of the expressions, we carry out the proof for $m = 2$. Again the generalization is straightforward. Using the notation of the previous section, we have that

$$\begin{aligned}\langle[\lambda]\rangle \cdot \langle[\mu]\rangle &= (\langle[\lambda_1]; [\lambda_2]\rangle \otimes \langle[\mu_1]; [\mu_2]\rangle)^{S(n+n',m)}\\ &= [([\lambda_1] \otimes [\lambda_2]V_1)^{S(n,m)} \otimes ([\mu_1] \otimes [\mu_2]V_2)^{S(n',m)}]^{S(n+n',m)}\\ &= [(([\lambda_1] \otimes [\lambda_2]V_1) \otimes ([\mu_1] \otimes [\mu_2]V_2))^{S(n,m)\times S(n',m)}]^{S(n+n',m)}\\ &= [([\lambda_1] \otimes [\lambda_2]) \otimes ([\mu_1] \otimes [\mu_2])(V_1 \otimes V_2)]^{S(n+n',m)}\\ &= (([\lambda_1] \otimes [\mu_1]) \otimes ([\lambda_2] \otimes [\mu_2])(V_1 \otimes V_2))^{S(n+n',m)}\\ &= (([\lambda_1] \otimes [\mu_1])^{S(n_1+n_1')} \otimes ([\lambda_2] \otimes [\mu_2])^{S(n_2+n_2')})^{S(n+n',m)}\\ &\qquad\qquad (V_1' \otimes V_2')^{S(n+n',m)},\end{aligned}$$

where V_i' is just a direct sum of V_i (certain number of times). Thus, we have shown that

$$\langle[\lambda]\rangle \cdot \langle[\mu]\rangle = ([\lambda_1] \cdot [\mu_1] \otimes [\lambda_2] \cdot [\mu_2]V_1' \otimes V_2')^{S(n+n',m)}.$$

By the Littlewood-Richardson rule (**6**), one obtains

$$[\lambda_i] \cdot [\mu_i] = \sum_j c_{ij}[\nu_j{}^i],$$

where $[\nu_j{}^i]$ is an irreducible representation of $S(n_i + n_i')$ and c_{ij} are non-negative integers. Therefore,

$$\begin{aligned}\langle[\lambda]\rangle \cdot \langle[\mu]\rangle &= \left(\sum_j c_{1j}[\nu_j{}^1] \otimes \sum_k c_{2k}[\nu_k{}^2](V_1' \otimes V_2')\right)^{S(n+n',m)}\\ &= \sum_j \sum_k c_{1j}c_{2k}([\nu_j{}^1] \otimes [\nu_k{}^2]V_{11})^{S(n+n',m)},\end{aligned}$$

where $V_1' \otimes V_2'$ is the direct sum of $V_{11} = V_{11}(\nu_j{}^1, \nu_k{}^2)$ a certain number of times. This yields the equation

$$\langle[\lambda]\rangle \cdot \langle[\mu]\rangle = \sum_j \sum_k c_{1j}c_{2k}\langle[\nu_j{}^1]; [\nu_k{}^2]\rangle$$

which generalizes the Littlewood-Richardson rule.

Example 6. The representation $\langle[21]; [2]; [\emptyset]\rangle \cdot \langle[2]; [1]; [1]\rangle$ of $S(9, 3)$ is the outer product representation. In this case,

$$\begin{aligned}\langle[21]; [2]; [\emptyset]\rangle \cdot \langle[2]; [1]; [1]\rangle &= \langle[41]; [3]; [1]\rangle + \langle[32]; [3]; [1]\rangle\\ &\quad + \langle[31^2]; [3]; [1]\rangle + \langle[2^21]; [3]; [1]\rangle + \langle[41]; [21]; [1]\rangle\\ &\quad + \langle[32]; [21]; [1]\rangle + \langle[31^2]; [21]; [1]\rangle + \langle[2^21]; [21]; [1]\rangle.\end{aligned}$$

References

1. A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. of Math. (2) *38* (1937), 533–550.
2. J. S. Frame, *Orthogonal group matrices of hyperoctahedral groups*, Nagoya Math. J. *27* (1966), 585–590.
3. J. S. Frame, G. de B. Robinson, and R. M. Thrall, *The hook graphs of the symmetric group*, Can. J. Math. *6* (1954), 316–324.
4. M. Osima, *On the representations of the generalized symmetric group*. I, Math. J. Okayama Univ. *4* (1954), 39–56.
5. B. M. Puttaswamaiah, *Alternating and generalized symmetric groups*, Thesis, University of Toronto, Toronto, 1963.
6. G. de B. Robinson, *Representation theory of the symmetric group*, Mathematical Expositions, No. 12 (Univ. Toronto Press, Toronto, 1961).
7. A. Young, *On quantitative substitutional analysis*. IV; V, Proc. London Math. Soc. (2) *31* (1930), 253–272; 273–288.

Carleton University,
Ottawa, Canada