NOTES ON HYPERSURFACES IN A RIEMANNIAN MANIFOLD

KENTARO YANO

1. Introduction. H. Liebmann **(3)** and W. Süss **(7)** proved

THEOREM A. *The only convex closed hypersurface with constant mean curvature in a Euclidean space is a sphere.*

Y. Katsurada **(1; 2)** gave the following generalization.

THEOREM B. *Let M be an orientable Einstein space which admits a proper conformal Killing vector field, that is, a vector field generating a local one-parameter group of conformal transformations which is not that of isometries, and S a closed orientable hypersurface in M whose first mean curvature is constant. If the inner product of the conformal Killing vector field and the normal to the hypersurface has fixed sign on S, then every point of S is umbilical.*

The present author **(9)** proved

THEOREM C. *Let M be an orientable Riemannian manifold which admits a proper homothetic Killing vector field, that is, a vector field generating a local one-parameter group of homothetic transformations which is not that of isometries, and S a closed orientable hypersurface in M such that the first mean curvature is constant and the Ricci curvature with respect to the normal is non-negative along it. If the inner product of the homothetic Killing vector field and the normal to the hypersurface has fixed sign on S, then every point of S is umbilical.*

To prove Theorem A, we need integral formulas of Minkowski for a hypersurface in a Euclidean space in which the position vector plays a very important role.

To prove Theorems B and C, we need integral formulas of Minkowski for a hypersurface in a Riemannian manifold in which the conformal or homothetic Killing vector field plays the same role as the position vector in a Euclidean space.

Let M be an n-dimensional orientable Riemannian manifold covered by a system of coordinate neighbourhoods (ξ^h) and g_{ji}, ∇_i, K_{kjih}, K_{ji}, and K, the positive definite fundamental tensor, the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ji}, the curvature tensor, the Ricci tensor, and the curvature scalar of M respectively, where here and in the following the indices $h, i, j, \ldots$ run over the range $1, 2, \ldots, n$.

Received January 4, 1966. Supported by National Science Foundation grant GP-3990.

https://doi.org/10.4153/CJM-1967-036-6 Published online by Cambridge University Press

Suppose that v^h is a proper conformal Killing vector field; then we have

$$\mathfrak{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \tag{1.1}$$

where $\mathfrak{L}$ denotes the operator of Lie derivation with respect to v^h, $v_i = g_{ih} v^h$, and ρ is a scalar function given by

$$\rho = (1/n)\nabla_i v^i.$$

For a conformal Killing vector field v^h, we have **(8)**

$$\mathfrak{L}K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki}, \tag{1.2}$$

$$\mathfrak{L}K_{ji} = -(n-2)\nabla_j \rho_i - \Delta\rho g_{ji}, \tag{1.3}$$

$$\mathfrak{L}K = -2(n-1)\Delta\rho - 2\rho K, \tag{1.4}$$

where

$$\rho_i = \nabla_i \rho, \qquad \rho^h = g^{hi}\rho_i, \qquad \Delta\rho = g^{ji}\nabla_j \nabla_i \rho.$$

When M is an Einstein space:

$$K_{ji} = (K/n)g_{ji}, \qquad K = \text{const.},$$

we have, for a conformal Killing vector field v^h,

$$\mathfrak{L}K_{ji} = (1/n)K\mathfrak{L}g_{ji} = (2/n)K\rho g_{ji}, \qquad \mathfrak{L}K = 0,$$

and consequently, from (1.3) and (1.4),

$$(2/n)K\rho g_{ji} = -(n-2)\nabla_j\rho_i - \Delta\rho g_{ji},$$
$$0 = -2(n-1)\Delta\rho - 2\rho K,$$

respectively, from which

$$\nabla_j \rho_i = -\frac{K}{n(n-1)}\rho g_{ji} \qquad \text{if } n > 2.$$

Thus if an Einstein space of dimension $n > 2$ admits a proper conformal vector field, then it admits a non-zero scalar function ρ which satisfies the above equation.

So, to obtain a generalization of Theorem B, we assume in this paper the existence of a non-constant scalar function v which satisfies similar partial differential equations and prove

THEOREM 1. *Let M be an orientable Riemannian manifold of dimension n which admits a non-constant scalar field v such that*

$$\nabla_j \nabla_i v = f(v) g_{ji}, \tag{1.5}$$

where f is a differentiable function of v and S a closed orientable hypersurface in M such that

(i) *its first mean curvature is constant,*

(ii) *$[K_{ji} + (n-1)f'(v)g_{ji}]C^jC^i \geqslant 0$ on S, where C^h is the unit normal to S,*

(iii) *the inner product $C^i\nabla_i v$ has fixed sign on S.*

Then every point of S is umbilical. (This generalization is due to the referee.)

We also prove the following Theorems 2 and 3, the first parts of which are special cases of Theorem 1.

THEOREM 2. *Let M be an orientable Riemannian manifold of dimension n which admits a non-constant scalar field v such that*

$$\nabla_j \nabla_i v = kvg_{ji}, \qquad k = \text{const.}, \tag{1.6}$$

and S a closed orientable hypersurface in M such that
(i) *its first mean curvature is constant,*
(ii) $[K_{ji} + (n-1)kg_{ji}]C^jC^i \geqslant 0$ *on S,*
(iii) *the inner product $C^i\nabla_i v$ has fixed sign on S.*
Then every point of S is umbilical. If, moreover, v is not constant on S, then S is isometric to a sphere.

THEOREM 3. *Let M be an orientable Riemannian manifold of dimension n which admits a non-constant scalar field v such that*

$$\nabla_j \nabla_i v = kg_{ji}, \qquad k = \text{const.}, \tag{1.7}$$

and S a closed orientable hypersurface in M such that
(i) *its first mean curvature is constant,*
(ii) $K_{ji}C^jC^i \geqslant 0$ *on S,*
(iii) *the inner product $C^i\nabla_i v$ has fixed sign on S.*
Then every point of S is umbilical. If, moreover, $v \neq$ const. on S, then S is isometric to a sphere.

The first part of Theorem 3 is a special case of Theorem C.
To prove that the hypersurface under consideration is isometric to a sphere, we use the following theorem of Obata **(4)**.

THEOREM D. *If a Riemannian manifold M is complete, of dimension $n \geqslant 2$, and if there exists a non-null function v such that*

$$\nabla_j \nabla_i v = -c^2 vg_{ji}, \qquad c = \text{const.}, \tag{1.8}$$

then M is isometric to a sphere of radius $1/c$.

If the manifold M in Theorem 2 is complete and $k = -c^2 < 0$, then M is isometric to a sphere according to this theorem of Obata and Theorem 2 refers to a hypersurface in an n-dimensional sphere.
If the manifold M in Theorem 3 is complete and $k \neq 0$, then the holonomy group of the complete Riemannian manifold M fixes a point and consequently, according to a theorem of Sasaki and Goto **(5)**, the manifold M is a Euclidean space. Thus Theorem 3 is identical with Theorem A.

https://doi.org/10.4153/CJM-1967-036-6 Published online by Cambridge University Press

2. General formulas. We consider a closed orientable hypersurface S in a Riemannian manifold M whose local parametric equations are

$$\xi^h = \xi^h(\eta^a), \tag{2.1}$$

η^a being parameters on S, where here and in the following the indices $a, b, c, \ldots$ run over the range $1, 2, \ldots, n-1$.

If we put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\eta^b, \tag{2.2}$$

then $B_b{}^h$ are $n-1$ linearly independent vectors tangent to S and the first fundamental tensor of S is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i. \tag{2.3}$$

We assume that $n-1$ vectors $B_1{}^h, B_2{}^h, \ldots, B_{n-1}{}^h$ give the positive orientation on S and we denote by C^h the unit normal vector to S such that

$$B_1{}^h, B_2{}^h, \ldots, B_{n-1}{}^h, C^h$$

give the positive orientation in M.

Denoting by ∇_c the operator of van der Waerden–Bortolotti covariant differentiation along S (cf. **6**, p. 254), we have the following equations of Gauss and of Weingarten:

$$\nabla_c B_b{}^h = h_{cb} C^h, \tag{2.4}$$

$$\nabla_c C^h = -h_c{}^a B_a{}^h, \tag{2.5}$$

where h_{cb} is the second fundamental tensor of S and $h_c{}^a = h_{cb} g^{ba}$. We also obtain the equations of Gauss and those of Codazzi in the form

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h = K_{dcba} - (h_{da} h_{cb} - h_{ca} h_{db}), \tag{2.6}$$

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i C^h = \nabla_d h_{cb} - \nabla_c h_{db}, \tag{2.7}$$

where K_{dcba} is the curvature tensor of the hypersurface S. From the equations of Codazzi, we have, by a transvection with g^{cb},

$$K_{kh} B_d{}^k C^h = \nabla_d h_c{}^c - \nabla_c h_d{}^c. \tag{2.8}$$

3. Formulas in M admitting a scalar field v such that $\nabla_j \nabla_i v = f(v) g_{ji}$. We now assume that the Riemannian manifold M admits a non-constant scalar field v such that

$$\nabla_j v_i = f(v) g_{ji}, \qquad v_i = \nabla_i v, \tag{3.1}$$

where $f(v)$ is a differentiable function of v, and put

$$v^h = B_a{}^h v^a + \alpha C^h \tag{3.2}$$

on the hypersurface S. From (3.1), we obtain by transvection with $B_c{}^j B_b{}^i$

$$\nabla_c v_b = f(v) g_{cb} + \alpha h_{cb}, \tag{3.3}$$

from which

$$\Delta v = (n-1) f(v) + \alpha h_c{}^c, \tag{3.4}$$

where Δ is the Laplacian operator on S: $\Delta = g^{cb} \nabla_c \nabla_b$.

From (3.1), we also obtain by transvection with $B_b{}^j C^i$

$$\nabla_b \alpha = - h_b{}^a v_a. \tag{3.5}$$

On the other hand, substituting (3.1) into the Ricci identity

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = - K_{kji}{}^h v_h,$$

we find that

$$- K_{kji}{}^h v_h = f'(v)(v_k g_{ji} - v_j g_{ki}),$$

from which

$$K_{ji} v^j = - (n-1) f'(v) v_i$$

and consequently

$$K_{ji} v^j C^i = - (n-1) f'(v) \alpha,$$

which can also be written as

$$K_{ji}(B_c{}^j v^c + \alpha C^j) C^i = - (n-1) f'(v) \alpha,$$

or, by virtue of (2.8),

$$(\nabla_c h_b{}^b - \nabla_b h_c{}^b) v^c + \alpha K_{ji} C^j C^i = - (n-1) f'(v) \alpha,$$

that is

$$\alpha K_{ji} C^j C^i + (n-1) f'(v) \alpha + v^c \nabla_c h_b{}^b - \nabla_b (h_c{}^b c^c) + f(v) h_b{}^b + \alpha h_c{}^b h_b{}^c = 0 \tag{3.6}$$

by virtue of (3.3).

We now assume that the hypersurface S is closed and apply Green's formula **(10)** to (3.4) and (3.6). We then obtain

$$(n-1) \int_S f(v)\, dS + \int_S \alpha h_c{}^c\, dS = 0 \tag{3.7}$$

and

$$\int_S [\alpha K_{ji} C^j C^i + (n-1) f'(v) \alpha + v^c \nabla_c h_b{}^b + f(v) h_b{}^b + \alpha h_c{}^b h_b{}^c] dS = 0 \tag{3.8}$$

respectively, where dS denotes the surface element of S.

If we assume, moreover, that the first mean curvature of S is constant:

$$[1/(n-1)] h_a{}^a = \text{const.},$$

then we obtain, from (3.7) and (3.8),

$$(n-1) \int_S f(v)\, dS + h_c{}^c \int_S \alpha\, dS = 0$$

https://doi.org/10.4153/CJM-1967-036-6 Published online by Cambridge University Press

and

$$\int_S \alpha K_{ji} C^j C^i \, dS + (n-1)\int_S f'(v)\alpha \, dS + h_b{}^b \int_S f(v) \, dS + \int_S \alpha h_c{}^b h_b{}^c \, dS = 0,$$

respectively. Eliminating $\int_S f(v) \, dS$ from these two equations, we find that

$$\int_S \alpha \{ [K_{ji} + (n-1)f'(v) g_{ji}] C^j C^i + [h_c{}^b h_b{}^c - [1/(n-1)] h_c{}^c h_b{}^b] \} dS = 0$$

or

$$(3.9) \quad \int_S \alpha \left[(K_{ji} + (n-1)f'(v) g_{ji}) C^j C^i + \left(h^{cb} - \frac{1}{n-1} h_t{}^t g^{cb} \right) \times \left(h_{cb} - \frac{1}{n-1} h_s{}^s g_{cb} \right) \right] dS = 0.$$

4. Proofs.

Proof of Theorem 1. Suppose that the three conditions of Theorem 1 are satisfied. Then, in the integral formula (3.9), we have

$$[K_{ji} + (n-1)f'(v) g_{ji}] C^j C^i \geqslant 0,$$

$$\left(h^{cb} - \frac{1}{n-1} h_t{}^t g^{cb} \right) \left(h_{cb} - \frac{1}{n-1} h_s{}^s g_{cb} \right) \geqslant 0,$$

and $\alpha = C^i \nabla_i v$ has fixed sign on S; hence

$$h_{cb} - [1/(n-1)] h_s{}^s g_{cb} = 0,$$

which shows that every point of S is umbilical.

Proof of Theorem 2. The first part of Theorem 2 is a special case of Theorem 1 with $f(v) = kv$, k being a constant.

We assume, moreover, that S is a hypersurface along which

$$(4.1) \qquad v \neq \text{const.}$$

Since S is umbilical, we put

$$(4.2) \qquad h_{cb} = \lambda g_{cb}, \qquad \lambda = \text{const.}$$

Then from (3.3) with $f(v) = kv$,

$$(4.3) \qquad \nabla_c \nabla_c v = (kv + \lambda\alpha) g_{cb}$$

and from (3.5)

$$(4.4) \qquad \nabla_b \alpha = -\lambda v_b;$$

hence

$$(4.5) \qquad \alpha + \lambda v = c = \text{const.}$$

Substituting this into (4.3), we find that

$$\nabla_c \nabla_b v = [kv + \lambda(c - \lambda v)] g_{cb}$$

https://doi.org/10.4153/CJM-1967-036-6 Published online by Cambridge University Press

or

$$(4.6) \qquad \nabla_c \nabla_b v = [-(\lambda^2 - k)v + \lambda c]g_{cb}.$$

Here $\lambda^2 - k \neq 0$. Because, if $\lambda^2 - k = 0$, then (4.6) becomes $\nabla_c \nabla_b v = \lambda c g_{cb}$ from which $\Delta v = (n-1)\lambda c$, which is impossible unless $v = \text{const.}$

Thus, $\lambda^2 - k$ being different from zero, we have, from (4.6),

$$(4.7) \qquad \nabla_c \nabla_b\left(v - \frac{\lambda c}{\lambda^2 - k}\right) = -(\lambda^2 - k)\left(v - \frac{\lambda c}{\lambda^2 - k}\right)g_{cb},$$

from which

$$\Delta\left(v - \frac{\lambda c}{\lambda^2 - k}\right) = -(n-1)(\lambda^2 - \kappa)\left(v - \frac{\lambda c}{\lambda^2 - k}\right),$$

and consequently

$$\lambda^2 - k > 0.$$

By Theorem D, equation (4.7) shows that the hypersurface S is isometric to a sphere. This completes the proof of Theorem 2.

Proof of Theorem 3. The first part of Theorem 3 is a special case of Theorem 1 with $f(v) = k = \text{const.}$

We assume that S is a hypersurface along which

$$(4.8) \qquad v \neq \text{const.}$$

Since S is umbilical, we put

$$(4.9) \qquad h_{cb} = \lambda g_{cb}, \qquad \lambda = \text{const.}$$

Then from (3.3) with $f(v) = k$,

$$(4.10) \qquad \nabla_c \nabla_b v = (k + \lambda\alpha)g_{cb}$$

and from (3.5)

$$(4.11) \qquad \nabla_b \alpha = -\lambda v_b,$$

from which

$$(4.12) \qquad \alpha + \lambda v = c = \text{const.}$$

Substituting (4.12) into (4.10), we find that

$$(4.13) \qquad \nabla_c \nabla_b v = (-\lambda^2 v + k + \lambda c)g_{cb}.$$

Here $\lambda \neq 0$. Because if $\lambda = 0$, then (4.13) becomes $\nabla_c \nabla_b v = k g_{cb}$ from which $\Delta v = (n-1)k$, which is impossible unless $v = \text{const.}$

Thus λ being different from zero, we have, from (4.13),

$$(4.14) \qquad \nabla_c \nabla_b\left(v - \frac{k + \lambda c}{\lambda^2}\right) = -\lambda^2\left(v - \frac{k + \lambda c}{\lambda^2}\right)g_{cb},$$

and consequently by Theorem D the hypersurface S is isometric to a sphere. This completes the proof of Theorem 3.

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University of California, Berkeley