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THE SECOND DUAL OF A C^*-TERNARY RING

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ABSTRACT. The Arens extension of the triple product of an associative triple system is studied. Using a representation theorem for C^*-ternary rings due to Zettl, it is shown that the second dual of a C^*-ternary ring is itself a C^*-ternary ring

§1 **Introduction.** The fact that the second dual of a Banach algebra can be made into a Banach algebra has played a useful role in the general theory of Banach algebras (Bonsall–Duncan [3]).

In particular the study of C^*-algebras has been partially reduced to the study of W^*-algebras by the following:

THEOREM A. (Sherman, Takeda, Tomita). *The second dual of a C^*-algebra is a C^*-algebra.*

The original proof of Theorem A was based on the universal representation and Gelfand–Naimark–Segal constructions. A later proof was based on the numerical range (Bonsall–Duncan [2]).

A (concrete) C^*-algebra is a norm-closed self-adjoint sub-algebra of $\mathcal{B}(H)$, the bounded linear operators on a complex Hilbert space H. Recently there has been interest in considering subspaces of $\mathcal{B}(H, K)$, the bounded linear operators from one Hilbert space H to another K, which are closed under a triple product of its elements, e.g. (1) $(A, B, C) \to AB^*C$, (2) $A \to AA^*A$. In the literature these spaces have been called ternary algebras (Hestenes [8]), and J^*-algebras (Harris [7]), respectively.

J^*-algebras are related to the study of infinite dimensional bounded symmetric domains, and ternary algebras provide an appropriate setting for the spectral theory of certain differential operators (Hestenes [9]). These spaces have also appeared naturally as the range of contractive projections on C^*-algebras (Friedman–Russo [6]).

A detailed study of the structure of ternary subalgebras of $\mathcal{B}(H, K)$ which are closed in the norm topology or in the weak operator topology has been undertaken by Zettl [12]. His main results are analogs of the representation theorems of Gelfand–Naimark and Sakai.

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The purpose of this paper is to develop an analog of Theorem A for a C^*-ternary ring, which is the abstract version of a norm closed ternary algebra of operators. In §2 we use a general construction to show how the second dual of an associative triple system can itself be made into an associative triple system. In §3 we prove that the second dual of a C^*-ternary ring is itself a C^*-ternary ring.

§2. **The second dual of an associative triple system.** Let M be a complex linear space endowed with a mapping $[.,.,.]: M\times M\times M\to M$ which is linear in the first and third variables and conjugate linear in the second variable. M is called an *associative triple system* (ATS) of the second kind (AT2) if the following is satisfied:

$$[uv[xyz]]=[u[yxv]z]=[[uvx]yz]. \tag{2.1}$$

Associative triple systems of the second kind have been studied by Loos [11] and Hestenes [8]. An associative triple system of the first kind (AT1) is a pair $(M,[.,.,.])$ in which $[.,.,.]$ is trilinear and in which (2.1) is replaced by

$$[uv[xyz]]=[u[vxy]z]=[[uvx]yz]. \tag{2.2}$$

These have been studied by Lister [10].

Any complex associative algebra A (resp. associative involutive algebra B) becomes an AT1 (resp. AT2) if we define $[xyz]=xyz$ (resp xy^*z). More generally any linear subspace of A (resp. B) which is closed under the triple product $[xyz]$ just defined is an AT1 (resp. AT2). We shall say that an AT1 (resp. AT2) $(M,[...])$ is embedded in A (resp. B) if there is a linear isomorphism ϕ of M into A (resp. B) satisfying $\phi([xyz])=\phi(x)\phi(y)\phi(z)$ (resp $\phi(x)\phi(y)^*\phi(z)$) for x, y, z in M.

It is known that an AT1 can be embedded in an associative algebra (Lister [10]) and that an AT2 can be embedded in an associative involutive algebra (Loos [11]).

Suppose an AT1 M is embedded in an associative algebra A. Then an elementary argument shows that M'', the second dual of M, considered as a subspace of A'', is closed under the triple product $F\circ G\circ H$ where $\circ$ denotes the Arens product on A''. Similarly, if an AT2 M is embedded in an associative involutive algebra B and the Arens multiplication on B'' is regular so that B'' is involutive [2; p. 107], then M'' is closed under the triple product $F\circ G^*\circ H$.

More generally, we have the following.

THEOREM 1. *Let M be an associative triple system. Then the triple product $[\ ]$ on M extends to a triple product $[\ ]''$ on M'' with the following properties*:

(a) *if* $[M,[\ ])$ *is* AT1, *then* $(M'',[\ ]'')$ *is* AT1.

(b) *if* $(M,[\ ])$ *is* AT1 *and is embedded in an associative algebra* A, *then* $(M'',[\ ]'')$ *is embedded in* A''

(c) *if* $(M, [\ ])$ *is* AT2 *and is embedded in an involutive associative algebra* B *with regular Arens multiplication on* B'', *then* $(M'', [\ ]'')$ *is an* AT2 *which is embedded in* B''.

REMARK. Although we believe it to be true we are unable to verify:

(d) if $(M, [\ ])$ is AT2, then $(M'', [\ ]'')$ is AT2.

This seems to require very deep properties of the Arens multiplication. The statements in Theorem 1 suffice for our purposes in this paper, i.e. Theorem 2.

Proof. Identifying M with its canonical image in M'', we shall extend the triple product on M to a function $\mu_3: M''\times M''\times M''\to M''$. Assume first that M is AT1.

The function μ_3 is obtained inductively by the following construction which is due to R. Arens [1]. Define:

$$\mu_0: M'\times M\times M\to M';\ \langle\mu_0(f, x, y), z\rangle = \langle f, [xyz]\rangle$$
$$\text{for}\quad f\in M',\ x, y, z\in M.$$
$$\mu_1: M''\times M'\times M\to M';\ \langle\mu_1(F, f, x), y\rangle = \langle F, \mu_0(f, x, y)\rangle$$
$$\text{for}\quad F\in M'',\ f\in M',\ x, y\in M.$$
$$\mu_2: M''\times M''\times M'\to M';\ \langle\mu_2(F, G, f), x\rangle = \langle F, \mu_1(G, f, x)\rangle$$
$$\text{for}\quad F, G\in M'',\ f\in M',\ x\in M.$$
$$\mu_3: M''\times M''\times M''\to M'';\ \langle\mu_3(F, G, H), f\rangle = \langle F, \mu_2(G, H, f)\rangle$$
$$\text{for}\quad F, G, H\in M'',\ f\in M'.$$

Clearly, μ_3 is an extension of $[.,.,.]$ which is linear in each variable. To prove (a), it remains to verify (2.2) for μ_3 i.e.,

$$\begin{aligned}\mu_3(F, G, \mu_3(H, K, L)) &= \mu_3(F, \mu_3(G, H, K), L)\\ &= \mu_3(\mu_3(F, G, H), K, L)\quad\text{for}\quad F, G, H, K, L\in M''.\end{aligned}\tag{2.3}$$

This is a straightforward but tedious application of the definition of μ_3.

The proof of (b) is entirely similar to that of (c). To prove (c) we define inductively functions μ_0^*, μ_1^*, μ_2^*, μ_3^* as before except that the formulas for μ_1^* and μ_2^* are complex conjugates of the corresponding formulas for μ_1 and μ_2. This makes μ_3^* an extension of $[\ ]$ which is linear in the first and third positions and conjugate linear in the second position. Suppose now that M is embedded in an associative involutive algebra B so that M'' is included in B''. To complete the proof of Theorem 1, it must be shown that

$$\mu_3^*(F, G, H) = F\circ G^*\circ H,\quad\text{for}\quad F, G, H\in M''\tag{2.4}$$

where $F\circ G$ and G^* denote the usual Arens multiplication and involution respectively on A''.

The usual Arens multiplication $F \circ G$ on A'' can be defined inductively as follows [1]:

$$\nu_0 : A' \times A \to A'; \ \langle \nu_0(f, x), y \rangle = \langle f, xy \rangle$$
$$\text{for} \quad f \in A', \ x, y \in A.$$
$$\nu_1 : A'' \times A' \to A'; \ \langle \nu_1(F, f), x \rangle = \langle F, \nu_0(f, x) \rangle$$
$$\text{for} \quad F \in A'', \ f \in A', \ x \in A.$$
$$\nu_2 : A'' \times A'' \to A''; \ \langle \nu_2(F, G), f \rangle = \langle F, \nu_1(G, f) \rangle$$
$$\text{for} \quad F, G \in A'', \ f \in A'.$$

Then $F \circ G = \nu_2(F, G)$; and $G^* \in A''$ is defined by

$$\langle G^*, f \rangle = \overline{\langle G, f^* \rangle}$$

where

$$\langle f^*, x \rangle = \overline{\langle f, x^* \rangle}.$$

We proceed to the proof of (2.4). Let $f \in M'$. We must show

$$\langle \mu_3^*(F, G, H), f \rangle = \langle F \circ G^* \circ H, f \rangle \tag{2.5}$$

By the above definitions, (2.5) is equivalent to each of the following statements:

$$\langle F, \mu_2^*(G, H, f) \rangle = \langle F, \nu_1(G^* \circ H, f) \rangle;$$
$$\langle \mu_2^*(G, H, f), x \rangle = \langle \nu_1(G^* \circ H, f), x \rangle \quad \text{for} \quad x \in M;$$
$$\langle G, \mu_1^*(H, f, x) \rangle = \overline{\langle G, \nu_1(H, \nu_0(f, x))^* \rangle};$$
$$\langle \mu_1^*(H, f, x), y \rangle = \langle \nu_1(H, \nu_0(f, x)), y^* \rangle \quad \text{for} \quad y \in M;$$
$$\langle H, \mu_0^*(f, x, y) \rangle = \langle H, \nu_0(\nu_0(f, x), y^*) \rangle;$$
$$\langle \mu_0^*(f, x, y), z \rangle = \langle \nu_0(\nu_0(f, x), y^*), z \rangle \quad \text{for} \quad z \in M;$$
$$\langle f, [xyz] \rangle = \langle f, xy^*z \rangle. \tag{2.6}$$

Since M is embedded in B, (2.6) holds, so that (2.4) is proved. This completes the proof of Theorem 1.

§3. C^*-**ternary rings.** In this section we show that the second dual of a C^*-ternary ring is itself a C^*-ternary ring.

If an ATS M has a norm satisfying

$$\|[xyz]\| \leq \|x\| \, \|y\| \, \|z\| \quad \text{for} \quad x, y, z \in M, \tag{3.1}$$

it is called a *normed* ATS. It is clear from Theorem 1 that the second normed dual of a normed ATS satisfies the norm inequality (3.1).

A C^*-*ternary ring* is a complete normed AT2 M satisfying $\|[xxx]\| = \|x\|^3$ for each x in M. If in addition M is the dual of a Banach space it is called a W^*-ternary ring. H. Zettl [12] has proved the following:

REPRESENTATION THEOREM (Zettl). *Let M be a C^*-ternary ring. Then there exists a linear map $T:M \to M$ with $T^2 = I$ and $T([xyz]) = [Tx, y, z] = [x, Ty, z] = [x, y, Tz]$ and there exist Hilbert spaces H, K and a linear isometry $U:M \to \mathcal{B}(H, K)$ such that $U(T[xyz]) = U(x)U(y)^*U(z)$.*

In the proof of this theorem, it is shown that a C^*-ternary ring M can be made into a Hilbert module over a C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-valued inner product given by

$$\langle x \mid y \rangle = a(Tx, y)$$

for some conjugate bilinear form $a:M \times M \to \mathcal{A}$ with $\|a\| \le 1$. Therefore, for $x \in M$,

$$\|x\|^2 = \|\langle x \mid x \rangle\| = \|a(Tx, x)\| \le \|Tx\| \, \|x\|.$$

And so, $\|x\| \le \|Tx\|$. Since $T^2 = I$, T is an isometry.

It follows immediately from the representation theorem that if we equip a C^*-ternary ring M with a new ternary product $[xyz]_T = T[xyz]$ then U is a linear isometry of M into $\mathcal{B}(H, K)$ which is a ternary isomorphism i.e.

$$U([xyz]_T) = U(x)U(y)^*U(z).$$

Let $\sigma:\mathcal{B}(H, K) \to \mathcal{B}(H \oplus K)$ be the map which takes the element a into the operator matrix $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$. Then σ is a linear isometry satisfying $\sigma(ab^*c) = \sigma(a)\sigma(b)^*\sigma(c)$ for a, b, c in $\mathcal{B}(H, K)$. Therefore the composition $\sigma \circ U$ is an isometric embedding of M with ternary product $[.,.,.]_T$ into the C^*-algebra $A = \mathcal{B}(H \oplus K)$. It follows that M'' with the ternary product $[.,.,.]''_T$ given by Theorem 1 is isometrically embedded in the C^*-algebra A''. Therefore by part (c) of Theorem 1, for $F \in M''$,

$$\|[F, F, F]''_T\| = \|F \circ F^* \circ F\| = \|F\|^3.$$

It is easy to show that

$$[F, G, H]''_T = T''([F, G, H]'') \quad \text{for} \quad F, G, H \in M'',$$

where $[F, G, H]''$ is the triple product on M''. Since T is an isometry, $\|F\|^3 = \|[F, F, F]''_T\| = \|T''[F, F, F]''\| = \|[F, F, F]''\|$. We have proved:

THEOREM 2. *The second dual of a C^*-ternary ring is a C^*-ternary ring.*

We conclude by giving an alternative proof of Theorem 2 which avoids the Arens product but uses the universal representation of a C^*-algebra. This proof is based on the following Lemma.

LEMMA. *Let $\mathcal{R}$ be a norm closed ternary subalgebra of $\mathcal{B}(H)$, let A be the C^*-algebra generated by $\mathcal{R}$ and let π be the universal representation of A. Then, identifying $\mathcal{R}$ with its canonical image in $\mathcal{R}''$, the map $\pi/\mathcal{R}$ extends to an isometry π'' of $\mathcal{R}''$ onto the closure $\mathcal{S}$ of $\pi(\mathcal{R})$ in the weak operator topology. The map π'' is a homeomorphism in the weak * topology of $\mathcal{R}''$ and the weak operator topology of $\mathcal{S}$.*

Proof. As noted by Zettl, $\mathcal{S}$ is a weakly closed ternary algebra and a Kaplansky density theorem holds: the unit ball of $\pi(\mathcal{R})$ is weakly dense in the unit ball of $\mathcal{S}$ [12]. By the Hahn Banach theorem and the properties of π each $f \in \pi(\mathcal{R})'$ is ultraweakly continuous so extends uniquely to an ultraweakly continuous functional $\tilde{f}$ on $\mathcal{S}$, which by the Kaplansky density theorem satisfies $\|f\| = \|\tilde{f}\|$. The map $f \to \tilde{f}$ is an isometry of $\pi(\mathcal{R})'$ onto the set $\mathcal{S}_*$ of all ultraweakly continuous linear functionals on $\mathcal{S}$. Its adjoint then gives an isometry of $\mathcal{S}$ onto $\pi(\mathcal{R})''$ which carries $\pi(\mathcal{R})$ onto the canonical image of $\pi(\mathcal{R})$ in $\pi(\mathcal{R})''$. We have used Dixmier [4: p. 41] and [5: §12.1]. This now yields the following:

Second Proof of Theorem 2. If M is a C^*-ternary ring and U and σ are as defined previously in this section, then M is isometric to the norm closed ternary subalgebra $\mathcal{R} \equiv \sigma(U(M))$ of $\mathcal{B}(H \oplus K)$. It follows that M'' is isometric to $\mathcal{R}''$, which by the lemma is a C^*-ternary ring.

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