Theorems connected with the Differentiation of a Circulant.

By Professor W. H. METZLER.

(*Read and Received 9th May 1919*).

1. In 1909 Dr Thomas Muir,* in a paper on the above topic, gave several theorems involving the derivation of a circulant, and it is the writer's purpose in this paper to extend these investigations with a number of other results.

2. Let the imaginary factors of the circulant C be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$, where $\alpha_k = a_1 + a_2\,\theta^k + a_3\,\theta^{2k} + \ldots + a_n\,\theta^{(n-1)k}$, and taking their products in pairs as follows: $\alpha_1 . \alpha_{2m}, \alpha_2 . \alpha_{2m-1}, \ldots \alpha_m . \alpha_{m+1}$, let them be denoted by $\alpha_{1,2m}, \alpha_2 . \alpha_{2m-1}, \ldots \alpha_{m,m+1}$ respectively. Let us denote the sum

$$\sum_1^{m_k} \alpha_{1,2m} . \alpha_{2,2m-1} \ldots \alpha_{k,2m-k+1} \quad \text{by} \quad \sum_1^{m_k}{}^j \alpha_j^{(k)},$$

then it is readily seen that

$$\text{(A)} \qquad \frac{\partial}{\partial a_i} \sum_1^{m_k}{}^j \alpha_j^{(k)} = \left(\sum_1^{m_{k-1}}{}^j \alpha_j^{(k-1)} \right) \left(\sum_1^{m-k+1}{}_{h_1} I_{h_1} \right)$$

where I_{h_1} represents $(\alpha_{h_1}\,\theta^{n-h_1(i-1)} + \alpha_{2m+1-h_1}\,\theta^{h_1(i-1)})$, and where for each $\alpha_j^{(k-1)}$ in $\Sigma^j \alpha_j^{(k-1)}$, the co-factor ΣI_{h_1} contains no α which is found in $\alpha_j^{(k-1)}$.

Again,

$$\text{(B)} \quad \frac{\partial^2}{\partial a_i^2} \sum_1^{m_k}{}^i \alpha^{(k)} = 2(m-k+1) \sum_1^{m_{k-1}}{}^j \alpha_j^{(k-1)} + \left(\sum_1^{m-h+1}{}_{h_1} I_{h_1} \right) \frac{\partial}{\partial a_i} \sum_1^{m_{k-1}}{}^j \alpha^{(k-1)}$$

$$= 2\,(m-k+1) \sum_1^{m_{k-1}}{}^j \alpha_j^{(k-1)} + 2 \left(\sum_1^{m_{k-2}}{}^j \alpha_j^{(k-2)} \right) \sum_1^{(m-k+2)_2} I_{h_1} I_{h_2}$$

where in each term of $\Sigma \alpha_j^{(k-2)}$ there is no α common to $\alpha_j^{(k-2)}$, I_{h_1}, and I_{h_2}.

* Muir, *Messenger of Maths.*, New Series, No. 460, August 1909.

In general,

$$(\mathrm{R}) \qquad \frac{\partial^r}{\partial a_i^r} \sum_1^{m_k} {}_i\, \alpha_j^{(k)} = 2\,(r-1)\,(m+k+1)\, \frac{\partial^{r-2}}{\partial a_i^{r-2}} \sum_1^{m_{k-1}} {}_j\, \alpha_j^{(k-1)}$$

$$+\left(\sum_1^{m-k+1} {}_{h_1}\ I_{h_1}\right) \frac{\partial^{r-1}}{\partial a_i^{r-1}} \left(\sum_1^{m_{k-1}} {}_j\, \alpha_j^{(k-1)}\right)$$

Taking the sum with respect to i in each case, we have

$$(\mathrm{A}') \qquad \sum_1^n \frac{\partial}{\partial a^i} \sum_1^{m_k} {}_j\ \alpha_j^{(k)} = 0$$

$$(\mathrm{B}') \qquad \sum_1^n {}_i\, \frac{\partial^2}{\partial a_i^2} \sum_1^{m_k} {}_j\ \alpha_j^{(k)} = 2\,.\,n\,.\,(m-k+1) \sum_1^{m_{k-1}} {}_j\ \alpha_j^{(k-1)}$$

$$(\mathrm{R}') \qquad \sum_1^n {}_i\, \frac{\partial^r}{\partial a_i^r} \sum_1^{m_k} {}_j\ \alpha_j^{(k)} = 2(r-1)\,(m-k+1) \sum_1^n {}_i\, \frac{\partial^{r-2}}{\partial a_i^{r-2}} \sum_1^{m_{k-1}} {}_j\ \alpha_j^{(k-1)}$$

since $\sum_1^n {}_i \left(\sum_1^{m-k+1} {}_{h_1}\ I_{h_1}\right) \frac{\partial^{r-1}}{\partial a_i^{r-1}} \left(\sum_1^{m_{k-1}} {}_j\ \alpha_j^{(k-1)}\right)$ vanishes.

From (R′) we have, when $r = 2p$,

$$(1) \qquad \sum_1^n {}_i\, \frac{\partial^{2p}}{\partial a_i^{2p}} \sum_1^{m_k} {}_j\ \alpha_j^{(k)} = 2\,.\,(2p-1)\,(m-k+1) \sum_1^n {}_j\, \frac{\partial^{2p-2}}{\partial a_i^{2p-2}} \left(\sum_1^{m_{k-1}} {}_j\ \alpha_j^{(k-1)}\right)$$

$$= 2^p.\, n\,.\,(m-k+1)\,(m-k+2)\,\ldots\,(m-k+p)$$

$$\times\,(2p-1)\,(2p-3)\,\ldots 3\,.\,1\,.\, \sum_1^{m_{k-p}} {}_j\ \alpha_j^{(k-p)},$$

and when $r = 2p+1$

$$(2) \qquad \sum_1^n {}_i\, \frac{\partial^{2p+1}}{\partial a_i^{2p+1}} \sum_1^{m_k} {}_j\ \alpha_j^{(k)} = 0.$$

3. It is known* that

for $n = 2m+1$

$$\text{I.} \quad \sum_1^{m_k} {}_j\, \alpha_j^{(k)} = (-1)^k \Sigma \begin{pmatrix} 12\ldots 2k \\ 12\ldots 2k \end{pmatrix},\ (k = 1,\ 2,\ \ldots m),$$

$$\text{II.} \quad S\,.\, \sum_1^{m_k} {}_j\, \alpha_j^{(k)} = (-1)^k\ \Sigma \begin{pmatrix} 12\ldots 2k+1 \\ 12\ldots 2k+1 \end{pmatrix},\ (k = 0,\ 1,\ \ldots m)\,;$$

and for $n = 2m+2$

---

* *Vide* author's paper in the *Mathematics Teacher*, December 1918.

III. $$S \,.\, S' \sum_{1}^{m_{k-1}}{}_j\, \alpha_j^{(k-1)} - \sum_{1}^{m_k}{}_j\, \alpha_j^{(k)}\,{}^{*} = (-1)^{k-1} \Sigma \begin{pmatrix} 12\ldots 2k \\ 12\ldots 2k \end{pmatrix},$$
$$(k = 1,\, 2,\, \ldots\, m+1),$$

IV. $$(S + S') \sum_{1}^{m_k}{}_j\, \alpha_j^{(k)} = (-1)^k \Sigma \begin{pmatrix} 12\ldots 2k+1 \\ 12\ldots 2k+1 \end{pmatrix}, \quad (k = 0,\ 1,\ \ldots,\ m)$$

where $S = a_1 + a_2 + a_3 + \ldots,$

$S' = a_1 - a_2 + a_3 - \ldots,$

and where $\Sigma \begin{pmatrix} 12\ldots k \\ 12\ldots k \end{pmatrix}$, (or Σ_k say), is the sum of the coaxial minors of order k.

4. For circulants of odd order, $n = 2m+1$, we have from (1) and I

(3) $$\sum_{1}^{n}{}_i \frac{\partial^{2p}}{\partial a_i^{2p}} \Sigma_{2k} = -2\,.\,(2p-1)\,(m-k+1) \sum_{1}^{n}{}_i \frac{\partial^{2p-2}}{\partial a_i^{2p-2}} \Sigma_{2k-2}$$
$$= (-1)^p\, 2^{ip}.\, n\,.\,(2p-1)\,(2p-3)\ \ldots 3\,.\,1\,.\,(m-k+1)$$
$$\ldots (m-k+p)\, \Sigma_{2k-2p}\,.$$

For $p = 1$ and $k = m$, this is

$$\sum_{1}^{n}{}_i \frac{\partial^2}{\partial a_i^2} \Sigma A_{11} = -2n\, \Sigma_{2m-2}$$
$$= -\frac{2n\, \Sigma_{2n-1}}{S}\,.$$

From (2) and I we have

(4) $$\sum_{1}^{n}{}_i \frac{\partial^{2p+1}}{\partial a_i^{2p+1}} \Sigma_{2k} = 0,$$ which, for $p = 1$ and $k = m$, is

$$\sum_{1}^{n}{}_i \frac{\partial}{\partial a_i} \Sigma A_{11} = 0.$$

From the relation $S\,.\,\Sigma_{2k} = \Sigma_{2k+1}$, we have

(A″) $$\frac{\partial}{\partial a_i} \Sigma_{2k+1} = \Sigma_{2k} + S\,.\,\frac{\partial}{\partial a_i} \Sigma_{2k}$$

(B″) $$\frac{\partial^2}{\partial a_i^2} \Sigma_{2k+1} = 2\frac{\partial}{\partial a_i} \Sigma_{2k} + S \frac{\partial^2}{\partial a_i} \Sigma_{2k}\,.$$

---

* It is to be observed that in this relation, when $k = m+1$, we must consider $\Sigma_j\, \alpha_j^{(m+1)}$ to be zero.

$$\text{(R}'')\qquad \frac{\partial^r}{\partial a_i^r}\Sigma_{2k+1}=r\frac{\partial^{r-1}}{\partial a_i^{r-1}}\Sigma_{2k}+S\frac{\partial^r}{\partial a_i^r}\Sigma_{2k}.$$

From (R″) we have, when $r=2p$,

$$(5)\qquad \sum_1^n{}_i\frac{\partial^{2p}}{\partial a_i^{2p}}\Sigma_{2k+1}=S.\sum_1^n{}_i\frac{\partial^{2p}}{\partial a_i^{2p}}\Sigma_{2k}$$

$$=(-1)^p\,2^p.\,n.\,S.(2p-1)(2p-3)\ldots 3.1.$$

$$\times(m-k+1)(m-k+2)\ldots(m-k+p)\,\Sigma_{2k-2p},$$

which, for $p=1$ and $k=m$, gives

$$\sum_1^n{}_i\frac{\partial^2}{\partial a_i^2}C=-2.n.\Sigma_{n-3},$$

$$\text{but}\qquad \sum_1^n{}_i\frac{\partial^2}{\partial a_i^2}C=n\sum_1^n{}_i\frac{\partial}{\partial \alpha_i}A_i,^*$$

$$\therefore\qquad \sum_1{}_i\frac{\partial A_i}{\partial a_i}=-2\Sigma_{n-3}.$$

When $r=2p+1$ we have

$$\sum_1^n{}_i\frac{\partial^{2p+1}}{\partial a_i^{2p+1}}\Sigma_{2k+1}=(2p+1)\sum_1^n{}_i\frac{\partial^{2p}}{\partial a_i^{2p}}\Sigma_{2k}$$

$$=(-1)^p.\,2^p.\,n.(2p+1)(2p-1)(2p-3)\ldots 3.1$$

$$\times(m-k+1)(m-k+2)\ldots(m-k+p)\,\Sigma_{2k-2p},$$

which, for $p=0$ and $k=m$, gives

$$\sum_1^n{}_i\frac{\partial}{\partial a_i}C=n\,\Sigma_{2m}=n\,\Sigma A_1\,\dagger,$$

where A_k is the signed complementary minor corresponding to the element a_k in the first row of C.

Since

$$\frac{\partial}{\partial a_i}\Sigma_{2k+1}=S.\frac{\partial}{\partial a_i}\Sigma_{2k}+\Sigma_{2k}\text{ and }\frac{\partial}{\partial a_j}\Sigma_{2k+1}=S.\frac{\partial}{\partial a_j}\Sigma_{2k}+\Sigma_{2k}$$

it follows that

$$(7)\qquad \frac{\partial}{\partial a_i}\Sigma_{2k+1}-\frac{\partial}{\partial a_j}\Sigma_{2k+1}=S\left(\frac{\partial}{\partial a_i}\Sigma_{2k}-\frac{\partial}{\partial a_j}\Sigma_{2k}\right),$$

---

* *Vide* Article 6 below.

† *Vide* Article 6 below.

or when $k=m$,

$$n\,(A_i - A_j) = S\left(\frac{\partial}{\partial a_i}\,\Sigma A_{11} - \frac{\partial}{\partial a_j}\,\Sigma A_{11}\right)$$

since $\dfrac{\partial}{\partial a_i}\,C = nA_i$.

But by Stern's theorem *

$$A_i - A_j = (-1)^{i+j-1}\,S\,.\,Q,$$

where Q is the determinant got from C by deleting the first row and the i^{th} and j^{th} columns and inserting a column of units in the first place.

Therefore

$$(8)\qquad \frac{\partial}{\partial a_i}\,\Sigma A_{11} - \frac{\partial}{\partial a_j}\,\Sigma A_{11} = (-1)^{i+j-1}\,.\,n\,.\,Q.$$

5. For circulants of even order, $n=2m+2$, we have from IV.

$$(\text{A}''')\quad (-1)^k\,\frac{\partial}{\partial a_i}\,\Sigma_{2k+1} = (1+(-1)^{i-1})\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)} + (S+S')\,\frac{\partial}{\partial a_i}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)}$$

$$(\text{B}''')\quad (-1)^k\,\frac{\partial_2}{\partial a_i^2}\,\Sigma_{2k+1} = 2(1+(-1)^{i-1})\,\frac{\partial}{\partial a_i}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)} + (S+S')\,\frac{\partial^2}{\partial a_i^2}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)}$$

$$(\text{R}''')\quad (-1)^k\,\frac{\partial^r}{\partial a_i^r}\,\Sigma_{2k+1} = r\,(1+(-1)^{i-1})\,\frac{\partial^{r-1}}{\partial a_i^{r-1}}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)}$$

$$+\,(S+S')\,\frac{\partial^r}{\partial a_i^r}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)}.$$

From (R''') we have, when $r=2p$

$$(9)\qquad \overset{n}{\underset{1}{\Sigma_i}}\,\frac{\partial^{2p}}{\partial a_i^{2p}}\,\Sigma_{2k+1} = (-1)^k\,(S+S')\overset{n}{\underset{1}{\Sigma_i}}\,\frac{\partial^{2p}}{\partial a_i^{2p}}\,\overset{m_k}{\underset{1}{\Sigma_j}}\,\alpha_j^{(k)}$$

$$= (-1)^p\,.\,2^p\,.\,n\,.\,(2p-1)\,(2p-3)\,\ldots\,3\,.\,1$$

$$\times\,(m-k+1)\,(m-k+2)\ldots(m-k+p)\,\Sigma_{2k-2p+1},$$

which, for $p=1$ and $k=m$, is

$$\overset{n}{\underset{1}{\Sigma_i}}\,\frac{\partial^2}{\partial a_i^2}\,\Sigma A_{11} = -\,2n\,\Sigma_{2m-1} = -\,2n\,\Sigma_{n-3}\,.$$

---

* *Crelle's Journal*, lxxii., pp. 374-380. *Cf.* Muir, *Mess. Math.*, New Series, No. 491, March 1912.

When $r = 2p+1$, we have

$$(10) \quad \sum_1^n{}_i \frac{\partial^{2p+1}}{\partial a_i^{2p+1}} \Sigma_{2k+1} = (-1)^k (2p+1) \sum_1^n{}_i \left(1 + (-1)^i\right) \frac{\partial^{2p}}{\partial a_i^{2p}} \sum_1^{m_k}{}_j \; \alpha_j^{(k)}$$

$$= (-1)^{k+p} 2^p \,.\, n \,.\, (2p+1)(2p-1) \ldots 3 \,.\, 1 \,.$$
$$\times (m-k+1)(m-k+2) \ldots$$
$$(m-k+p) \frac{\Sigma_{2k-2p+1}}{\Sigma a_{11}} .$$

which, when $p = 0$ and $k = m$ is

$$\sum_1^n{}_i \frac{\partial}{\partial a_i} \Sigma A_{11} = n \frac{\Sigma A_{11}}{\Sigma a_{11}} .$$

From III. we have

$$(\mathrm{A^{iv}}) \quad (-1)^{k-1} \frac{\partial}{\partial a_i} \Sigma_{2k} = (S' + (-1)^{i-1} S) \sum_1^{m_{k-1}}{}_j \; \alpha_j^{(k-1)}$$
$$\times S \,.\, S' \frac{\partial}{\partial a_i} \sum_1^{m_{k-1}}{}_j \; \alpha_j^{(k-1)} - \frac{\partial}{\partial a_i} \sum_1^{m_k}{}_j \; \alpha_j^{(k)}$$

$$(\mathrm{B^{iv}}) \quad (-1)^{k-1} \frac{\partial^2}{\partial a_i^2} \Sigma_{2k} = 2 (S' + (-1)^{i-1} S) \frac{\partial}{\partial a_i} \Sigma \, \alpha_j^{(k-1)}$$
$$\times S \,.\, S' \frac{\partial^2}{\partial a_i^2} \Sigma \, \alpha_j^{(k-1)} - \frac{\partial^2}{\partial a_i^2} \Sigma \, \alpha_j^{(k)} + 2 (-1)^{i-1} \Sigma \, \alpha_j^{(k-1)}$$

$$(\mathrm{R^{iv}}) \quad (-1)^{k-1} \frac{\partial^r}{\partial a_i^r} \Sigma^{2k} = r (S' + (-1)^i S) \frac{\partial^{r-1}}{\partial a_i^{r-1}} \Sigma \alpha_j^{(k-1)} + S \,.\, S' \frac{\partial^r}{\partial a_i^r} \Sigma \alpha_j^{(k-1)}$$
$$- \frac{\partial^r}{\partial a_i^r} \Sigma \alpha_j^{(k)} + r (r-1) (-1)^i \frac{\partial^{r-2}}{\partial a_i^{r-2}} \Sigma \alpha_j^{(k-1)}.$$

From ($\mathrm{R^{iv}}$) we have when $r = 2p$

$$(11) \quad (-1)^{k-1} \sum_1^n{}_i \frac{\partial^{2p}}{\partial a_i^{2p}} \Sigma_{2k} = 2p \sum_1^n{}_i (S' + (-1)^i S) \frac{\partial^{2p-1}}{\partial a_i^{2p-1}} \Sigma \alpha_j^{(k-1)}$$
$$+ \sum_1^n{}_i S . S' \frac{\partial^{2p}}{\partial a_i^{2p}} \Sigma \alpha_j^{(k-1)} - \sum_1^n{}_i \frac{\partial^{2p}}{\partial a_i^{2p}} \Sigma \alpha_j^{(k)}$$
$$+ 2p (2p-1) \sum_1^n{}_i (-1)^i \frac{\partial^{2p-2}}{\partial a_i^{2p-2}} \Sigma \alpha_j^{(k-1}$$
$$= (-1)^{k-p} 2^p . \, n \,.\, (2p-1)(2p-3) \ldots$$
$$3 \,.\, 1 \,.\, (m-k+2) \ldots (m-k+p)$$
$$\times \left\{ p \frac{\Sigma_{2k-2p+1}}{\Sigma a_{11}} - (m-k+p+1) \Sigma_{2k-2p} \right\}$$

https://doi.org/10.1017/S0013091500035458 Published online by Cambridge University Press

which, when $p=1$ and $k=m+1$, is

$$\sum_1^n{}_i \frac{\partial^2}{\partial a_i} C = -2n\,\Sigma_{2m-1} = -2n\,\Sigma_{n-3}.$$

When $r=2p+1$ we have

$$(12)\qquad (-)^{k-1}\sum_1^n{}_i \frac{\partial^{2p+1}}{\partial a_i^{2p+1}}\Sigma_{2k} = (2p+1)\sum_1^n{}_i\,(S'+(-1)S)\frac{\partial^{2p}}{\partial a_i^{2p}}\Sigma\alpha_j^{(k-1)}$$

$$+\sum_1^n{}_i\,S.S'\frac{\partial^{2p+1}}{\partial a_i^{2p+1}}\Sigma\alpha_j^{(k-1)} - \sum_1^n{}_i \frac{\partial^{2p+1}}{\partial a_i^{2p+1}}\Sigma\alpha_j^{(k)}$$

$$+(2p+1)\,2p\sum_1^n{}_i\,(-1)^i\frac{\partial^{2p-1}}{\partial a_i^{2p-1}}\Sigma\alpha_j^{(k)}$$

$$=(-1)^{k-p-1}2^p.\,n.\,S'.\,(2p+1)\,(2p-1)\ldots$$

$$3.1.(m-k+2)\ldots$$

$$(m-k+p+1)\frac{\Sigma_{2k-2p+1}}{\Sigma a_{11}}$$

which, when $p=0$, and $k=m+1$, is

$$\Sigma\frac{\partial}{\partial a_i}C = \frac{n.S'\Sigma A_{11}}{\Sigma a_{11}} = n\Sigma A_1,$$

when $p=0$ and $k=1$, it is

$$\sum_1^n{}_i \frac{\partial}{\partial a_i}\Sigma_2 = n.S',$$

when $p=1$ and $k=m+1$, it is

$$\sum_1^n{}_i \frac{\partial^2}{\partial a_i^2}C = 2n\left(\frac{\Sigma_{2m+1}-\Sigma a_{11}\,\Sigma_{2m}}{\Sigma a_{11}}\right)$$

$$= -2n\frac{S.S'}{\Sigma a_{11}}.\Sigma_{2m-1} = -2n\frac{S.S'}{S+S'}.\Sigma_{n-3},$$

but $$\sum_1^n{}_i \frac{\partial^2}{\partial a_i^2}C = n\sum_1^n{}_i\frac{\partial}{\partial a_i}A_i. \qquad \text{(Art. 6.)}$$

$$\therefore\quad \sum_1^n{}_i\frac{\partial A_i}{\partial a_i} = -2.\frac{S.S'}{S+S'}\Sigma_{n-1},$$

6. The relations given without proof in this article are either well known or are readily obtained from those which are.

$$\frac{\partial C}{\partial a_i} = n\,A_i;\quad \frac{\partial C}{\partial a_i} = n\,A_j,$$

https://doi.org/10.1017/S0013091500035458 Published online by Cambridge University Press

$$\therefore \frac{\partial A_i}{\partial a_j} = \frac{\partial A_j}{\partial a_i}, \quad \sum_1^n i \frac{\partial C}{\partial a_i} = n \sum_1^n i \, A_i = n \, \Sigma A_i \text{ and } \sum_1^n i \frac{\partial^2 C}{\partial a_i^2} = n \sum_1^n i \frac{\partial}{\partial a_i} A_i .$$

Since $\quad C = \Sigma a_1 \, \Sigma A_1 \quad$ and $\quad \dfrac{\partial C}{\partial a_i} = \Sigma A_1 + \Sigma a \left(\dfrac{\partial}{\partial a_i} \Sigma A_1 \right)$,

$$\therefore \quad \frac{\partial}{\partial a_i} \Sigma A_1 = \frac{n A_i - \Sigma A_1}{\Sigma a_1} \quad \text{and} \quad \Sigma \frac{\partial}{\partial a_i} \Sigma A_1 = 0.$$

Since $\quad C = \alpha_1, \alpha_2 \ldots \alpha_n$

$$\frac{\partial C}{\partial a_1} = \frac{C}{\alpha_1} \frac{\partial \alpha_1}{\partial a_1} + \frac{C}{\alpha_2} \frac{\partial \alpha_2}{\partial a_1} + \ldots$$

$$\frac{\partial C}{\partial a_2} = \frac{C}{\alpha_1} \frac{\partial \alpha_1}{\partial a_2} + \frac{C}{\alpha_2} \frac{\partial \alpha_2}{\partial a_2} + \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \quad \left(a_k \frac{\partial}{\partial a_1} + a_{k+1} \frac{\partial}{\partial a_2} + \ldots a_{k-1} \frac{\partial}{\partial a_n} \right) C = C \left(1 + \theta + \theta^2 + \ldots \theta^{n-1} \right) = 0$$

(13) $\quad$ or $\Sigma a_k \dfrac{\partial C}{\partial a_1} = 0, \; k \neq 1.$

By Euler's theorem for homogeneous functions we have

(14) $\qquad \Sigma a_1 \dfrac{\partial}{\partial a_1} C = nC$, and

(15) $\qquad \left(\Sigma a_1 \dfrac{\partial}{\partial a_1} \right) \Sigma A_1 = (n-1) \, \Sigma A_1$,

(16) $\qquad \left(\Sigma a_1 \dfrac{\partial}{\partial a_1} \right) A_k = (n-1) \, A_k$.

7. It is readily seen that

$$\frac{\partial A_k}{\partial a_h} = \begin{vmatrix} 1 & 2 \\ k & h-1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ k & h-2 \end{vmatrix} + \ldots + \begin{vmatrix} 1 & h \\ k & 1 \end{vmatrix} + \begin{vmatrix} 1 & h+1 \\ k & n \end{vmatrix} + \ldots + \begin{vmatrix} 1 & n \\ k & h+1 \end{vmatrix}$$

Where $\begin{vmatrix} 1 & j \\ k & h \end{vmatrix}$ represents the minor formed from the circulant by deleting the 1^{st} and j^{th} rows and the k^{th} and h^{th} columns, where $\begin{vmatrix} 1 & i+1 \\ k & h-1 \end{vmatrix} = 0$, when $k = h - i$, and where each term having an odd number of inversions is to have a negative sign.

Since for circulants any minor

$$\begin{vmatrix} r & s & t & \ldots \\ u & v & w & \ldots \end{vmatrix} \text{ is equal to } \begin{vmatrix} r+a & s+a & t+a & \ldots \\ u-a & v-a & w-a & \ldots \end{vmatrix}$$

where when any row number becomes greater than n it must be reduced by n, and where any column number that becomes zero or negative must be increased by n, it is seen that for circulants of even order $n = 2m$ we have

$A_{\substack{k\\k}} = A_{\substack{1\\2k-1}} = A_{\substack{m+k\\m+k}}$ $(k = 1, 2, \ldots m)$, and therefore the primary minors corresponding to elements in the odd places of the first row have primary minors corresponding to elements along the principal diagonal equal to them, but those in the even places have not. It follows that $\Sigma A_{11} = 2\Sigma A_{2k-1}$ $(k = 1, 2 \ldots m)$,

but $$\Sigma A_1 = \Sigma A_{2k-1} + \Sigma A_{2k},$$

$$\Sigma^i \frac{\partial}{\partial a_i} \Sigma A_1 = 0, \text{ and } \Sigma^i \frac{\partial}{\partial a_i} \Sigma A_{11} = n \frac{\Sigma A_{11}}{\Sigma a_{11}},$$

(17) $$\therefore \quad \Sigma^i \frac{\partial}{\partial a_i} \Sigma A_{2k-1} = -\Sigma^i \frac{\partial}{\partial a_i} \Sigma A_{2k}$$

(18) and $$\Sigma^i \frac{\partial}{\partial a_i} \Sigma A_{2k-1} = \frac{n}{2} \frac{\Sigma A_{11}}{\Sigma a_{11}}.$$

In this case of circulants of odd order for every signed primary minor corresponding to elements in the first row there is a primary minor corresponding to some elements along the principal diagonal.

8. If we denote $a_k \frac{\partial}{\partial a_1} + a_{k+1} \frac{\partial}{\partial a_2} + \ldots + a_n \frac{\partial}{\partial a_{n-k+1}}$

$$+ a_1 \frac{\partial}{\partial a_{n-k+2}} + \ldots + a_{k-1} \frac{\partial}{\partial a_n}$$

by $\Sigma a_k \frac{\partial}{\partial a_1}$, then

(19) $$\left(\Sigma a_k \frac{\partial}{\partial a_1}\right) A_h = -A_{h-k+1}, \text{ (or } A_{n+h-k+1} \text{ if } k > h)$$

$$\left\{ \begin{array}{l} h = 1, 2, \ldots n \\ k = 2, 3, \ldots n \end{array} \right\}$$

The truth of this is seen on observing that $\left(\Sigma a_k \frac{\partial}{\partial a_1}\right) A_h$ is the sum of n determinants which are obtained by increasing the subscripts of the elements in the first, second, and so on, columns of A_h by $k - 1$. Of these all vanish, having identical columns, except the $(h-k+1)^{\text{th}}$ (or the $\overline{n+h-k}^{\text{th}}$ if $h < k$), and it is $-A_{h-k+1}$ (or $-A_{n+h-k-1}$).

If $h = k$ then (16) becomes

$$\left(\Sigma a_k \frac{\partial}{\partial a_1}\right) A_k = -A_1\,, \quad (k = 2, 3 \ldots n).$$

From the foregoing we have

(20) $\left(\Sigma o_k \frac{\partial}{\partial a_1}\right)\Sigma A_1 = -\Sigma A_1$, or it may be seen as follows:

$$C = \Sigma a_1 \, \Sigma A_1$$

$$a_k \frac{\partial}{\partial a_1} C = a_k \, \Sigma A_1 + \Sigma a_1 \left(a_k \frac{\partial}{\partial a_1} \Sigma A_1 \right)$$

$$a_{k+1} \frac{\partial}{\partial a_2} C = a_{k+1} \Sigma A_1 + \Sigma a_1 \left(a_{k+1} \frac{\partial}{\partial a_2} \Sigma A_1 \right)$$

....

$$\therefore \quad \left(\Sigma a_k \frac{\partial}{\partial a_1}\right) C = \Sigma a_1 \, \Sigma A_1 + \Sigma a_1 \left(\Sigma a_k \frac{\partial}{\partial a_1} \Sigma A_1\right),$$

or $\qquad \Sigma a_k \frac{\partial}{\partial a_1} \Sigma A_1 = -\Sigma A_1$, since $\Sigma a_k \frac{\partial}{\partial a_1} C = 0.$

From $\dfrac{\partial C}{\partial a_k} = n A_k$ we get

$$a_1 \frac{\partial^2 C}{\partial a_k \, \partial a_1} = n a_1 \frac{\partial A_k}{\partial a_1}, \quad a_2 \frac{\partial^2 C}{\partial a_k \, \partial a_2} = n \, a_2 \frac{\partial A_k}{\partial a_2}, \ \ldots$$

and therefore

(21) $$a_1 \frac{\partial^2 C}{\partial a_k \, \partial a_1} + a_2 \frac{\partial^2 C}{\partial a_k \, \partial a_2} + \ldots = n \left(a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + \ldots \right) A_k$$
$$= n(n-1) A_k$$

Also

(22) $$a_k \frac{\partial^2 C}{\partial a_k \, \partial a_1} + a_{k+1} \frac{\partial^2 C}{\partial a_k \, \partial a_2} + \ldots = n \left(a_k \frac{\partial}{\partial a_1} + a_{k+1} \frac{\partial}{\partial a_2} + \ldots \right) A_k$$
$$= -n A_1 .$$

Again, from $C = \Sigma a_1 \, \Sigma A_1$ we get

$$a_1 \frac{\partial^2 C}{\partial a_1^{\,2}} = 2 a_1 \frac{\partial}{\partial a_1} \Sigma A_1 + (\Sigma a_1) \left(a_1 \frac{\partial^2}{\partial a_1^{\,2}} \Sigma A_1 \right),$$

$$a_2 \frac{\partial^2 C}{\partial a_1 \, \partial a_2} = a_2 \frac{\partial}{\partial a_2} \Sigma A_2 + a_2 \left(\frac{\partial}{\partial a_1} \Sigma A_1 \right) + (\Sigma a_1) \left(a_2 \frac{\partial^2}{\partial a_1 \, \partial a_2} \Sigma A_1 \right)$$

....

and therefore

$$\left(a_1 \frac{\partial^2}{\partial a_1^{\,2}} + a_2 \frac{\delta^2}{\partial a_1 \, \partial a_2} + a_3 \frac{\partial^2}{\partial a_1 \, \partial a_3} + \ldots \right) C = \left(\Sigma a_1 \frac{\partial}{\partial a_1}\right) \Sigma A_1$$
$$+ \Sigma a_1 \left(\frac{\partial}{\partial a_1} \Sigma A_1 \right) + \Sigma a_1 \left(a_1 \frac{\partial^2}{\partial a_1^{\,2}} + a_2 \frac{\partial^2}{\partial a_1 \, \partial a_2} \ldots \right) \Sigma A_1$$

or

$$n(n-1)A_1 = (n-1)\Sigma A_1 + (nA_1 - \Sigma A_1) + \Sigma a_1\left(a_1\frac{\partial^2}{\partial a_1{}^2} + a_2\frac{\partial^2}{\partial a_1\,\partial a_2}\ldots\right)\Sigma A_1,$$

$$(23)\quad \therefore\quad \Sigma a_1\left(a_1\frac{\partial^2}{\partial a_1{}^2} + a_2\frac{\partial^2}{\partial a_1\,\partial a_2} + \ldots\right)\Sigma A_1 = (n-2)(nA_1 - \Sigma A_1),$$

and also

$$(24)\qquad \left(\Sigma a_1\frac{\partial^2}{\partial a_1{}^2}\right)C = \Sigma a_1\left(\Sigma a_1\frac{\partial^2}{\partial a_1{}^2}\right)\Sigma A_1 + 2(n-1)\Sigma A_1,$$

and

$$(25)\quad \Sigma a_1\left(a_k\frac{\partial^2}{\partial a_k\,\partial a_1} + a_{k+1}\frac{\partial^2}{\partial a_k\,\partial a_2} + \ldots\right)\Sigma A_1 = 2\Sigma A_1 - n(A_1 + A_k).$$

From

$$\Sigma^i\frac{\partial^2}{\partial a_k\,\partial a_i}C = n\left(\frac{\partial}{\partial a_k}\Sigma A_1\right) = n\frac{\partial}{\partial a_k}\Sigma A_1 + \Sigma a_1\left(\Sigma^i\frac{\partial^2}{\partial a_k\,\partial a_i}\Sigma A_i\right) + \Sigma^i\frac{\partial}{\partial a_i}\Sigma A_1$$

and $\qquad \Sigma^i\dfrac{\partial^2}{\partial a_k\,\partial a_1}C = n\left(\Sigma^i\dfrac{\partial}{\partial a_i}\right)A_k$

it follows that

$$(26)\qquad \frac{\partial}{\partial a_k}\Sigma A_1 = \left(\Sigma^i\frac{\partial}{\partial a_i}\right)A_k,\ \text{and}$$

$$(27)\qquad \Sigma^i\frac{\partial^2}{\partial a_k\,\partial a_i}\Sigma A_1 = 0,\ \text{since}\ \Sigma^i\frac{\partial}{\partial a_i}\Sigma A_1 = 0.$$

10. Another form for writing the circulant is

$$C = a_1A_1 + a_2A_2 + \ldots + a_nA_n,$$

and therefore

$$\frac{\partial C}{\partial a_k} = a_1\frac{\partial A_1}{\partial a_k} + a_2\frac{\partial A_2}{\partial a_k} + \ldots + a_k\frac{\partial A_k}{\partial a_k}A_k + \ldots + a_n\frac{\partial A_n}{\partial a_k}.$$

$$(28)\quad \text{or}\quad \Sigma^i a_i\frac{\partial}{\delta a_k}A_i = (n-1)A_k.$$

Also

$$(29)\qquad \Sigma^i a_i\frac{\partial^2}{\partial a_k^2}A_i = (n-2)\frac{\partial}{\partial a_k}A_k$$

$$(30)\quad \therefore\quad n\Sigma^i a_i\frac{\partial^2}{\partial a_k^2}A_i = (n-2)\frac{\partial^2 C}{\partial a_k^2}.$$

https://doi.org/10.1017/S0013091500035458 Published online by Cambridge University Press

Taking the sum of both sides of (28) with respect to k gives

$$a_1\left(\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\dots\right)A_1+a_2\left(\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\dots\right)A_2+\dots=(n-1)\,\Sigma A_1$$

(31) or $\Sigma^i\, a_i\, \Sigma^k \dfrac{\partial}{\partial a_k} A_i = (n-1)\,\Sigma A_1$,

and therefore

(32) $$\Sigma^i\, a_i\, \Sigma^k \frac{\partial}{\partial a_k} A_1 = \left(\Sigma a_1 \frac{\partial}{\partial a_1}\right)\Sigma A_1 .$$

11. As an illustration of some of the foregoing relations let us find the value of the Hessian of a circulant. Taking for convenience sake, the case when the order is five, though the process is obviously general, we have

$$H(C)\,.\,C$$

$$=\begin{vmatrix} \frac{\partial^2 C}{\partial a_1^2} & \frac{\partial^2 C}{\partial a_1\,\partial a_2} & \frac{\partial^2 C}{\partial a_1\,\partial a_3} & \frac{\partial^2 C}{\partial a_1\,\partial a_4} & \frac{\partial^2 C}{\partial a_1\,\partial a_5} \\ \frac{\partial^2 C}{\partial a_2\,\partial a_1} & \frac{\partial^2 C}{\partial a_2^2} & \frac{\partial^2 C}{\partial a_2\,\partial a_3} & \frac{\partial^2 C}{\partial a_2\,\partial a_4} & \frac{\partial^2 C}{\partial a_2\,\partial a_5} \\ \frac{\partial^2 C}{\partial a_3\,\partial a_1} & \frac{\partial^2 C}{\partial a_3\,\partial a_2} & \frac{\partial^2 C}{\partial a_3^2} & \frac{\partial^2 C}{\partial a_3\,\partial a_4} & \frac{\partial^2 C}{\partial a_3\,\partial a_5} \\ \frac{\partial^2 C}{\partial a_4\,\partial a_1} & \frac{\partial^2 C}{\partial a_4\,\partial a_2} & \frac{\partial^2 C}{\partial a_4\,\partial a_3} & \frac{\partial^2 C}{\partial a_4^2} & \frac{\partial^2 C}{\partial a_4\,\partial a_5} \\ \frac{\partial^3 C}{\partial a_5\,\partial a_1} & \frac{\partial^2 C}{\partial a^5\,\partial a_2} & \frac{\partial^2 C}{\partial a_5\,\partial a_3} & \frac{\partial^2 C}{\partial a_5\,\partial a_4} & \frac{\partial^2 C}{\partial a_2^5} \end{vmatrix}\;\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_5 & a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

$$=\begin{vmatrix} n(n-1)A_1 & n(n-1)A_2 & n(n-1)A_3 & n(n-1)A_4 & n(n-1)A_5 \\ -n\,A_5 & -n\,A_1 & -n\,A_2 & -n\,A_3 & -n\,A_4 \\ -n\,A_4 & -n\,A_5 & -n\,A_1 & -n\,A_2 & -n\,A_3 \\ -n\,A_3 & -n\,A_4 & -n\,A_5 & -n\,A_1 & -n\,A_2 \\ -n\,A_2 & -n\,A_3 & -n\,A_4 & -n\,A_5 & -n\,A_1 \end{vmatrix}$$

$$=(-1)^{\frac{1}{2}(n+1)(n-2)}\,(n-1)\,n^n\,C^{n-1}.$$

Therefore

(33) $$H(C)=(-1)^{\frac{1}{2}(n+1)(n-2)}\,(n-1)\,n^n\,C^{n-2}.$$