Glasgow Math. J. **51** (2009) 219–241. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004643. Printed in the United Kingdom

CONSTANT SCALAR CURVATURE HYPERSURFACES WITH SECOND-ORDER UMBILICITY

ANTONIO GERVASIO COLARES
Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici, 60455-760 Fortaleza-CE, Brazil
e-mail: gcolares@mat.ufc.br

and FERNANDO ENRIQUE ECHAIZ-ESPINOZA
Instituto de Matemática, Universidade Federal de Alagoas, Campus A.C. Simões, 57455-760 Maceió-AL, Brazil
e-mail: echaiz@pos.mat.ufal.br

(Received 22 April 2008; accepted 21 August 2008)

Abstract. We extend the concept of umbilicity to higher order umbilicity in Riemannian manifolds saying that an isometric immersion is k-umbilical when $AP_{k-1}(A)$ is a multiple of the identity, where $P_k(A)$ is the kth Newton polynomial in the second fundamental form A with $P_0(A)$ being the identity. Thus, for $k = 1$, one-umbilical coincides with umbilical. We determine the principal curvatures of the two-umbilical isometric immersions in terms of the mean curvatures. We give a description of the two-umbilical isometric immersions in space forms which includes the product of spheres $S^k(\frac{1}{\sqrt{2}}) \times S^k(\frac{1}{\sqrt{2}})$ embedded in the Euclidean sphere S^{2k+1} of radius 1. We also introduce an operator ϕ_k which measures how an isometric immersion fails to be k-umbilical, giving in particular that $\phi_1 \equiv 0$ if and only if the immersion is totally umbilical. We characterize the two-umbilical hypersurfaces of a space form as images of isometric immersions of Einstein manifolds.

2000 *Mathematics Subject Classification.* Primary 53C42, 53C40; Secondary 53B20, 53B25.

1. Introduction. Let $x : \mathbf{M}^n \to \overline{\mathbf{M}}^{n+1}$, $n \geq 2$, be an isometric immersion of a Riemannian manifold $\mathbf{M}^n$ in a Riemannian manifold $\overline{\mathbf{M}}^{n+1}$. We know that x is totally umbilical if for each $p \in \mathbf{M}^n$ the second fundamental form $A_p : T_p\mathbf{M} \to T_p\mathbf{M}$ is a multiple of the identity on $T_p\mathbf{M}$. That is, if $\lambda_1(p), \ldots, \lambda_n(p)$ are the eigenvalues of A_p, then $S_1 = \sum_{i=1}^n \lambda_i$ is constant and so

$$A_p = \frac{S_1}{n} I,$$

I the identity of $T_p\mathbf{M}$.

We extend the concept of umbilicity to higher order umbilicity, calling k-umbilicity, for $k = 1, \ldots, n$. We say that an isometric immersion x is k-umbilical (or has umbilicity of order k) if, at each point $p \in \mathbf{M}$, AP_{k-1} is a multiple of the identity. Here, P_k is the Newton polynomial in the second fundamental form A on $T_p\mathbf{M}$ given, inductively, by

$P_0 = I$, $P_k = S_k I - P_{k-1}A$. If $AP_{k-1} \equiv 0$ we say that x is k-totally geodesic. Here,

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}\ .$$

The kth mean curvature H_k is given by

$$H_k = \frac{1}{\binom{n}{k}} S_k.$$

Thus, one-umbilical immersion is the known totally umbilical immersion. A k-umbilical isometric immersion in a Riemannian manifold satisfies (cf. Theorem 5.5): $AP_{k-1}(A)$ is a Codazzi tensor if and only if S_k is constant (known for $k = 1$). An interesting but different concept of k-umbilicity was introduced and developed in [**2**] and [**7**]. To study k-umbilicity we define an operator ϕ_k on tangent spaces of the immersion given by

$$\phi_k(X) = \frac{k}{n} S_k(A)X - AP_{k-1}(A)X.$$

We will prove that $\phi_k = 0$ if and only if the immersion is k-umbilical (cf. Remark 5.7). For $k = 1$ this was studied in [**1**].

An example of a two-umbilical embedding is given by $S^k(\frac{1}{\sqrt{2}}) \times S^k(\frac{1}{\sqrt{2}}) \to S^{2k+1}(1)$.

We prove that if x is k-umbilical, then

$$H_{k+1} = H_1\, H_k.$$

But there exist hypersurfaces satisfying the condition $H_{k+1} = H_1\, H_k$ that are not k-umbilical. For example, consider

$$x : SO(3) \longrightarrow \quad S^4(r),$$

letting

$$g \longmapsto g \begin{pmatrix} \frac{r\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{-r\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} g^{-1},$$

where $S^4(r)$ is the Euclidean sphere. We see that $\lambda_1 = 0$, $\lambda_2 = \frac{\sqrt{3}}{2}$ and $\lambda_3 = -\frac{\sqrt{3}}{2}$ and so AP_1 is not a multiple of the identity, hence x is not two-umbilical. Clearly, $H_3 = H_1 H_2$, because $S_1 = 0 = S_3$. However, the condition $H_1 H_k = H_{k+1}$ characterizes the k-umbilical isometric immersions whose principal curvatures never vanish (cf. Theorem 7.1).

We will show that for $n \geq 3$, if x is k-umbilical in a space form $\mathbf{M}^{n+1}(c)$, then $S_k$is constant. For $n = 2$ this is not true because every immersion x of $M^3(c)$ is two-umbilical, since the eigenvalues of AP_1 are equal to the Gaussian curvature of x. This should not be surprise, since it would happen with the concept of umbilicity extended to one-dimensional immersion in $M^2(c)$: every curve would be umbilical.

In this paper, we will study k-umbilical isometric immersions in Riemannian manifolds. We concentrate mainly on two-umbilical isometric immersions. First, we will prove the following theorem (cf. Theorem 8.1) on the determination of the principal curvatures of the two-umbilical isometric immersions in a Riemannian manifold:

Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ *(*$n \geq 3$*) be any two-umbilical isometric immersion.*

(a) *If its principal curvatures are distinct, then they are given by*

$$\lambda_{i_1} = \cdots = \lambda_{i_r} = ((n-(r+1))/(n-2r))S_1$$

and

$$\lambda_{i_{r+1}} = \cdots = \lambda_{i_n} = -((r-1)/(n-2r))S_1,$$

where $r \in \{0, 1, 2, \ldots, [[\frac{n}{2}]]^1\}$ *or* $r \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}$*, according to whether n is odd or even, respectively.*

(b) *If its principal curvatures are equal, then n is even and its principal curvatures are given by*

$$\lambda_{i_1} = \cdots = \lambda_{i_{\frac{n}{2}}} = \sqrt{2/n}\,\sqrt{-S_2}$$

and

$$\lambda_{i_{\frac{n}{2}+1}} = \cdots = \lambda_{i_n} = -\sqrt{2/n}\,\sqrt{-S_2}\,.$$

We give a description of an infinite family of two-umbilical hypersurfaces in the sphere $S^{n+1}(1)$ (cf. Theorem 9.3):

There exists a countably infinite family of two-umbilical hypersurfaces in the Euclidean sphere $S^{n+1}(1)$ *: for any* $n \geq 4$ *and for every* $m \in \{2, \ldots, n-2\}$*, the Clifford's hypersurface*

$$S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow S^{n+1}(1) \quad \textit{is two-umbilical if and only if} \quad r = \sqrt{\frac{n-m-1}{n-2}}.$$

We will see that for $r = \sqrt{\frac{n-m-1}{n-2}}$ the hypersurface $S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow S^{n+1}(1)$ with the metric induced from $S^{n+1}(1)$ is an Einstein manifold. In fact, $S_2 = -\frac{n}{2}$ and $\mathrm{Ric}(X, Y) = (n - 1 + \frac{2S_2}{n}) < X, Y >$, when $r = \sqrt{\frac{n-m-1}{n-2}}$. That is, for $r = \sqrt{\frac{n-m-1}{n-2}}$, the two-umbilical Clifford hypersurfaces are examples of Einstein manifolds which admit isometric immersion in $S^{n+1}(1)$.

In **[8]**, Fialkow gave a classification of the Einstein hypersurfaces in space forms. By using our methods we reprove part of Theorem 7.1 of **[8]**:

Every connected Einstein hypersurface in a space form has at most two distinct principal curvatures (cf. Theorem 4.1*).*

The classical reference **[4]** has characterizations of Einstein manifolds under many distinct aspects.

Here, we will prove a characterization of Einstein hypersurfaces in space forms in terms of the second-order umbilicity (cf. Theorem 8.3):

[1] $[[x]]$ is the largest integer not exceeding x.

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

Let $\mathbf{M}^n$ *be a connected Riemannian manifold and* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}(c)$, $n \geq 3$, *be an isometric immersion. Then*

$$\mathbf{M}^n \text{ is Einstein if and only if } x \text{ is two-umbilical.}$$

Moreover, in this case $Ric(X, Y) = (c(n-1) + \frac{2S_2}{n}) < X, Y >$, *with* S_2 *constant.*

As a consequence, this yields ϕ_2 as a measure of how much $\mathbf{M}^n$ fails to be an Einstein hypersurface (cf. Remark 8.4).

Our methods offer the possibility of studying k-umbilical isometric immersions in more general ambient spaces, as in Theorem 8.1.

By using Theorem 8.1, the paper ends with a description of the two-umbilical hypersurfaces in a space form (cf. Theorem 9.6):

Let $\mathbf{M}^n$ *be a two-umbilical hypersurface in* $\overline{\mathbf{M}}^{n+1}(c)$, $n > 2$. *Then*

(a) $\mathbf{M}$ *is two-totally geodesic or*
(b) $\mathbf{M}$ *is one-umbilical or*
(c) *if* $c > 0$, *then* $\mathbf{M}$ *is locally a standard product embedding of*

$$S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow S^{n+1}(1),$$

where $r = \sqrt{\frac{n-m-1}{n-2}}$. *In particular, when the embedding is minimal we have*

$$S^k\left(\frac{1}{\sqrt{2}}\right) \times S^k\left(\frac{1}{\sqrt{2}}\right) \hookrightarrow S^{2k+1}(1),$$

where $n = 2k$;

(d) *if* $c < 0$, *then* $\mathbf{M}$ *is geodesic hyperspheres, horospheres, totally geodesic hyperplanes and their equidistant hypersurfaces, tubes around totally geodesic subspaces of dimension at least one (in another words, it is locally a standard product embedding of* $S^k \times \mathbb{H}^{n-k}$*);*
(e) *if* $c = 0$, *then* $\mathbf{M}$ *is locally hyperspheres, hyperplanes or a standard product embedding given by* $S^k \times \mathbb{R}^{n-k}$.

2. Preliminaries. In this section, we fix notation and recall basic concepts that will be extended to self-adjoint operators in the next section.

Definition 2.1. Given any integer k, the function $S_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$S_k(x_1, x_2, \ldots, x_n) := \begin{cases} 1, & k = 0, \\ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}, & \forall k \in \{1, 2, \ldots, n\}, \\ 0, & \forall k \in \mathbb{Z} \setminus \{0, \cdots, n\} \end{cases} \tag{2.1}$$

will be called an elementary k-symmetric polynomial.

Definition 2.2. Given any integer k, let S_k be the k-symmetric polynomial as given in Definition 2.1. We define the jth partial derivative of S_k by the following recurrence relations:

$$\frac{\partial}{\partial x_j} S_0(x_1, \dots, x_n) = 0;$$
$$\frac{\partial}{\partial x_j} S_1(x_1, \dots, x_n) = 1;$$
$$\frac{\partial}{\partial x_j} S_r(x_1, \dots, x_n) = S_{r-1}(x_1, \dots, x_n) - x_j \frac{\partial}{\partial x_j} S_{r-1}(x_1, \dots, x_n), \quad \forall r \geq 2. \tag{2.2}$$

From these relations, we can show by induction that

$$\frac{\partial}{\partial x_j} S_{r+1}(x_1, \dots, x_n) = \sum_{i=0}^{r} (-1)^i S_{r-i}(x_1, \dots, x_n)(x_j)^i. \tag{2.3}$$

PROPOSITION 2.3. *Given any integer k, let $S_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the elementary k-symmetric polynomial. Then*

$$S_k(x_1, \dots, \widehat{x_j}, \dots, x_n) = \sum_{i=0}^{k} (-1)^i S_{k-i}(x_1, \dots, x_n)(x_j)^i, \tag{2.4}$$

where $\widehat{x_j}$ indicates that x_j has been excluded, that is,

$$S_k(x_1, \dots, \widehat{x_j}, \dots, x_n) = S_k(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n).$$

Proof. By differentiation of $S_k(x_1, \dots, x_n)$ with respect to x_j we get

$$\frac{\partial}{\partial x_j} S_k(x_1, \dots, x_n) = S_{k-1}(x_1, \dots, \widehat{x_j}, \dots, x_n), \tag{2.5}$$

where $\widehat{x_j}$ denotes that x_j has been excluded. The proof follows after comparing (2.3) and (2.5). □

PROPOSITION 2.4 (Euler's identity). *Given any integer k, let $S_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the elementary k-symmetric polynomial. Then*

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} S_{k+1}(x_1, \dots, x_n) = (k+1) S_{k+1}(x_1, \dots, x_n), \tag{2.6}$$

or equivalently,

$$\sum_{j=1}^{n} x_j S_k(x_1, \dots, \widehat{x_j}, \dots, x_n) = (k+1) S_{k+1}(x_1, \dots, x_n). \tag{2.7}$$

Proof. See [**11**]. □

3. r-Newton Operators. Throughout what follows, V stands for an n-dimensional real vector space equipped with an inner product.

DEFINITION 3.1. We define the elementary r-symmetric polynomial by

$$\begin{aligned} S_r : \mathcal{L}(V) &\longrightarrow \mathbb{R} \\ B &\longrightarrow S_r(B) := S_r(\lambda_1, \lambda_2, \dots, \lambda_n), \end{aligned}$$

where $B \in \mathcal{L}(V)$ is a self-adjoint linear operator and we let $\lambda_1, \dots, \lambda_n$ denote the set of its associated eigenvalues.

DEFINITION 3.2. Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator and we let $\{\lambda_i\}_{i\in\mathbb{Z}}$ denote all its eigenvalues and let $\{v_i\}_{i\in\mathbb{Z}}$ be its associated orthonormal eigenvectors, i.e., $Bv_i = \lambda_i v_i$. We make use of the convention that $\lambda_i = 0$ if $i \in \mathbb{Z} \setminus \{1, 2, \dots, n\}$ and $v_i \begin{cases} \neq 0_V, & \text{if } i \in \{1, 2, \dots, n\}; \\ = 0_V, & \text{if } i \in \mathbb{Z} \setminus \{1, 2, \dots, n\}. \end{cases}$
We define

$$B_i := \begin{cases} B, & \forall i \in \mathbb{Z} \setminus \{1, 2, \dots, n\} \\ B|_{span\{v_i\}^{\perp}}, & \forall i = 1, 2, \dots, n. \end{cases}$$

DEFINITION 3.3. Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator. Denote by B_i the operator defined in Definition 3.2. We define

$$S_r(B_i) := \begin{cases} 0, & \text{if } r \in \mathbb{Z} \setminus \{0, 1, \dots, n\} \text{ and } i \in \mathbb{Z}; \\ 1, & \text{if } r = 0 \text{ and } i \in \mathbb{Z}; \\ S_r(B), & \text{if } r \in \{1, \dots, n\} \text{ and } i \in \mathbb{Z} \setminus \{1, 2, \dots, n\}; \\ S_r(\lambda_1, \dots, \lambda_{i-1}, \widehat{\lambda_i}, \lambda_{i+1}, \dots, \lambda_n), & \text{if } r \in \{1, \dots, n\} \text{ and } i \in \{1, 2, \dots, n\}, \end{cases}$$

where $\widehat{\lambda_i}$ means that the term λ_i is excluded, that is,

$$S_r(\lambda_1, \dots, \lambda_{i-1}, \widehat{\lambda_i}, \lambda_{i+1}, \dots, \lambda_n) = S_r(\lambda_1, \dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n).$$

PROPOSITION 3.4. *Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator, $\{\lambda_i\}$, $1 \le i \le n$, its eigenvalues and let $\{v_i\}$, $1 \le i \le n$, be its associated orthonormal eigenvectors. Then,*
(a) $S_n(B_i) = 0\ ;\ \forall i \in \{1, \dots, n\}$;
(b) $S_r(B_i) = \dfrac{\partial}{\partial \lambda_i} S_{r+1}(B)$;
(c) $S_{r+1}(B_i) = S_{r+1}(B) - \lambda_i S_r(B_i)$;
(d) $\dfrac{\partial}{\partial \lambda_j} S_r(B_i) = \dfrac{\partial}{\partial \lambda_i} S_r(B_j)$.

Proof.
(a) This is immediate from the fact that $S_n(B_i)$ is a product of n terms where one of them is zero.
(b) Immediate from Definition 3.3 and equations (2.3) and (2.4).
(c) In the expression (2.2), we apply Definition 3.3 and the above item (b).
(d) This is a consequence of the item (b).

□

DEFINITION 3.5. Given any integer r, an operator

$$P_r : \{B \in \mathcal{L}(V);\ B \text{ is self-adjoint}\} \longrightarrow \{B \in \mathcal{L}(V);\ B \text{ is self-adjoint}\}$$

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

given by

$$P_r(B) := \begin{cases} I, & r = 0; \\ \sum_{j=0}^{r}(-1)^j S_{r-j}(B)B^j, & \forall r \in \{1, 2, \ldots, n-1\}; \\ \mathcal{O}, & r \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}. \end{cases}$$

is called an r-Newton operator associated to B, where I and $\mathcal{O}$ are the identity and the null operators, respectively.

PROPOSITION 3.6.

$$P_{r+1}(B) = S_{r+1}(B)I - BP_r(B), \tag{3.1}$$

or equivalently,

$$P_{r+1}(B) = S_{r+1}(B)I - P_r(B)B, \tag{3.2}$$

for each $r = 0, 1, 2, \ldots, n-1$.

Proof. The proof of (3.1) is by induction on r and (3.2) is justified by the fact that $P_r(B)$ is a polynomial and $P_r(B)B = BP_r(B)$. □

The next proposition is a summary of important relations about $P_r(B)$ and $S_r(B)$.

PROPOSITION 3.7. *Let* $B \in \mathcal{L}(V)$ *be a self-adjoint operator. Then*

(a) $P_n(B) = \mathcal{O}$, *where* $\mathcal{O}$ *is the null operator in* V*;*
(b) $\textit{trace}\,(BP_r(B)) = nS_{r+1}(B) - \textit{trace}\,(P_{r+1}(B))$*;*
(c) $\textit{trace}\,(B^2P_r(B)) = \textit{trace}\,(S_{r+1}(B)B) - \textit{trace}\,(BP_{r+1}(B))$*;*
(d)
  (d.1) $\textit{trace}\,(P_r(B)) = (n-r)S_r(B)$*;*
  (d.2) $\textit{trace}\,(P_r(B)) = \sum_{j=0}^{r}(-1)^j S_{r-j}(B)\textit{trace}\,(B^j)$*;*
(e) $\textit{trace}\,(BP_r(B)) = (r+1)S_{r+1}(B)$ (Newton's formula)*;*
(f) $\textit{trace}\,(B^2P_r(B)) = S_1(B)S_{r+1}(B) - (r+2)S_{r+2}(B)$*;*
(g) $P_r(B)$ *and* B *have the same eigenvectors;*
(h) *The eigenvalues of* $P_r(B)$ *are* $\dfrac{\partial}{\partial \lambda_j}S_r(B)$, *where* λ_j *is an eigenvalue of* B*;*
(i) $P_r(B)v_i = S_r(B_i)v_i$, *where* v_i *is an eigenvector of* B*;*
(j) $\textit{trace}\,(P_r(B)) = \displaystyle\sum_{i=1}^{n} S_r(B_i)$*;*
(k) $\textit{trace}\,(BP_r(B)) = \displaystyle\sum_{i=1}^{n} \lambda_i S_r(B_i)$*;*
(l) $\textit{trace}\,(B^2P_r(B)) = \displaystyle\sum_{i=1}^{n} \lambda_i^2 S_r(B_i)$.

Proof. See [**3**]. □

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

DEFINITION 3.8. Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator. Denote by B_i the operator given in Definition 3.2. We define

$$S_r(B_i, B_j) := \begin{cases} 0 & \text{if } r \in \mathbb{Z} \setminus \{0, 1, \dots, n\},\ \forall\, i, j \in \mathbb{Z}; \\ 1 & \text{if } r = 0,\ \forall\, i, j \in \mathbb{Z}; \\ S_r(B) & \text{if } r \in \{1, \dots, n\},\ \forall\, i, j \in \mathbb{Z} \setminus \{1, 2, \dots, n\}; \\ S_r(B_i) & \text{if } r, i \in \{1, \dots, n\},\ j \in \mathbb{Z} \setminus \{1, 2, \dots, n\}; \\ S_r(B_j) & \text{if } r, j \in \{1, \dots, n\},\ i \in \mathbb{Z} \setminus \{1, 2, \dots, n\}; \\ S_r(B_i) & \text{if } r \in \{1, \dots, n\},\ j = i \in \{1, 2, \dots, n\}; \\ S_r(\lambda_1, \dots, \widehat{\lambda_i}, \dots, \widehat{\lambda_j}, \dots, \lambda_n) & \text{if } r \in \{1, \dots, n\},\ j \neq i \in \{1, 2, \dots, n\}. \end{cases}$$

where $\widehat{\lambda_i}$ means that the term λ_i is excluded and we are denoting

$$S_r(B_i, B_j) := S_r\left(B\big|_{span\{v_i,\, v_j\}^{\perp}} \right).$$

Note that $S_r(B_i, B_j)$ is an extension of Definition 3.3. We next show a few relations involving $S_r(B_i, B_j)$.

PROPOSITION 3.9. *Let $B \in \mathcal{L}(V)$ be a self-adjoint operator, $\lambda_1, \dots, \lambda_n$ the eigenvalues of B and let $v_1, \dots, v_n$ be its associated orthonormal eigenvectors. Then*

(a) $S_r(B_i, B_j) = S_r(B_j, B_i)$;

(b) $S_r(B_i, B_j) = \dfrac{\partial^2}{\partial\lambda_i \partial\lambda_j} S_{r+2}(B), \quad \forall i, j \in \{1, \dots, n\}$;

(c) $S_{n-1}(B_i, B_j) = 0 = S_n(B_i, B_j), \quad \forall i, j \in \{1, \dots, n\}$;

(d)

$$S_{r+1}(B_i, B_j) = S_{r+1}(B_i) - \lambda_j S_r(B_i, B_j)$$

and

$$S_{r+1}(B_i, B_j) = S_{r+1}(B_j) - \lambda_i S_r(B_i, B_j);$$

(e) $S_{r+1}(B_i) - S_{r+1}(B_j) = (\lambda_j - \lambda_i) S_r(B_i, B_j)$;

(f) $\sum_{\substack{i=1 \\ i \neq k}}^{n} \lambda_i S_r(B_i, B_k) = (r+1) S_{r+1}(B_k)$ (Euler's identity);[2]

(g) *With the definition of $S_r(B_i, B_j, B_k)$, we have (analogous to part (d)):*

$$S_{r+1}(B_i, B_j, B_k) = S_{r+1}(B_i, B_j) - \lambda_k S_r(B_i, B_j, B_k),$$
$$S_{r+1}(B_i, B_j, B_k) = S_{r+1}(B_i, B_k) - \lambda_j S_r(B_i, B_j, B_k),$$
$$S_{r+1}(B_i, B_j, B_k) = S_{r+1}(B_j, B_k) - \lambda_i S_r(B_i, B_j, B_k);$$

(h) $S_{r+1}(B_i, B_k) - S_{r+1}(B_k, B_j) = (\lambda_j - \lambda_i) S_r(B_i, B_j, B_k)$;

[2]It should be noted that if we make $k = 0$ in this expression we get Proposition 2.4.

(i) *For any* $j \in \{1, \ldots, n\}$, $r \in \{0, \ldots, n\}$ *we have* [3]

$$[(n-1)-r]\; S_r(B_j) = \sum_{\substack{i=1\\ i\neq j}}^{n} S_r(B_i, B_j).$$

Proof.

(a) Direct from the equality

$$S_r(B_i, B_j) = S_r\left(\left(B\big|_{\operatorname{span}\{v_i\}^\perp}\right)\Big|_{\operatorname{span}\{v_j\}^\perp}\right) = S_r\left(\left(B\big|_{\operatorname{span}\{v_j\}^\perp}\right)\Big|_{\operatorname{span}\{v_i\}^\perp}\right).$$

(b) We have

$$\begin{aligned}\frac{\partial^2}{\partial\lambda_i\partial\lambda_j} S_{r+2}(B) &= \frac{\partial}{\partial\lambda_i}\left(\frac{\partial}{\partial\lambda_j} S_{r+2}(B)\right),\\ &= \frac{\partial}{\partial\lambda_i}\left(S_{r+1}\left(B_{|_{\operatorname{span}\{e_j\}^\perp}}\right)\right) = S_r\left(\left(B_{|_{\operatorname{span}\{e_i\}^\perp}}\right)_{|_{\operatorname{span}\{e_j\}^\perp}}\right).\end{aligned}$$

The last equality comes from Proposition 3.4(b) and Definition 3.2.

(c) It is immediate from part (b) and Proposition 3.4(b).

(d) Use that the statement of Proposition 3.4(c) remains valid if we replace B by $B_{|_{\operatorname{span}\{v_j\}^\perp}}$.

(e) Immediate from part (d).

(f) First use notation of Definition 3.3 in relation (2.7) and finally replace B by $B_{|_{\operatorname{span}\{v_j\}^\perp}}$.

(g) We obtain our result by replacing B by $B_{|_{\operatorname{span}\{v_i,v_j\}^\perp}}$, $B_{|_{\operatorname{span}\{v_i,v_k\}^\perp}}$ and $B_{|_{\operatorname{span}\{v_j,v_k\}^\perp}}$, successively, in Proposition 3.4(c).

(h) Immediate from part (g).

(i) Only replace B by $B_{|_{\operatorname{span}\{v_j\}^\perp}}$ in Proposition 3.7(e) and (k).

□

4. First Applications. Here, we give some applications of the result obtained in Section 3, including another proof of a result of Fialkow in [**8**] about Einstein manifolds. A Riemannian manifold $\mathbf{M}^n$ is an Einstein manifold if its Ricci tensor satisfies: for any X, Y tangent to M^n, $\operatorname{Ric}(X, Y) = \lambda\langle X, Y\rangle$, where λ is a real function.

THEOREM 4.1 ([**8**] part of Theorem 7.1). *Let* $\mathbf{M}^n$ *be a connected Riemannian manifold and* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}(c)$, $n \geq 3$, *an isometric immersion. If* $\mathbf{M}^n$ *is an Einstein manifold, then the maximum number of distinct principal curvatures of* x *is two.*

Proof. From Gauss equation

$$\operatorname{Ric}(X, Y) = c(n-1)\langle X, Y\rangle + \langle AP_1(A)X, Y\rangle,$$

[3]This is a version of Proposition 3.7 (d.1).

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

where A is the shape operator. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis which diagonalizes A at a point (that is, $Ae_i = \lambda_i e_i$). Since $\mathbf{M}^n$ is Einstein we get

$$\lambda = c(n-1) + \lambda_i S_1(A_i), \quad \text{for } i = 1, 2, \ldots, n. \tag{4.1}$$

If $\lambda_1 = \cdots = \lambda_n = \rho$, clearly $\lambda_i S_1(A_i) = (n-1)\rho^2$. From (4.1) we get $\rho^2 = \dfrac{\lambda}{n-1} - c$. Since $\rho \in \mathbb{R}$ it follows that

(a) $\lambda > c(n-1)$ or
(b) $\lambda = c(n-1)$.

If x has at least two distinct principal curvatures $\lambda_i \neq \lambda_j$, from (4.1) we have that $\lambda_i S_1(A_i) = \lambda_j S_1(A_j)$; from Proposition 3.4(c) we get $S_2(A_j) - S_2(A_i) = 0$. From Proposition 3.9(d) we get $(\lambda_i - \lambda_j)S_1(A_i, A_j) = S_2(A_j) - S_2(A_i) = 0$. Then $S_1(A_i, A_j) = 0$. By Proposition 3.9(d) we get $S_1(A_i) = S_1(A_i, A_j) + \lambda_j$ and hence $\lambda_i S_1(A_i) = \lambda_i S_1(A_i, A_j) + \lambda_i \lambda_j$. But we see that $S_1(A_i, A_j) = 0$ and so $\lambda_i S_1(A_i) = \lambda_i \lambda_j$. In other words, from (4.1) we get $\lambda_i \lambda_j = \lambda - c(n-1)$.
Now, for any principal curvature λ_k with $k \neq i, j$, by (4.1) we have

$$\lambda_k S_1(A_k) = \lambda - c(n-1).$$

From Proposition 3.4(c) we get

$$\lambda_k(S_1(A) - \lambda_k) = \lambda - c(n-1).$$

We have seen that $\lambda_i \lambda_j = \lambda - c(n-1)$. Thus

$$\lambda_k^2 - S_1(A)\lambda_k + \lambda_i \lambda_j = 0.$$

Since $S_1(A) = S_1(A_i, A_j) + \lambda_i \lambda_j$, and $S_1(A_i, A_j) = 0$ we get

$$\lambda_k^2 - (\lambda_i + \lambda_j)\lambda_k + \lambda_i \lambda_j = 0.$$

Therefore, the above equality shows that each λ_k must be λ_i or λ_j. □

THEOREM 4.2. *Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator. For any $r, k \in \mathbb{Z}$ we have*

$$trace[P_{r-1}(B)\, B\, P_k(B)] = \sum_{j=0}^{k} (r + k - 2j)\, S_{r+k-j}(B)\, S_j(B).$$

Proof. Multiply relation (3.2) on both sides by $P_k(B)$ to get

$$P_r(B)P_k(B) = S_r(B)P_k(B) - P_{r-1}(B)\, B\, P_k(B).$$

Next, taking trace and by Proposition 3.7(d.1) we get

$$\text{trace}(P_r(B)\, P_k(B)) = (n-k)\, S_r(B)\, S_k(B) - \text{trace}(P_{r-1}(B)\, B\, P_k(B)).$$

It follows, by interchanging the roles of r and k, that

$$\text{trace}(P_k(B)\, P_r(B)) = (n-r)\, S_k(B)\, S_r(B) - \text{trace}(P_{k-1}(B)\, B\, P_r(B)).$$

Hence,

$$(r-k)S_r(B)S_k(B) = \mathrm{trace}(P_{r-1}(B)\ B\ P_k(B)) - \mathrm{trace}(P_{k-1}(B)\ B\ P_r(B)). \tag{4.2}$$

For fixed r, the proof is by induction on k.
Taking $k = 1$ in (4.2), by Proposition 3.7(d.1) we get

$$\mathrm{trace}(P_{r-1}(B)\ B\ P_1(B)) = (r+1)S_{r+1}(B) + (r-1)S_1(B)S_r(B).$$

By the induction hypothesis, we then have

$$\mathrm{trace}[P_{r-1}(B)\ B\ P_k(B)] = \sum_{j=0}^{k}(r+k-2j)\ S_{r+k-j}(B)\ S_j(B).$$

Replacing k for $k+1$ in (4.2) we get

$$\mathrm{trace}[P_{r-1}(B)\ B\ P_{k+1}(B)] = (r-k-1)S_r(B)S_{k+1}(B) + \mathrm{trace}[P_k(B)\ B\ P_r(B)].$$

Again by the induction hypothesis, we then have

$$\mathrm{trace}[P_{r-1}(B)\ B\ P_{k+1}(B)] = \\ (r-k-1)S_r(B)S_{k+1}(B) + \sum_{j=0}^{k}(r+1+k-2j)\ S_{r+1+k-j}(B)\ S_j(B).$$

Therefore,

$$\mathrm{trace}[P_{r-1}(B)\ B\ P_{k+1}(B)] = \sum_{j=0}^{k+1}(r+1+k-2j)\ S_{r+1+k-j}(B)\ S_j(B). \qquad \square$$

From now onwards $\|\cdot\|$ means the *Hilbert–Schmidt norm*, that is, if B is any linear operator

$$\|B\| := \sqrt{\mathrm{trace}\ (B^* \circ B)},$$

where B^* means the adjoint to the operator B.

Corollary 4.3. *Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator. For any $r \in \mathbb{Z}$ we have*

$$\|P_r(B)\|^2 = (n-r)\ S_r(B)^2 - 2\sum_{j=0}^{r-1}(r-j)\ S_j(B)\ S_{2r-j}(B)\ .$$

Proof. Multiply relation (3.2) on the right by $P_r(B)$ to get

$$P_r(B)P_r(B) = S_r(B)P_r(B) - P_{r-1}(B)\ B\ P_r(B).$$

Now the proof follows as a consequence of taking the trace and also by Proposition 3.7(d.1) and by Theorem 4.2. $\square$

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

COROLLARY 4.4. *Let $B \in \mathcal{L}(V)$ be a self-adjoint linear operator. For any $r \in \mathbb{Z}$ we have*

$$\|B\, P_{r-1}(B)\|^2 = r\, S_r(B)^2 - 2\sum_{j=0}^{r-1} (r-j)\, S_j(B)\, S_{2r-j}(B).$$

Proof. By (3.1),

$$P_r(B)^2 = BP_{r-1}(B)BP_{r-1}(B) - 2S_r(B)\, BP_{r-1}(B) + S_r(B)^2 I.$$

Now take the trace of this expression. Then, by Proposition 3.7(e) and since $BP_{r-1}(B) = P_{r-1}(B)B$, we have

$$\|P_r(B)\|^2 = \|B\, P_{r-1}(B)\|^2 + (n-2r)S_r(B)^2. \tag{4.3}$$

Corollary 4.3 finishes the proof. □

5. k-umbilicity in Riemannian Manifolds. Let $\mathbf{M}^n$ and $\overline{\mathbf{M}}^{n+1}$ be Riemannian manifolds of dimension n and $n+1$, respectively. Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion and denote its shape operator at a point q in $\mathbf{M}^n$ by $A : T_q\mathbf{M} \to T_q\mathbf{M}$ (by abuse of language A is also called the second fundamental form).

DEFINITION 5.1. An isometric immersion $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ is said to be k-umbilical at $q \in \mathbf{M}^n$, $k = 1, \dots, n-1$, if

$$AP_{k-1}(A) = \lambda I, \tag{5.1}$$

where $\lambda = \lambda(k)$ is a real function and I is the identity map of $T_q\mathbf{M}$.

Even though we do not know any term in (5.1), by Proposition 3.7(e) we can show that $\lambda(k) = \frac{k}{n} S_k(A)$. Hence, another way to define k-umbilicity is

$$AP_{k-1}(A) = \frac{k}{n} S_k(A)\, I. \tag{5.2}$$

By Proposition 3.6, we can get an equivalent definition of k-umbilicity:

$$P_k(A) = \left(1 - \frac{k}{n}\right) S_k(A)\, I. \tag{5.3}$$

DEFINITION 5.2. We say that an isometric immersion $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ is k-umbilical when it is k-umbilical at every point of $\mathbf{M}$.

DEFINITION 5.3. An isometric immersion $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ is said to be k-totally geodesic if

$$A\, P_{k-1}(A) = 0.$$

PROPOSITION 5.4. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion.*

x is k-totally geodesic if and only if x is k-umbilical with $S_k(A) = 0$.

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

Proof. Let x be k-umbilical such that $S_k(A) = 0$. Then by (5.2) x is k-totally geodesic.

Conversely, let x be k-totally geodesic. Then $AP_{k-1}(A) = 0 = 0\,I$. Thus every k-totally geodesic is k-umbilical. By Proposition 3.7(e) we have that $S_k(A) = 0$. □

It is known that the second fundamental form A of a totally umbilical isometric immersion $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ satisfies the Codazzi equation if and only if S_1 is constant. We generalize this fact for k-umbilical isometric immersions with $AP_{k-1}(A)$ in place of A. As a $(1, 1)$ symmetric tensor, $AP_{k-1}(A)$ is said to be Codazzi if $(\nabla_X AP_{k-1}(A))Y = (\nabla_Y AP_{k-1}(A))X$, where

$$(\nabla_X AP_{k-1}(A))(Y) = \nabla_Y (AP_{k-1}(A)(X)) - AP_{k-1}(A)(\nabla_X Y),$$

for any X, Y tangents to $\mathbf{M}$.

THEOREM 5.5. *Let $\mathbf{M}^n$ be a connected Riemannian manifold and $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ a k-umbilical isometric immersion. Then*

$$AP_{k-1}(A) \textit{ is Codazzi if and only if } S_k(A) \textit{ is constant.}$$

Proof. For any X, Y tangents to $\mathbf{M}^n$

$$(\nabla_X AP_{k-1}(A))(Y) = \nabla_Y (AP_{k-1}(A)(X)) - AP_{k-1}(A)(\nabla_X Y).$$

Since the immersion is k-umbilical

$$(\nabla_X AP_{k-1}(A))(Y) - (\nabla_Y AP_{k-1}(A))(X) = X\left(\frac{k}{n} S_k(A)\right)Y - Y\left(\frac{k}{n} S_k(A)\right)X. \tag{5.4}$$

If $AP_{k-1}(A)$ is Codazzi, then the left-hand side of (5.4) is zero. Hence,

$$X\left(\frac{k}{n} S_k(A)\right)Y - Y\left(\frac{k}{n} S_k(A)\right)X = 0.$$

Now if X and Y are chosen to be linearly independent, we get

$$X\left(\frac{k}{n} S_k(A)\right) = 0 = Y\left(\frac{k}{n} S_k(A)\right);$$

thus, $\frac{k}{n} S_k(A)$ is constant at every point of $\mathbf{M}^n$ and by the connectedness of $\mathbf{M}^n$ it follows that $\frac{k}{n} S_k(A)$ is constant in $\mathbf{M}^n$.

Conversely, if $\frac{k}{n} S_k(A)$ is constant in $\mathbf{M}^n$, then the right-hand side of (5.4) is zero. Thus,

$$(\nabla_X AP_{k-1}(A))(Y) - (\nabla_Y AP_{k-1}(A))(X) = 0.$$

Hence, $AP_{k-1}(A)$ is Codazzi. □

Now we are going to introduce an operator, defined on tangent spaces, which measures how much an isometric immersion fails to be k-umbilical.

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

DEFINITION 5.6. Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion and A its shape operator. For each $p \in \mathbf{M}$, the k-umbilicity operator

$$\phi_k : T_p\mathbf{M} \longrightarrow T_p\mathbf{M}$$

is defined by

$$\phi_k(X) := \frac{k}{n} S_k(A)\, X - AP_{k-1}(A)\, X, \quad \forall X \in T_p\mathbf{M},$$

where P_i is the ith Newton operator and $S_j(A)$ is the jth symmetric function associated to A.

REMARK 5.7. We must note that when $k = 1$ the operator $\phi_1(X) = \phi(X) = HX - AX$ was used in [**1**], where $\phi_1 \equiv 0$ if and only if the immersion is totally umbilical. This fact extends to k-umbilical immersions: by (5.2) $\phi_k \equiv 0$ if and only if the immersion is k-umbilical; in another words, the operator ϕ_k gives a measure of how much an isometric immersion fails to be k-umbilical.

PROPOSITION 5.8. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion and let A be its shape operator. Then the map ϕ_k satisfies the following:*

(a) *ϕ_k is self-adjoint;*

(b) *ϕ_k is simultaneously diagonalizable with A and if $\{e_1, \ldots, e_n\}$ is an orthonormal basis which diagonalizes A we have $\phi_k(e_i) = \mu_i e_i$, where*

$$\mu_i = S_k(A_i) - \left(1 - \frac{k}{n}\right) S_k.$$

Proof.

(a) Since A is self-adjoint, it follows that ϕ_k is self-adjoint, too.

(b) The proof follows by using that $Ae_i = \lambda_i\, e_i,\ i = 1, 2, \ldots, n$ and Proposition 3.7(i). □

PROPOSITION 5.9. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion and let A be its shape operator. Let ϕ_k be given by the Definition 5.6 and $\|.\|$ denotes the Hilbert–Schmidt norm. Then*

1. $\|\phi_k\|^2 = \sum_{i=1}^{n} \mu_i^2$, *where μ_i was defined in Proposition 5.8(b);*
2. $\|\phi_k\|^2 = \dfrac{k(n-k)}{n} S_k(A)^2 - 2\sum_{j=0}^{k-1} (k-j)\, S_j(A) S_{2k-j}(A);$
3. $\|\phi_k\|^2 = \dfrac{1}{n} \sum_{1 \leq i < j \leq n} (\lambda_i\, S_{k-1}(A_i) - \lambda_j\, S_{k-1}(A_j))^2.$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis such that $Ae_i = \lambda_i\, e_i$ for $i = 1, \ldots, n$. Here,

$$\|\phi_k\|^2 = \operatorname{trace}\left(\phi_k^* \circ \phi_k\right),$$

where ϕ_k^* is the adjoint of ϕ_k.

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

1. The proof follows from Proposition 5.8(a) and (b).
2. By part (1), Proposition 5.8(b) and Proposition 3.7(j) and (d.1) we get

$$\|\phi_k\|^2 = \sum_{i=1}^{n} S_k(A_i)^2 - \frac{(n-k)^2}{n} S_k(A)^2 \ ;$$

in another words,

$$\|\phi_k\|^2 = \operatorname{trace} \left(P_k(A)^2 - \frac{(n-k)^2}{n^2} \, S_k(A)^2 \, I \right).$$

From Propositions 3.6 and 3.7 we get

$$\|\phi_k\|^2 = ((n-k) S_k(A)^2 - \operatorname{trace}(P_{k-1}(A) \ A \ P_k(A))) - \frac{(n-k)^2}{n} \ S_k(A)^2.$$

Finally use Theorem 4.2.
3. We can see that $2S_2\big(AP_{k-1}(A)\big) = S_1 \left(AP_{k-1}(A)\right)^2 - \|AP_{k-1}(A)\|^2$. Our result then follows from Proposition 3.7(e) and Corollary 4.4. □

6. Characterizations of k-umbilical immersions. Here, $\overline{\mathbf{M}}^{n+1}$ will be a Riemannian manifold.

PROPOSITION 6.1. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion, A its shape operator and $\{e_1, \ldots, e_n\}$ an orthonormal basis which diagonalizes A at a point $q \in M$. The immersion is k-umbilical at q if and only if in Newton's Formula, $\sum_{j=1}^{n} \lambda_j S_{k-1}(A_j) = kS_k(A)$, each term in the sum is equal to $\frac{k}{n}$ $S_k(A)$ at q.*

Proof. The proof follows by using (5.2) and Proposition 3.7(i). □

REMARK 6.2. We see from above that an isometric immersion $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ is k-umbilical if and only if

$$\lambda_j \ S_{k-1}(A_j) = \frac{k}{n} \ S_k(A) \quad , \forall j.$$

COROLLARY 6.3. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion, A its shape operator and $\{e_1, \ldots, e_n\}$ an orthonormal basis which diagonalizes A at $q \in M$. The immersion is k-umbilical if and only if*

$$\sum_{j=0}^{k-1} (-1)^j \ S_j(A) \ \lambda_i^{k-j} + (-1)^k \ \frac{k}{n} \ S_k(A) = 0 \quad at\ q \in \mathbf{M}. \tag{6.1}$$

Proof. From definition of A_i and (2.4) we get

$$S_k(A_i) = \sum_{j=0}^{k} (-1)^j \ S_{k-j}(A) \ (\lambda_i)^j. \tag{6.2}$$

The proof now follows from identity (6.2) and Remark 6.2. □

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

REMARK 6.4. Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion. Then
x is one-umbilical if and only if $\lambda_i = \frac{1}{n} S_1$;
x is two-umbilical if and only if $\lambda_i^2 - S_1\lambda_i + \frac{2}{n} S_2 = 0$;
x is three-umbilical if and only if $\lambda_i^3 - S_1\lambda_i^2 + S_2\lambda_i - \frac{3}{n} S_3 = 0$.

7. Consequences of the k-umbilicity. It is clear that every one-umbilical immersion is a k-umbilical immersion, but the converse is not true.

THEOREM 7.1. *Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ be an isometric immersion and let A be its shape operator. If x is k-umbilical at $q \in \mathbf{M}$, then*

$$H_{k+1} = H_1\ H_k \text{ at } q.$$

The converse is true if all its principal curvatures are different from zero. Moreover, we have the following identity

$$S_1(A)S_{k+1}(A) - (k+2)S_{k+2}(A) = \left(\frac{n-k}{n}\right) S_k(A)\,\|A\|^2. \tag{7.1}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis which diagonalizes A at q and λ_i the eigenvalue corresponding to e_i. From the k-umbilicity of x and Remark 6.2 we have $\lambda_j^2\ S_{k-1}(A_j) = \frac{k}{n}\ S_k(A)\ \lambda_j$. By summing on j we have $\sum_{j=1}^n \lambda_j^2\ S_{k-1}(A) = \frac{k}{n}\ S_k(A)(\sum_{j=1}^n \lambda_j)$. Using Proposition 3.7(f) and (l) we get

$$S_1(A)S_k(A) - (k+1)S_{k+1}(A) = \frac{k}{n} S_1(A)S_k(A);$$

or equivalently

$$\binom{n}{k+1}\frac{S_{k+1}(A)}{\binom{n}{k+1}} = \frac{n-k}{k+1}\left(\frac{S_1(A)}{n}\right)\ \frac{S_k(A)}{\binom{n}{k}}\ \binom{n}{k}.$$

Hence,

$$H_{k+1} = H_1 H_k.$$

For the converse, suppose that there is a point $q \in M$ such that $H_{k+1} = H_1\ H_k$ and $\lambda_i \neq 0$ for each i at q. Thus,

$$(k+1)S_{k+1}(A) = \left(\frac{n-k}{n}\right) S_1(A)S_k(A);$$

reordering the last equality and making use of Proposition 3.7(f) we can rewrite it as

$$\operatorname{trace}\,(A^2P_{k-1}(A)) = \frac{k}{n} S_1(A)S_k(A);$$

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

by Proposition 3.7(l), we get

$$\sum_{j=1}^{n}\left(\lambda_j^2 S_{k-1}(A_j) - \lambda_j \frac{k}{n} S_k(A)\right) = 0.$$

Now, let $\{v_1, \ldots, v_n\}$ be a basis of T_qM and consider the following linear combination:

$$\sum_{j=1}^{n}\left(\lambda_j^2 S_{k-1}(A_j) - \lambda_j \frac{k}{n} S_k(A)\right) v_j = 0.$$

Thus, $\lambda_j(\lambda_j S_{k-1}(A_j) - \frac{k}{n} S_k(A)) = 0$, for each j. The proof now follows from Remark 6.2. □

PROPOSITION 7.2. *Every k-umbilical isometric immersion* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ *with a zero principal curvature at a point p is k-totally geodesic at p and has at least* $n - k + 1$ *principal curvatures equal to zero at p.*

Proof. First we will prove that at p

$$S_j(A) = 0, \quad \forall j \geq k \quad \text{and} \quad S_j(A_i) = 0, \quad \forall j \geq k, \forall i. \tag{7.2}$$

By hypothesis we can suppose $\lambda_\alpha = 0$, for some α. Substitute λ_i for $\lambda_\alpha = 0$ into (6.1) to obtain $S_k(A) = 0$. Since $H_{k+1} = H_1 H_k$, it follows that $S_{k+1}(A) = 0$. From (7.1) we have $S_{k+2}(A) = 0$. Now, by Propositions 6.1 and 3.4(c) we get $S_k(A_i) = 0$, $\forall i$; again by Proposition 3.4(c), we have $S_{k+1}(A_i) = 0, \forall i$ and thus $S_{k+2}(A_i) = 0$, $\forall i$. The proof of (7.2) follows from a recursive process using the same arguments.

Now, we are going to show that there exist at least $n - k + 1$ principal curvatures equal to zero. We had seen that $S_n(A) = 0$; then at least one principal curvature is null; denote it by $\lambda_{j_1} = \lambda_\alpha = 0$. We also had seen that $S_{n-1}(A_j) = 0$, for any j. Thus

$$S_{n-1}(A_{j_1}) = \prod_{\substack{i=1 \\ i \neq j_1}}^{n} \lambda_i = 0,$$

and hence there exists another null principal curvature and we denote it by $\lambda_{j_2} = 0$. Now, by Proposition 3.9(d) one gets $S_r(A_{j_1}, A_\theta) = 0$. Taking $r = n - 2$ and $\theta = j_2$ we get

$$S_{n-2}(A_{j_1}, A_{j_2}) = \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{n} \lambda_i = 0;$$

therefore, there exists another null principal curvature which we will denote by $\lambda_{j_3} = 0$. Again, by Proposition 3.9(g), one gets $S_r(A_{j_1}, A_{j_2}, A_\theta) = 0$. Taking $r = n - 3$ and $\theta = j_3$ we get

$$S_{n-3}(A_{j_1}, A_{j_2}, A_{j_3}) = \prod_{\substack{i=1 \\ i \neq j_1, j_2, j_3}}^{n} \lambda_i = 0.$$

Therefore, there exists another null principal curvature which we will denote by $\lambda_{j_4} = 0$.

Continuing in this fashion, we will show that there exist $n - k + 1$ null principal curvatures. □

COROLLARY 7.3. *Let* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$, $n \geq 3$ *be a k-umbilical isometric immersion. If* $H_n = 0$ *at one point, then* $H_j = 0$ *at the same point,* $\forall j \geq k$.

8. Two-umbilical Isometric Immersions. Here, we obtain the principal curvatures $\lambda_1, \ldots, \lambda_n$, of a two-umbilical isometric immersion in a Riemannian manifold $\overline{\mathbf{M}}^{n+1}$, in terms of S_1 and S_2, as roots of $\lambda_i^2 - S_1\lambda_i + \frac{2}{n}S_2 = 0$ (cf. Remark 6.4).

THEOREM 8.1 (Determination of the principal curvatures of the two-umbilical isometric immersions). *Let* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$ $(n \geq 3)$ *be any two-umbilical isometric immersion.*

(a) *If its principal curvatures are distinct, then they are given by*

$$\lambda_{i_1} = \cdots = \lambda_{i_r} = ((n - (r + 1))/(n - 2r))S_1$$

and

$$\lambda_{i_{r+1}} = \cdots = \lambda_{i_n} = -((r - 1)/(n - 2r))S_1,$$

where $r \in \{0, 1, 2, \ldots, [[\frac{n}{2}]]^4\}$ *or* $r \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}$, *according to whether n is odd or even, respectively.*

(b) *If its principal curvatures are equal, then n is even and its principal curvatures are given by*

$$\lambda_{i_1} = \cdots = \lambda_{i_{\frac{n}{2}}} = \sqrt{2/n}\,\sqrt{-S_2}$$

and

$$\lambda_{i_{\frac{n}{2}+1}} = \cdots = \lambda_{i_n} = -\sqrt{2/n}\,\sqrt{-S_2}\,.$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis which diagonalizes A. Since x is two-umbilical, by Remark 6.4 each λ_i satisfies $\lambda_i^2 - S_1(A)\lambda_i + \frac{2}{n}S_2(A) = 0$; then each λ_i has at most two distinct principal curvatures:

$$\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_r} = \frac{S_1(A) + \sqrt{S_1(A)^2 - \frac{8}{n}S_2(A)}}{2};$$

$$\lambda_{i_{r+1}} = \lambda_{i_{r+2}} = \cdots = \lambda_{i_n} = \frac{S_1(A) - \sqrt{S_1(A)^2 - \frac{8}{n}S_2(A)}}{2}.$$

We then obtain

$$S_1(A) = \frac{nS_1(A) + (2r - n)\sqrt{S_1(A)^2 - \frac{8}{n}\,S_2(A)}}{2}.$$

[4] $[[x]]$ is the largest integer not exceeding x.

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

Suppose that $r \neq \frac{n}{2}$. Then $\sqrt{S_1(A)^2 - \frac{8}{n}S_2} = \frac{(n-2)S_1(A)}{n-2r}$. Now we have

(i) If $n \geq 2$, $S_1 \geq 0$ and $r < \frac{n}{2}$, then the principal curvatures are given by

$$\lambda_{i_1} = \cdots = \lambda_{i_r} = \left[\frac{n-(r+1)}{n-2r}\right] S_1(A),$$
$$\lambda_{i_{r+1}} = \cdots = \lambda_{i_n} = -\left(\frac{r-1}{n-2r}\right) S_1(A).$$

(ii) If $n \geq 2$, $S_1(A) \leq 0$ and $r > \frac{n}{2}$, then the principal curvatures are given by

$$\lambda_{i_1} = \cdots = \lambda_{i_r} = \left[\frac{n-(r+1)}{n-2r}\right] (-S_1(A)),$$
$$\lambda_{i_{r+1}} = \cdots = \lambda_{i_n} = -\left(\frac{r-1}{n-2r}\right) (-S_1(A)).$$

Now suppose that $r = \frac{n}{2}$ and $n \geq 4$. Then $(n-2)S_1(A) = 0$ and we obtain

$$\lambda_{i_1} = \cdots = \lambda_{i_{\frac{n}{2}}} = \sqrt{\frac{2}{n}}\sqrt{-S_2(A)},$$
$$\lambda_{i_{\frac{n}{2}+1}} = \cdots = \lambda_{i_n} = -\sqrt{\frac{2}{n}}\sqrt{-S_2(A)}.$$

$\square$

Corollary 8.2. *For any odd integer n, every minimal two-umbilical isometric immersion of* $\mathbf{M}^n$ *is one-totally geodesic.*

From Remark 6.4 we can see that any two-umbilical immersion has no more that two principal curvatures, and Theorem 4.1 says that the maximum number of principal curvatures of any Einstein hypersurface immersed in a space form is two. Then arise a question: Is any two-umbilical manifold immersed in a space form an Einstein manifold? The answer is yes and it will be proved in the next theorem.

Theorem 8.3 (A characterization of Einstein hypersurfaces). *Let* $\mathbf{M}^n$ *be a connected Riemannian manifold and* $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}(c)$, $n \geq 3$, *an isometric immersion. Then*

$$\mathbf{M}^n \text{ is Einstein if and only if } x \text{ is two-umbilical.}$$

Moreover, in this case $Ric(X, Y) = (c(n-1) + \frac{2S_2}{n}) < X, Y >$, *with* S_2 *constant.*

Proof. Suppose $\mathbf{M}^n$ is Einstein. From Gauss equation

$$\mathrm{Ric}(X, Y) - c(n-1)\langle X, Y\rangle = \langle AP_1(A)X, Y\rangle;$$

since $\mathbf{M}^n$ is Einstein, then $\mathrm{Ric}(X, Y) = \lambda\langle X, Y\rangle$, hence

$$\langle AP_1(A)X, Y\rangle = (\lambda - c(n-1))\langle X, Y\rangle.$$

Thus, $AP_1(A) = (\lambda - c(n-1))\, I$ and the proof follows by using (5.1).

Now suppose x is two-umbilical. By Gauss equation

$$\mathrm{Ric}(X, Y) = (c(n-1) + \frac{2S_2(A)}{n})\ \langle X, Y\rangle;$$

therefore, $\mathbf{M}^n$ is an Einstein manifold. Moreover, because $\mathbf{M}^n$ is connected $(c(n-1) + \frac{2S_2(A)}{n})$ is constant, hence S_2 is constant. $\square$

REMARK 8.4. Let $x : \mathbf{M}^n \longrightarrow \overline{\mathbf{M}}^{n+1}$, $n \geq 2$ be an isometric immersion and $X, Y \in T\mathbf{M}$. The Gauss equation and the definition of ϕ_2 (Definition 5.6, for $k = 2$) give that

$$\mathrm{Ric}(X, Y) - \overline{\mathrm{Ric}}\,(X, Y) = \frac{2S_2(A)}{n}\ \langle X, Y\rangle - \langle \phi_2 X, Y\rangle.$$

From this we can see that the operator ϕ_2 gives a measure of how much a manifold $\mathbf{M}$ immersed isometrically in a space form $\overline{\mathbf{M}}^{n+1}(c)$ fails to be an Einstein hypersurface.

9. Examples and Description of two-umbilical Hypersurfaces. An example of two-totally geodesic immersion which is not one-totally geodesic is given by

$$S^1(r) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}.$$

EXAMPLE 9.1 (Clifford's hypersurfaces). Given $n_1, n_2 \in \mathbb{N}$ and $r_1, r_2 > 0$. Consider

$$S^{n_1}(r_1) = \{u \in \mathbb{R}^{n_1+1} \ :\ \|u\| = r_1\},$$
$$S^{n_2}(r_2) = \{u \in \mathbb{R}^{n_2+1} \ :\ \|u\| = r_2\}$$

and

$$S^{n_1}(r_1) \times S^{n_2}(r_2) = \{(u, v) \in \mathbb{R}^{n_1+n_2+2} \ :\ u \in S^{n_1}(r_1), v \in S^{n_2}(r_2)\}.$$

$S^{n_1}(r_1) \times S^{n_2}(r_2)$ is a hypersurface of $S^{n_1+n_2+1}(1) \subset \mathbb{R}^{n_1+n_2+2}$ with $r_1^2 + r_2^2 = 1$ and it is called a Clifford's hypersurface.

PROPOSITION 9.2. *For any $n \in \mathbb{N}$, $0 < r < 1$ and fixed $m \in \{1, \ldots, n-1\}$, the Clifford's hypersurface $S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \longrightarrow S^{n+1}(1)$ has its principal curvatures given by*

$$\lambda_1 = \cdots = \lambda_{n-m} = \frac{\sqrt{1-r^2}}{r},$$
$$\lambda_{n-m+1} = \cdots = \lambda_n = \frac{-r}{\sqrt{1-r^2}}.$$

THEOREM 9.3 (Examples of two-umbilical hypersurfaces in $S^{n+1}(1)$). *There exists a countably infinite family of two-umbilical hypersurfaces in the Euclidean sphere $S^{n+1}(1)$. More precisely: for any $n \geq 4$ and for every $m \in \{2, \ldots, n-2\}$, the Clifford's hypersurface*

$$S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow S^{n+1}(1) \quad \textit{is two-umbilical if and only if} \quad r = \sqrt{\frac{n-m-1}{n-2}}.$$

Furthermore, the norm of its second fundamental form A, the associated polynomial S_k and its Ricci curvature are given by

$$S_k = \left(\frac{m-1}{n-m-1}\right)^{\frac{k}{2}} \sum_{i=0}^{k} (-1)^i \left(\frac{n-m-1}{m-1}\right)^i \binom{n-m}{k-i} \binom{m}{i};$$
$$\|A\|^2 = \frac{n+m(n-4)(n-m)}{(n-m-1)(m-1)};$$
$$Ric(e_j) = n-2, \quad j = 1, 2, \ldots, n.$$

Proof. By Proposition 5.9(2) we obtain

$$\|\phi_2\|^2 = \left(\frac{n-2}{n}\right) S_2(A_\eta)^2 - S_1(A_\eta)\, S_3(A_\eta) - 2S_4(A_\eta);$$

a short calculation shows that

$$\|\phi_2\|^2 = \frac{-m(m-n)}{2nr^4(r^2-1)^2}\,((n-2)r^2 - (n-m-1)).$$

The radius r is obtained as a consequence of Remark 5.7 and we then conclude the proof. □

REMARK 9.4. A straightforward computation shows that the Clifford's hypersurface with the above radius r has

$$S_1 = \frac{2m-n}{m-1}\sqrt{\frac{m-1}{n-m-1}},$$
$$S_2 = \frac{-n}{2!},$$
$$S_3 = \frac{-(n-2)(2m-n)}{3!(m-1)}\sqrt{\frac{m-1}{n-m-1}};$$

we then show that the hypersurfaces in Theorem 9.3 satisfy the condition $H_3 = H_1 H_2$ given in Theorem 7.1.

REMARK 9.5. If in Theorem 9.3 we make $n = 2\eta$ and $m = \eta + j$, we obtain a family of two-umbilical embedding:

$$S^{\eta-j}(r) \times S^{\eta+j}(\sqrt{1-r^2}) \hookrightarrow S^{2\eta+1}(1),$$

where $r = \sqrt{\frac{\eta-j-1}{2\eta-2}}$, $\eta \geq 2$ and $j \in \{0, 1, 2, \ldots, \eta-2\}$.
In this case, $S_1(A) = \frac{2j}{\eta+j-1}\sqrt{\frac{\eta+j-1}{\eta-j-1}}$.

It is worth noting that $S_1(A) = 0 \Longleftrightarrow j = 0 \Longleftrightarrow r = \sqrt{\frac{1}{2}}$. Therefore, for every $\eta \in [2, \infty) \cap \mathbb{N}$ we have a minimal two-umbilical embedding

$$S^\eta\left(\frac{1}{\sqrt{2}}\right) \times S^\eta\left(\frac{1}{\sqrt{2}}\right) \longrightarrow S^{2\eta+1}(1).$$

https://doi.org/10.1017/S0017089508004643 Published online by Cambridge University Press

Thus, we have obtained a countably infinite family of minimal two-umbilical embeddings in the Euclidean sphere. Furthermore, for each $j = 1, 2, \dots, \eta$

$$S_{2j-1}(A) = 0;$$
$$S_{2j}(A) = (-1)^j \binom{\eta}{j}.$$

THEOREM 9.6 (Description of the two-umbilical hypersurfaces in a space form). *Let* $\mathbf{M}^n$ *be a two-umbilical hypersurface in* $\overline{\mathbf{M}}^{n+1}(c)$, $n > 2$. *Then*

(a) $\mathbf{M}$ *is two-totally geodesic or*

(b) $\mathbf{M}$ *is one-umbilical or*

(c) *if* $c > 0$, *then* $\mathbf{M}$ *is locally a standard product embedding of*

$$S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow S^{n+1}(1),$$

where $r = \sqrt{\frac{n-m-1}{n-2}}$. *In particular, when the embedding is minimal we have*

$$S^k\left(\frac{1}{\sqrt{2}}\right) \times S^k\left(\frac{1}{\sqrt{2}}\right) \hookrightarrow S^{2k+1}(1),$$

where $n = 2k$;

(d) *If* $c < 0$, *then* $\mathbf{M}$ *is geodesic hyperspheres, horospheres, totally geodesic hyperplanes and their equidistant hypersurfaces, tubes around totally geodesic subspaces of dimension at least one* (*in another words, it is locally a standard product embedding given by* $S^k \times \mathbb{H}^{n-k}$);

(e) *if* $c = 0$, *then* $\mathbf{M}$ *is locally hyperspheres, hyperplanes or a standard product embedding given by* $S^k \times \mathbb{R}^{n-k}$.

Proof. Theorem 8.1 on the determination of principal curvatures of two-umbilical hypersurfaces says that to know any two-umbilical hypersurface we need only to know $S_1(A)$ or $S_2(A)$. There are two cases to consider:

$$S_1(A)\, S_2(A) = 0 \qquad \text{or} \qquad S_1(A)\, S_2(A) \neq 0,$$

where A is the shape operator of x. We proceed with the study of each case.

(a) Suppose $S_1(A)\, S_2(A) = 0$, with $S_2 = 0$ at one point. Since S_2 is constant (Theorem 8.3) then $S_2 \equiv 0$. By Proposition 5.4 we have that $\mathbf{M}$ is two-totally geodesic. Now, we will show that $S_1 \equiv 0$; in fact if Theorem 8.1(b) is valid then we get $S_1 \equiv 0$; if Theorem 8.1(a) is valid then we obtain $S_2 = \frac{-n(r-1)[n-(r+1)]}{2(n-2r)^2} S_1^2$, which implies either $S_1 \equiv 0$ or $r = 1$. In the first case we obtain one-totally geodesic hypersurfaces, in the other case we obtain two-totally geodesic hypersurfaces (only one mean curvature is non-null).

(b) Suppose $S_1(A)S_2(A) \neq 0$. Trivially any one-umbilical hypersurface (so it is two-umbilical and not minimal) satisfies this condition. Thus (b) is satisfied.

(c) Suppose $S_1(A)S_2(A) \neq 0$ and $\mathbf{M}$ is not one-umbilical. Theorem 8.1(b) cannot hold, because $\mathbf{M}$ would be minimal which is a contradiction. Since Theorem 8.1(a) is valid we can see that $S_2 = \frac{-n(r-1)[n-(r+1)]}{2(n-2r)^2} S_1^2$ and by Theorem 8.3 we get that $S_1(A)$ is constant. In this case the immersion is isoparametric. By using the Gauss–Codazzi equations, Cartan **[5]** proved that $\mathbf{M}$ is locally a standard product embedding of two spheres with appropriate radii. By Theorem 9.3 we get the radii. If $S_2 \neq 0$ but $S_1 = 0$, we can see that only Theorem 8.1(b) can hold, and so $S_2(A) < 0$. If you suppose that $S_1 = 0$ and Theorem 8.1(a) is valid, then all its principal curvatures are equal to zero, and it follows that $S_2(A) = 0$, a contradiction. Hence, our hypersurface is minimal with two principal curvatures of multiplicity greater than two; the proof follows as a consequence of the corollary given in [**9**, Page 153] and Theorem 9.3.

(d) Using the Gauss–Codazzi equations, Cartan [**5**] proved that an isoparametric hypersurface $x : \mathbf{M}^n \to \mathbf{M}^{n+1}(-1)$ is either one-umbilical or has exactly two constant principal curvatures (see also [**6**]). This leads to the above classification.

As a consequence, all two-umbilical hypersurfaces in hyperbolic spaces are open parts of homogeneous hypersurfaces.

(e) By the same argument as given in (d). (See another proof in [**10**]). □

REFERENCES

1. H. Alencar and M. do Carmo, Hypersurfaces with constant mean curvature in spheres, *Proc. Am. Math. Soc.* **120**(4) (1994), 1223–1229.

2. A. C. Asperti and M. Dajczer, N-dimensional submanifolds of $\mathbb{R}^{N+1}$ and S^{N+2}, *Illinois J. Math.* **28**(4) (1984), 621–645.

3. J. L. Barbosa and A. G. Colares, Stability of hypersurfaces with constant r-mean curvature, *Ann. Global Anal. Geom.* **15** (1997), 277–297.

4. A. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, 1987).

5. E. Cartan, Familles de Surfaces Isoparamétriques dans les Espaces à Courbure Constant, *Annali di Mat.* **17** (1938), 177–191.

6. M. Dajczer and M. do Carmo, Rotation hypersurfaces in spaces of constant curvature, *Trans. Am. Math. Soc.* **277**(2) (1983), 685–709.

7. M. Dajczer and L. Florit, On Chen's basic equality, *Illinois J. Math.* **42**(1) (1998) 97–106.

8. A. Fialkow, Hypersurfaces of a space of constant curvature, *Ann. Math., 2nd Ser.* **39**(4) (1938), 762–785.

9. T. Otsuki, Minimal hypersurfaces in a Riemannian manifold of constant curvature, *Am. J. Math.* **92** (1970), 145–173.

10. H. Reckziegel, On the problem whether the image of a given differentiable map into a Riemannian manifold is contained in a submanifold with parallel second fundamental form, *J. Reine Angew. Math.* **325** (1981), 87–104.

11. R. Walter, Compact hypersurfaces with a constant higher mean curvature function, *Math. Ann.* **270** (1985), 125–145.