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Self-maps of the Grassmannian of Complex Structures

Dedicated to Professor Boju Jiang on his 65th birthday

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Abstract. Let CS_n be the flag manifold $\mathrm{SO}(2n)/\mathrm{U}(n)$. We give a partial classification for the endomorphisms of the cohomology ring $H^*(CS_n; Z)$ which is very close to a homotopy classification of all selfmaps of CS_n. Applications concerning the geometry of the space are discussed.

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1. Introduction

Let $\mathrm{O}(2n) = \mathrm{O}^+(2n) \sqcup \mathrm{O}^-(2n)$ be the orthogonal group of order $2n$ with $\mathrm{O}^+(2n)$, the connected component that contains the identity I_{2n}. Its subspace $G_n = \{J \in \mathrm{O}^+(2n) \mid J^2 = -I_{2n}\}$ is known as *the Grassmannian of complex structures on the* $2n$-*dimensional Euclidean space* R^{2n}. It is the space of minimal geodesics form I_{2n} to $-I_{2n}$ on $\mathrm{O}^+(2n)$ [12]. It serves as the classifying space for all complex n-bundles whose real reductions are trivial [4]. It has two connected components

$$CS_n = \{AJ_0A^\tau \mid A \in \mathrm{O}^+(2n)\}; \qquad CS_n^- = \{AJ_0A^\tau \mid A \in \mathrm{O}^-(2n)\},$$

where

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (n \text{ copies}),$$

both diffeomorphic to the homogeneous space $\mathrm{O}^+(2n)/\mathrm{U}(n)$.

For a topological space X let $[X, X]$ be the set of homotopy classes of self-maps of X, and $\mathrm{End}(H^*(X))$, the set of all endomorphisms of the integral cohomology ring. Sending a self-map to the induced endomorphism gives rise to a representation

$$l_X\colon [X, X] \to \mathrm{End}(H^*(X)), \quad f \to f^*$$

in view of the obvious monoid structure on the both sets. According to the rational homotopy theory, if X is a flag manifold (i.e. a homogeneous space G/K with G is a compact connected Lie group and $K \subset G$, a Borel subgroup), this representation is 'nearly faithful' in the sense that it has finite kernel and finite cokernel.

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Therefore, the problem of determining End($H^*(X)$), for a flag manifold X, is a step toward a homotopy classification of all self-maps of X. This problem has been studied in some detail for the complex Grassmannians ([10]), and for some compact Lie groups module its maximal torus ([3, 11]). This paper studies the problem for CS_n, with an intention to devote to geometry in the applications.

The ring $H^*(CS_n)$ can be described as follows. Let γ_n be the complex n-bundle obtained by furnishing the trivial real bundle $CS_n \times R^{2n} \to CS_n$ the complex structure

$$K : CS_n \times R^{2n} \to CS_n \times R^{2n}, \quad K(J, v) = (J, Jv),$$

and let $1 + c_1 + \cdots + c_n$ be its total Chern class.

THEOREM 1 (cf. [4]). *The classes $c_i \in H^{2i}(CS_n)$, $i \leqslant n-1$, are all divisible by 2. Further, if we let $d_i = \frac{1}{2}c_i$, then $d_1, \ldots, d_{n-1}$ form a simple system of generators for $H^*(CS_n)$, and are subject to the relations*

$$R_i\colon d_i^2 - 2d_{i-1}d_{i+1} + \cdots + (-1)^{i-1}2d_1d_{2i-1} + (-1)^i d_{2i} = 0, \quad 1 \leqslant i \leqslant n-1,$$

with $d_s = 0$, $s \geqslant n$, being understood.

Let f be an endomorphism of $H^*(CS_n)$ and let f^N be the N-iteration of f defined inductively by $f^N = f \circ f^{N-1}$, $N > 0$. Since d_1 is the only generator in dimension 2, f sends d_1 to a multiple of itself. The main results of this paper are

THEOREM 2. *If $f(d_1) = ad_1$ with $a \neq 0$, then $f(d_i) = a^i d_i$, $i \leqslant n-1$.*

THEOREM 3. *If $f(d_1) = 0$, there exists $N \geqslant 1$ so that*

$$f^N(d_i) = 0, \quad \textit{for all } i \neq 2[n/2] - 1.$$

In Theorem 3, the conclusion $f^N(d_i) = 0$ cannot be extended to $i = 2[n/2] - 1$. In Section 8, we shall present examples of self-maps f of CS_n so that the induced endomorphisms f^* satisfy $f^*(d_i) = 0$, $i \leqslant n-2$, and $f^{*N}(d_{n-1}) \neq 0$ for all $N > 0$.

We turn to applications of the previous results. Let CS_n^0 be the rationalization of CS_n. From the minimal model for CS_n given in Lemma 7.3, one can show that the monoid of homotopy classes $[CS_n^0, CS_n^0]$ is anti isomorphic to the monoid of endomorphisms of $H^*(CS_n; Q)$ (cf. Theorem 1.1 in [6]). Thus, Theorem 2 offers a complete classification on self-homotopy equivalencies of CS_n^0. In particular, Theorem 2 implies

COROLLARY 1. *Any space in the genus of CS_n* (see [7] for the definition) *is homotopy equivalent to CS_n.*

The self-map f_n of CS_n by

$$f_n(J) = \begin{cases} -J & \text{if } n \text{ is even} \\ -\widetilde{I_{2n}}\, J \widetilde{I_{2n}} & \text{if } n \text{ is odd,} \end{cases} \quad J \in CS_n, \ \widetilde{I_{2n}} = (-1) \oplus I_{2n-1},$$

is clearly a fixed point free involution (note that, the involution on G_n by $J \to -J$ exchanges the two connected components precisely when n is odd). Our next result implies that, cohomologically, f_n is the only fixed point free self-map of CS_n unless $n = 4$.

THEOREM 4. *Let f is a self-map of CS_n with Lefschetz number $L(f) = 0$, and let f^* be the induced endomorphism.*

(1) *If $n \neq 4$ then $f^*(d_i) = (-1)^i d_i$, $1 \leqslant i \leqslant n-1$;*
(2) *If $n = 4$ there are the two additional possibilities*

$$(f^*(d_1), f^*(d_2), f^*(d_3)) = (0, 0, \textit{either} - d_3 \textit{ or } - d_3 + d_1 d_2).$$

COROLLARY 2. *If $n \neq 4$, any self-map f of CS_n with $f^*(d_1) \neq -d_1$ has a fixed point.*

COROLLARY 3. *Any self-map of CS_n has a periodic point of order 2.*

The natural inclusion $R^{2(n-1)} \subset R^{2n}$ induces a smooth fiber bundle $CS_n \xrightarrow{p} S^{2(n-1)}$ over the $2(n-1)$ sphere $S^{2(n-1)}$ (See Section 8). If p admits a cross-section, say s, then the composed map

$$f\colon CS_n \xrightarrow{p} S^{2(n-1)} \xrightarrow{\tau} S^{2(n-1)} \xrightarrow{s} CS_n,$$

where τ is antipodal, is clearly fixed point free (hence $L(f) = 0$), and satisfies $f^*(d_i) = 0$ for $i < n-1$ (since f factors through the sphere). On the other hand, $S^{2(n-1)}$ admits an almost complex structure if and only if when p has a cross-section. Thus Theorem 4 implies the classical result, originally due to Borel and Serre [1]:

COROLLARY 4. *If $n \neq 2, 4$, $S^{2(n-1)}$ does not admit any almost complex structure.*

The existence of various kinds of geodesics is a central topic in geometry. For a Riemannian manifold M and an isometry g on M, a nontrivial geodesic σ is called *g-invariant* if there exists a period c so that $g \circ \sigma(t) = \sigma(t + c)$, $t \in R$. The case $g = \mathrm{id}$ corresponds to the classical notation of closed geodesic.

THEOREM 5. *If f is a self-homotopy equivalence of CS_n , the induced action $f_* \otimes 1$ on the odd dimensional rational homotopy group $\pi_{\mathrm{odd}}(CS_n) \otimes Q$ is the identity.*

Since $\dim(\pi_{\mathrm{odd}}(CS_n) \otimes Q) \geqslant 2$ for $n \geqslant 5$ (by Lemma 7.3), a combination of Theorem 5 with the results in [9] gives

COROLLARY 6. *With respect an arbitrary Riemannian metric on CS_n, $n \geqslant 5$, any isometry has infinitely many invariant geodesics.*

The paper is arranged as follows: After preliminary discussions in Sections 2, 3 and 4, Theorems 2 and 3 will be established in Sections 5 and 6. Section 7 is devoted to proofs of Theorems 4 and 5.

Finally we remark that results similar to Theorems 2 and 3 hold for the flag manifold $HS_n = \mathrm{Sp}(n)/\mathrm{U}(n)$, i.e. *the Grassmannian of quaternionic structures on* C^{2n}. For ignoring the grading, the two algebras $H^*(CS_{n+1}; Q)$ and $H^*(HS_n; Q)$ are isomorphic.

2. The Cohomology Ring

A sequence $\lambda = (i_1, \dots, i_r)$ of integers will be called *a strict partition* λ *of* i if

$$0 < i_1 < \cdots < i_r \quad \text{and} \quad i_1 + \cdots + i_r = i.$$

For a $i > 0$ let $P(i)$ be the set of all strict partitions of i and, for a $\lambda = (i_1, i_2, \dots, i_r) \in P(i)$, put $d_\lambda = d_{i_1} d_{i_2} \cdots d_{i_r}$. From Theorem 1 we have:

LEMMA 2.1. $H^{\mathrm{odd}}(CS_n) = 0$ *and the set of monomials* $\{d_\lambda \mid \lambda \in P(i)\}$ *forms a basis for the* Z*-module* $H^{2i}(CS_n)$.

As for the multiplicative structure we grade the polynomial ring $Z[d_1, d_2, \dots, d_{n-1}]$ by assigning $\deg(d_i) = 2i$. Theorem 1 also tells

LEMMA 2.2 (The first description of the ring $H^*(CS_n)$).

$$H^*(CS_n) = Z[d_1, d_2, \dots, d_{n-1}]/\langle R_r, r = 1, 2, \dots, n-1 \rangle .$$

More explicitly, the relations R_r's can be written as follows

$$R_1\colon d_1^2 - d_2 = 0;$$

$$R_2\colon d_2^2 - 2d_1d_3 + d_4 = 0;$$

$$R_3\colon d_3^2 - 2d_2d_4 + 2d_1d_5 - d_6 = 0;$$

$$\vdots$$

$$R_{n-2}\colon d_{n-2}^2 - 2d_{n-3}d_{n-1} = 0;$$

$$R_{n-1}\colon d_{n-1}^2 = 0.$$

from the first $[(n+1)/2] - 1$ relations one finds that each d_{2i} can be expressed as a polynomial g_{2i} in d_{odd}'s. For instance, the first four such polynomials are

$$g_2(= d_2) = d_1^2;$$

$$g_4(= d_4) = 2d_1d_3 - d_1^4;$$

$$g_6(= d_6) = 2d_1d_5 + d_3^2 - 4d_1^3d_3 + 2d_1^6;$$

$$g_8(= d_8) = 2d_1d_7 + 2d_3d_5 - 6d_1^2d_3^2 + 8d_1^5d_3 - 4d_1^3d_5 - 3d_1^8.$$

Consequently, substituting d_{2i}'s by g_{2i}'s in the remaining $n - [(n+1)/2]$ relations yields some equations in d_{odd}'s. Let $k = [n/2]$. Remaining d_{2i} instead of the g_{2i}'s in d_{odd}'s for the sake of simplicity, these equations are

$$l_r\colon d_r^2 - 2d_{r-1}d_{r+1} + 2d_{r-2}d_{r+2} - \cdots + 2(-1)^{r-1}d_{2r-2k+1}d_{2k-1} = 0,$$
$$k \leqslant r \leqslant 2k-1,$$

when n is even; and are

$$l_r\colon d_r^2 - 2d_{r-1}d_{r+1} + 2d_{r-2}d_{r+2} - \cdots + 2(-1)^{r-2}d_{2r-2k}d_{2k} = 0,$$
$$k+1 \leqslant r \leqslant 2k,$$

when n is odd. Thus we get

LEMMA 2.3 (The second description of the ring $H^*(CS_n)$).

$$H^*(CS_n) = Z[d_1, d_3, \ldots, d_{2k-1}]/\langle\langle l_r;\ r = \left[\frac{n+1}{2}\right], \ldots, n-1\rangle\rangle.$$

For a $\lambda \in P(i)$, let $D_\lambda \in Z[d_1, d_3, \ldots, d_{2k-1}]$ be obtained from d_λ by substituting d_{2j} by g_{2j}. Lemma 2.1 gives

LEMMA 2.4 (Basis Theorem). $H^{\text{odd}}(CS_n) = 0$ *and the set of monomials* $\{D_\lambda \mid \lambda \in P(i)\}$ *is a basis for* $H^{2i}(CS_n)$.

3. The Hard Lefschetz Theorem

Let f be an endomorphism of $H^*(CS_n)$, and let $k = [n/2]$. According to Lemma 2.4 f is given by

$$f(d_{2i-1}) = a_{2i-1}d_{2i-1} + \sum_{\lambda \in Q(2i-1)} a_\lambda D_\lambda, \qquad 1 \leqslant i \leqslant k, \tag{3.1}$$

where

$$a_{2i-1}, \quad a_\lambda \in Z \quad \text{and} \quad Q(2i-1) = P(2i-1)\backslash\{2i-1\}.$$

The leading coefficient of the polynomial $f(d_{2i-1})$ gives rise to a sequence $(a_1, a_3, \ldots, a_{2k-1})$ which will be termed as the *character sequence of f.*

Since, in the second description of the ring $H^*(CS_n)$, the first relation appears in degree $4[(n+1)/2] > \deg(d_{2k-1})$, f can be regarded as an endomorphism of the free algebra $Z[d_1, d_3, \ldots, d_{2k-1}]$, defined by $(3.1)_i$, that preserves the ideal generated by l_r's.

Let M be a m-dimensional compact Kaehler manifold with Kaehler class $u \in H^2(M; Q)$. The hard Lefschetz theorem states:

LEMMA 3.1. *If* $0 \leqslant r \leqslant m$, *multiplication by* u^{m-r} *gives an isomorphism*

$$H^r(M; Q) \to H^{2m-r}(M; Q).$$

The use of this theorem in the proof of next result is adopted from Hoffman [10].

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LEMMA 3.2. *Suppose that* $f(d_{2t-1}) = a^{2t-1}d_{2t-1}$, $1 \leqslant t < i$, $a \neq 0$. *Then we have either* $f(d_{2i-1}) = a^{2i-1}d_{2i-1}$ *or* $a_{2i-1} = -a^{2i-1}$.

Proof. For a $\lambda \in Q(2i-1)$, D_λ is a polynomial in d_{2t-1}'s, $t < i$, of homogeneous degree $2(2i-1)$. It follows from the assumption that

$$f(D_\lambda) = a^{2i-1}D_\lambda, \quad \lambda \in Q(2i-1).$$

Since CS_n is a Kaehler manifold of complex dimension $m = (n(n-1))/2$ with Kaehler class d_1, $\{d_1^{m-4i+2}D_\lambda \mid \lambda \in P(2i-1)\}$ is a basis for $H^{2m-4i+2}(CS_n; Q)$ by Lemmas 2.4 and 3.1. Thus, if we define a matrix

$$N = (N_{\lambda\mu})_{\lambda,\mu \in P(2i-1)}$$

by the relations

$$d_1^{m-4i+2}D_\lambda D_\mu = N_{\lambda,\mu}d_1^m, \ \ N_{\lambda,\mu} \in Q \tag{3.2}$$

in $H^{2m}(CS_n; Q) = Q$, then N is nonsingular by the Poincare duality.

For $\mu \in Q(2i-1)$ applying f to

$$d_1^{m-4i+2}D_\mu d_{2i-1} = N_{\mu,2i-1}d_1^m$$

gives

$$a^{m-2i+1}d_1^{m-4i+2}D_\mu\left(a_{2i-1}d_{2i-1} + \sum_{\lambda \in Q(2i-1)} a_\lambda D_\lambda\right) = N_{\mu,2i-1}a^m d_1^m.$$

Rewriting everything as a multiple of d_1^m by using (3.2) we get

$$N_{\mu,2i-1}(a_{2i-1} - a^{2i-1}) + \sum_{\lambda \in Q(2i-1)} N_{\mu,\lambda}a_\lambda = 0. \tag{3.3}$$

Similarly applying f to $d_1^{m-4i+2}d_{2i-1}^2 = N_{2i-1,2i-1}d_1^m$ yields

$$N_{2i-1,2i-1}(a_{2i-1}^2 - a^{2(2i-1)}) + 2a_{2i-1}\sum_{\lambda \in Q(2i-1)} N_{\lambda,2i-1}a_\lambda + $$
$$+ \sum_{\mu,\lambda \in Q(2i-1)} N_{\mu,\lambda}a_\mu a_\lambda = 0. \tag{3.4}$$

Multiplying (3.3) by a_λ, summing over $\lambda \in Q(2i-1)$, and subtracting the resulting equation from (3.4) gives rise to

$$N_{2i-1,2i-1}(a_{2i-1}^2 - a^{2(2i-1)}) + (a_{2i-1} + a^{2i-1})\sum_{\lambda \in Q(2i-1)} N_{\lambda,2i-1}a_\lambda = 0. \tag{3.5}$$

If $a_{2i-1} = -a^{2i-1}$, we are done. Assume next $a_{2i-1} + a^{2i-1} \neq 0$. Dividing (3.5) by $a_{2i-1} + a^{2i-1}$ gives

$$N_{2i-1,2i-1}(a_{2i-1} - a^{2i-1}) + \sum_{\lambda \in Q(2i-1)} N_{\lambda,2i-1}a_\lambda = 0.$$

Combining this with (3.3) for all $\mu \in Q(2i-1)$ gives a system

$$\sum_{\mu \in P(2i-1)} N_{\lambda\mu}(a_\mu - \delta_{\mu,2i-1}a^{2i-1}) = 0, \quad \lambda \in P(2i-1),$$

where $\delta_{\mu,2i-1}$ is the Kronecker delta. The nonsingularity of N implies

$$a_\mu = \delta_{\mu,2i-1}a^{2i-1}, \quad \text{i.e. } f(d_{2i-1}) = a^{2i-1}d_{2i-1}. \qquad \square$$

COROLLARY 3.3. *If* $f(d_1) = ad_1$ *with* $a \neq 0$, *then* $a_{2i-1} = \pm a^{2i-1}$, $i \leqslant k$.

Proof. Since, in the second description of the ring $H^*(CS_n)$, the first relation appears in degree $4[(n+1)/2] > \deg(d_{2k-1})$, from $(3.1)_i$ we find that the character sequence of f^2 is $(a_1^2, a_3^2, \ldots, a_{2k-1}^2)$. It now follows from Lemma 3.2 that $a_{2i-1}^2 = a^{2(2i-1)}$. $\square$

For a $d \in H^{2r}(CS_n)$ define the rational $M(d) \in Q$ by the relation

$$dd_1^{m-r} = M(d)d_1^m$$

on $H^{2m}(CS_n)$. In particular, the number $M(d_i)$ is the ratio of the degree of the special Schubert variety corresponding to d_i by the degree of CS_n [5].

Put $e_i = e_i(1, \ldots, n-1)$, where $e_i(t_1, \ldots, t_{n-1})$ is the ith elementary polynomial in $t_1, \ldots, t_{n-1}$. The following computation has been made in [5, Proposition 4]

LEMMA 3.3. $M(d_i) = 4^{i-1}e_i/e_1(e_1-1)\cdots(e_1-i+1)$.

We shall need the following consequence of Lemma 3.3.

LEMMA 3.4. *If* $f(d_1) = ad_1$ *with* $a \neq 0$, *then* $f(d_3) \neq -a^3d_3 + 4a^3d_1^3$.

Proof. Assume not. Applying f to the relation $d_3d_1^{m-r} = M(d_3)d_1^m$ gives

$$(-a^3d_3 + 4a^3d_1^3)a^{m-3}d_1^{m-3} = M(d_3)a^md_1^m. \tag{3.7}$$

Rewriting everything as a multiple of d_1^m, by using (3.6) we get $M(d_3) = 2$. This implies that $8e_3 = e_1(e_1-1)(e_1-2)$ by Lemma 3.3. From the Newton's formula we have

$$\tfrac{8}{3}\left(s_3 + \tfrac{1}{2}(s_1^2 - 3s_2)s_1\right) = s_1(s_1-1)(s_1-2), \tag{3.8}$$

where $s_k = 1^k + \cdots + (n-1)^k$. With

$$s_3 = \left[\tfrac{1}{2}n(n-1)\right]^2, \qquad s_2 = \tfrac{1}{6}(n-1)n(2n-1), \qquad \text{and} \quad s_1 = \tfrac{1}{2}n(n-1),$$

(3.8) turns out to be:

$$24 = (n^2 - 17n + 42)(n-1)n.$$

However, this has no solution in n. $\square$

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4. The g-Sequences

A sequence of m integers $(s_1, \dots, s_m)$ will be called a *g-sequence* of length m if, for every integer r with $k+1 \leqslant r \leqslant 2k-2$, the products $s_i s_{r-i}$ are independent of $i \leqslant [\frac{r}{2}]$. In other words, the following inductive strings of relations:

$$s_1 s_k = s_2 s_{k-1} = \cdots = s_{[\frac{k+1}{2}]} s_{k+1-[\frac{k+1}{2}]};$$
$$s_2 s_k = s_3 s_{k-1} = \cdots = s_{[\frac{k+2}{2}]} s_{k+2-[\frac{k+2}{2}]};$$
$$\vdots$$
$$s_{k-3} s_k = s_{k-2} s_{k-1};$$
$$s_{k-2} s_k = s_{k-1}^2$$

hold among the entries s_i's. We classify all such sequences in

LEMMA 4.1. *A g-sequence of length $m \geqslant 3$ belongs to one of the three types:*

Type 1: $(s_1, s_1 q, \dots, s_1 q^{m-1})$ *with* $s_1 q \neq 0$*;*
Type 2: $(s_1, s_2, \dots, s_{[\frac{m}{2}]}, 0, \dots, 0)$ *with* $s_1^2 + s_2^2 + \cdots + s_{[\frac{m}{2}]}^2 \neq 0$*;*
Type 3: $(0, 0, \dots, 0, s_m)$.

Proof. The proof is done by induction on m. If $m = 3$ then $s_1 s_3 = s_2^2$. The sequence (s_1, s_2, s_3) is of type 1 when $s_2 \neq 0$; belongs to type 2 if $s_2 = 0$ but $s_1 \neq 0$; and agrees with type 3 in the remaining case. The inductive procedure can be carried out easily, by the observation that if $(s_1, \dots, s_{m+1})$ is of length $m+1$, then, beside

(1) $s_1 s_k = s_2 s_{k-1} = \cdots = s_{[\frac{k+1}{2}]} s_{k+1-[\frac{k+1}{2}]}$, one has
(2) the subsequence $(s_2, \dots, s_{m+1})$ is a g-sequence of length m, therefore, falls into one of the three types by the inductive hypothesis. □

By considering f as an endomorphism of the free algebra $Z[d_1, d_3, \dots, d_{2k-1}]$ preserving the ideal generated by l_r's, we have, in $Z[d_1, d_3, \dots, d_{2k-1}]$, that

$$f(l_r) = x_{r,r} l_r + x_{r,r-1} l_{r-1} + \cdots + x_{r,k} l_k, \quad k \leqslant r \leqslant 2k-1 \tag{4.1}$$

when $n = 2k$ and that

$$f(l_r) = x_{r,r} l_r + x_{r,r-1} l_{r-1} + \cdots + x_{r,k+1} l_{k+1}, \quad k+1 \leqslant r \leqslant 2k \tag{4.2}$$

when $n = 2k+1$. Clearly we can assume that the polynomial $x_{r,s}$ has the homogeneous degree $\deg(x_{r,s}) = 4(r-s)$. In particular $x_{r,r}$ is an integer. This is the observation that brings g-sequences into our consideration.

LEMMA 4.2. *Let $(a_1, \dots, a_{2k-1})$ be the character sequence of f. If $n = 2k$ (resp. $n = 2k+1$), then $(a_1, \dots, a_{2k-1})$ (resp. $(a_3, \dots, a_{2k-1})$) is a g-sequence.*

Proof. Suppose that $n = 2k$ (resp. $n = 2k + 1$). For an r with $k \leqslant r \leqslant 2k - 1$ (resp. with $k + 1 \leqslant r \leqslant 2k - 1$) comparing the coefficient of $d_{2t-1}d_{2s-1}$, $s + t = r + 1$; $1 \leqslant s, t \leqslant k$, in $(4.1)_r$ (resp. $(4.2)_r$) gives

$$a_{2t-1}a_{2(r-t)+1} = x_{r,r}, \quad s + t = r + 1; \quad 1 \leqslant s, t \leqslant k \tag{4.3}$$

Lemma 4.1 for $n = 2k$ (resp. for $n = 2k + 1$) is verified by $(4.3)_r$ with $k \leqslant r \leqslant 2k - 3$ (resp. with $k + 1 \leqslant r \leqslant 2k - 3$). □

5. The Proof of Theorem 2

Assume in this section that $f(d_1) = ad_1 \neq 0$. Combining Lemma 4.1, Lemma 4.2 with Corollary 3.3 we find that the sequence $(a_1, \ldots, a_{2k-1})$ agrees with

$$(a, aq, \ldots, aq^{k-1}), \quad q = \pm a^2$$

when $n = 2k$; and agrees with

$$(a, a_3, a_3q, \ldots, a_3q^{k-2}), \qquad q = \pm a^2,\ a_3 = \pm a^3$$

when $n = 2k + 1$. We proceed further by showing the following lemma:

LEMMA 5.1. *Assume as the above. Then*

(1) $q = a^2$, *and*
(2) $a_3 = a^3$ *when* $n = 2k + 1$.

Proof. Suppose, otherwise, that $q = -a^2$. From $(4.3)_{2k-2}$ we find

$$x_{2k-2,2k-2} = -a^{4k-4}.$$

The relation $(4.1)_{2k-2}$ (resp. $(4.2)_{2k-2}$) becomes

$$f(l_{2k-2}) = -a^{4k-4}l_{2k-2} + x_{2k-2,2k-3}l_{2k-3} + \cdots + \\ + \begin{cases} x_{2k-2,k}l_k, & \text{if } n = 2k, \\ x_{2k-2,k+1}l_{k+1}, & \text{if } n = 2k+1. \end{cases} \tag{5.1}$$

If k is even comparing the coefficient of d_{k-1}^4 on both sides of (5.1) gives

$$a_{k-1}^4 = -a^{4k-4}. \tag{5.2}$$

If k is odd comparing the coefficient of $d_{k-2}^2 d_k^2$ we get

$$4a_{k-2}^2 a_k^2 = -4a^{4k-4} + \begin{cases} e & \text{if } n = 2k; \\ 0 & \text{if } n = 2k+1, \end{cases} \tag{5.3}$$

where $e \in Z$ is the coefficient of d_{k-2}^2 in $x_{2k-2,k}$, which is seen to be 0 by examining the coefficient of $d_{k-2}^3 d_{k+2}$ in (5.1). The contradictions in (5.2) or (5.3) verify (1).

For (2), assume that $a_3 = -a^3$. Then the character sequence of f is

$$(a, -a_3, \ldots, -a^{2k-1}),$$

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and the relation $(4.2)_{k+1}$ turns to be

$$f(l_{k+1}) = a^{2(k+1)} l_{k+1}.$$

Comparing the coefficient of d_{2k-1} one gets

$$2a_{2k-1}(f(d_3) - 2f(d_1)f(d_2)) = 2a^{2(k+1)}(d_3 - 2d_1 d_2).$$

With $d_2 = d_1^2$ and $a_{2k-1} = -a^{2k-1}$ we find

$$f(d_3) = -a^3 d_3 + 4a^3 d_1^3$$

This contradiction to Lemma 3.4 establishes (2). □

Proof of Theorem 2. With $f(d_1) = ad_1$, $a \neq 0$, $a_{2i-1} = a^{2i-1}$ by Lemma 5.1. It follows from Lemma 3.2 that

$$f(d_{2i-1}) = a^{2i-1} d_{2i-1}, \quad i \leqslant k.$$

Consequently $f(d_{2i}) = a^{2i} d_{2i}$, since $d_{2i} = g_{2i} \in Z[d_1, d_3, \ldots, d_{2k-1}]$ is of homogeneous degree $4i$. □

6. The Proof of Theorem 3

Theorem 3 can be easily deduced from

LEMMA 6.1. *If $f(d_1) = 0$, then the g-sequence $(a_1, \ldots, a_{2k-1})$ when $n = 2k$ (resp. $(a_3, \ldots, a_{2k-1})$ when $n = 2k + 1$) must be of type* 3.

Proof of Theorem 3. With $f(d_1) = 0$ the character sequence is $(0, \ldots, 0, a_{2k-1})$ by Lemma 6.1. Assume that $f^{m_i}(d_t) = 0$ for some m_i and $1 \leqslant t < i < 2k - 1$. We proceed to show $f^{m_i+1}(d_i) = 0$.

If i is even, d_i is the polynomial g_i in $d_1, \ldots, d_{i-2}$. $f^{m_i}(d_i) = 0$ follows from $f^{m_i}(d_t) = 0$, $t < i$. If i is odd, then $a_i = 0$ implies that $f(d_i)$ is a polynomial in $d_1, \ldots, d_{i-2}$. Again $f^{m_i}(d_t) = 0$, $t < i$, implies $f^{m_i+1}(d_i) = 0$.

Summarizing $f^N(d_i) = 0$, $i < 2k - 1$, for some N. It remains to show $f^N(d_{2k}) = 0$ when $n = 2k + 1$. However this follows directly from the relation

$$R_k\colon d_{2k} = 2d_1 d_{2k-1} - 2d_2 d_{2k-2} + \cdots + (-1)^{i-1} 2d_{k-1} d_{k+1} + d_k^2. \quad \square$$

The proof of Lemma 6.1 for even n is straightforward.

Proof of Lemma 6.1 *for* $n = 2k$. With $a_1 = 0$ the g-sequence $(a_1, \ldots, a_{2k-1})$ cannot be type 1 by Lemma 4.1. Suppose, on the contrary, that it is of type 2. Then from $(4.3)_r$ we find $x_{r,r} = 0$, $r \leqslant 2k - 1$, or equivalently, $(4.1)_r$ becomes

$$f(l_r) = x_{r,r-1} l_{r-1} + \cdots + x_{r,k} l_k, \quad k \leqslant r \leqslant 2k - 1. \qquad (6.1)_r$$

Applying f to both sides of $(6.1)_r$, substituting $(6.1)_s$, $k + 1 \leqslant s \leqslant r$, in the right hand side of the resulting equality yield

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$$f^2(l_r) = y_{r,r-2}l_{r-2} + \cdots + y_{r,k}l_k, \quad k \leqslant r \leqslant 2k-1,$$

where $y_{r,s}$ are certain polynomials in $x_{t,i}$'s and $f(x_{r,j})$'s. Repeating this procedure we find the iterated endomorphism f^k satisfies $f^k(l_r) = 0$, $k \leqslant r \leqslant 2k-1$, hence induces a ring homomorphism g: $H^*(CS_n) \to Z[d_1, \ldots, d_{2k-1}]$ so that the diagram

$$\begin{array}{ccc} Z[d_1, d_3, \ldots, d_{2k-1}] & \xrightarrow{f^k} & Z[d_1, d_3, \ldots, d_{2k-1}] \\ p \downarrow & \nearrow g & \\ H^*(CS_n) & & \end{array},$$

commutes, where p is the obvious quotient map. Since CS_n has finite dimension, and since the ring $Z[d_1, d_3, \ldots, d_{2k-1}]$ is a domain, $g = 0$. Thus $f^k(d_{2i-1}) = 0$, and consequently $a^k_{2i-1} = 0$, $i \leqslant k$. This contradiction verifies Lemma 6.1 for $n = 2k$. □

We complete the proof of Theorem 3 by establishing Lemma 6.1 for odd n.

DEFINITION. The sequence $(c_1, \ldots, c_{2k})$ whose entries are defined by the relations

$$c_1 = c_2 = 1; \qquad c_{2i-1} = 2c_{2i-2}, \quad i \leqslant k;$$
$$c_{2i} = 2c_1c_{2i-1} - 2c_2c_{2i-2} + \cdots + (-1)^{i-2}2c_{i-1}c_{i+1} + (-1)^{i-1}c_i^2, \quad i \leqslant k$$

will be called the *h-sequence* of length $2k$.

It is obvious that if $(c_1, \ldots, c_{2k})$ is the h-sequence of length $2k$ and if $k' \leqslant k$, then the subsequence $(c_1, \ldots, c_{2k'})$ is the h-sequence of length $2k'$. It is also clear that all h-sequences are classified by their lengths. For instance it is straightforward to see that the first ten entries in a h-sequence of length $\geqslant 10$ are given by

1, 1, 2, 3, 6, 10, 20, 35, 70, 146.

It is, indeed, a trivial exercise from the definition that

ASSERTION 1. *If $(c_1, \ldots, c_{2k})$ is a h-sequence, then $c_i > 0$, $i \leqslant 2k$.*

Again we use d_{2i} to represent the polynomial g_{2i}. Consider the graded homomorphism of free algebras

$$\beta : Z[d_1, d_3, \ldots, d_{2k-1}] \to Z[d_1]$$

defined by

$$\beta(d_1) = d_1; \quad \beta(d_{2i-1}) = 2\beta(d_1)\beta(d_{2i-2}), \quad 2 \leqslant i \leqslant k;$$

h-sequences plays the role in writing $\beta(d_i)$ as a multiple of d_1^i.

ASSERTION 2. *Let $(c_1, \ldots, c_{2k})$ be the h-sequence of length $2k$. Then β is given by $\beta(d_i) = c_i d_1^i$, $i \leqslant 2k$.*

What we need is the following variation of β.

ASSERTION 3. *If $\alpha : Z[d_1, d_3, \ldots, d_{2k-1}] \to Z[d_1]$ is the homomorphism defined by*

$$\alpha(d_1) = d_1; \qquad \alpha(d_{2i-1}) = 2\alpha(d_1)\alpha(d_{2i-2}), \quad 2 \leqslant i < k;$$

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and

$$\alpha(d_{2k-1}) = 4\alpha(d_1)\alpha(d_{2k-2}),$$

then

(1) $\alpha(d_i) = c_i d_1^i, \quad 1 \leqslant i \leqslant 2k-2; \quad \alpha(d_{2k-1}) = 2c_{2k-1}d_1^{2k-1};$
(2) $\alpha(d_{2k}) = (2c_{2k-1} + c_{2k})d_1^{2k}.$

Proof. The two homomorphisms α and β are related by

$$\alpha(d_{2i-1}) = \beta(d_{2i-1}), \quad 2 \leqslant i < k; \quad \text{and} \quad \alpha(d_{2k-1}) = 2\beta(d_{2k-1}).$$

(1) follows from Assertion 2. Finally since $d_{2k} = 2d_1d_{2k-1} + h$ with

$$h = -2d_2d_{2k-2} + \cdots + (-1)^{i-2}2d_{k-1}d_{k+1} + (-1)^{i-1}d_k^2,$$

a polynomial in $d_1, \ldots, d_{2k-3}$, we get

$$\begin{aligned}\alpha(d_{2k}) &= 2\alpha(d_1)\alpha(d_{2k-1}) + \beta(h)\\ &= 4c_{2k-1}d_1^{2k} + \beta(d_{2k} - 2d_1d_{2k-1}) = (2c_{2k-1} + c_{2k})d_1^{2k}.\end{aligned}$$ □

In the next result the homomorphisms α is applied to simplify some polynomial equalities in $Z[d_1, \ldots, d_{2k-1}]$ to equalities in $Z[d_1]$

LEMMA 6.2. *If $f(d_1) = 0$, then*

(1) *in the relation $(4.2)_{2k}$, $x_{2k,2k} = 0$; and*
(2) *the g-sequence $(a_3, \ldots, a_{2k-1})$ cannot be of type* 1.

Proof. Recall from Section 2 that the polynomial l_{2k} is given by

$$d_{2k}^2 = (2d_1d_{2k-1} - 2d_2d_{2k-2} + \cdots + (-1)^{i-2}2d_{k-1}d_{k+1} + (-1)^{i-1}d_k^2)^2.$$

From this we find that, with $f(d_1) = 0$, $f(l_{2k})$ is independent of d_{2k-1}. Thus comparing the coefficient of d_{2k-1} in $(4.2)_{2k}$ gives

$$\begin{aligned}0 &= x_{2k,2k}(4d_1d_{2k} - 4d_1^2d_{2k-1}) + x_{2k,2k-1}(d_{2k-1} - 4d_1d_{2k-2}) +\\ &\quad + x_{2k,2k-2}(-2d_{2k-3} + 4d_1d_{2k-4}) + \cdots \pm x_{2k,k+1}(2d_3 - 4d_1d_2).\end{aligned}$$

Applying the ring homomorphism α to this equality yields

$$0 = x_{2k,2k}(4\alpha(d_1)\alpha(d_{2k}) - 4\alpha(d_1^2)\alpha(d_{2k-1})),$$

i.e. $x_{2k,2k}c_{2k}d_1^{2k+1} = 0$ by Assertion 3. $x_{2k,2k} = 0$ follows from $c_{2k} > 0$.

For (2) the relation $(4.2)_{2k}$ takes the form

$$f(l_{2k}) = x_{2k,2k-1}l_{2k-1} + x_{2k,2k-2}l_{2k-2} + \cdots + x_{2k,k+1}l_{k+1} \tag{6.2}$$

by (1). Assume on the contrary that

$$a_{2i-1} = a_3q^{i-2} \neq 0, \quad 2 \leqslant i \leqslant k.$$

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Let $b_{j,i} \in Z$ be the coefficient of $d_{2i-1}d_{2(2k-j-i)+1}$, $1 \leqslant i \leqslant (2k-j-i+1)/2$, in $x_{2k,j}$.

If k is odd examining the coefficient of d_k^4 in (6.2) gives $a_k^4 = 0$.

If k is even we get

$$a_{k-1}^2 a_{k+1}^2 = b_{k+1,\frac{k}{2}} \text{ (by comparing the coefficient of } d_{k-1}^2 d_{k+1}^2 \text{ in (6.2))}$$

$$= 0 \text{ (by comparing the coefficient of } d_{k-1}^3 d_{k+3} \text{ in (6.2)).}$$

This contradiction to $a_3 q \neq 0$ verifies (2). □

Proof of Lemma 6.1 *for* $n = 2k+1$. With $f(d_1) = 0$ the g-sequence $(a_3, \ldots, a_{2k-1})$ is of either type 2 or 3 by (2) of Lemma 6.2. If it is of type 2,

$$x_{r,r} = 0, \quad k+1 \leqslant r \leqslant 2k-1$$

by $(4.3)_r$, and $x_{2k,2k} = 0$ by (1) of Lemma 6.2. The same argument as that in the proof of Lemma 6.1 for $n = 2k$ yields the contradiction $a_{2i-1} = 0$, $i \leqslant 2k-1$. □

7. The Proofs of Theorem 4 and 5

For a topological space X and an odd prime $p > 1$, let

$$\mathrm{St}_p^{2t(p-1)}\colon H^q(X; Z_p) \to H^{q+2t(p-1)}(X; Z_p)$$

be the Steenrod mod-p operators. The naturality of these operators imposes a bunch of restrictions on those endomorphisms of $H^*(X)$ that are induced by self-maps. This, besides Theorems 2 and 3, underlies the proof of Theorem 4.

For an integer $k > 1$ let $D(k)$ be the set of all odd primes p such that $1 < p < 2k-1$ and that p is prime to $2k-1$. As examples

$$D(3) = \{3\};\ D(4) = \{3, 5\};\ D(5) = \{5, 7\};\ \ldots,\ \text{etc.}$$

Obviously $D(k) \neq \phi$ for all $k > 2$.

For a self-map f of CS_n, we let $(a_1, \ldots, a_{2k-1})$ be the character sequence of the induced endomorphism f^*. Again we set $k = [n/2]$.

LEMMA 7.1. *If* $a_1 = 0$, *then* $(a_1, \ldots, a_{2k-1}) \equiv (0, \ldots, 0) \bmod p$, $p \in D(k)$.

Proof. If $a_1 = 0$, $(a_1, \ldots, a_{2k-1})$ is a g-sequence of type 3 by Lemma 6.1. It remains to show $a_{2k-1} \equiv 0 \bmod p$, $p \in D(k)$.

The action of St_p^* on the universal Chern classes c_i's is given by (cf. [1])

$$St_p^{2t(p-1)} c_i \equiv (i + t(p-1)) c_{i+t(p-1)} + h \bmod p,$$

where h is a polynomial decomposable in c_j, $j < i + t(p-1)$. Since the generators d_i's are related with the Chern classes of γ_n by the formula $c_i(\gamma_n) = 2d_i$ (Theorem 1), this implies that

$$St_p^{2t(p-1)} d_i \equiv (2k-1) d_{2k-1} + h' \bmod p \quad \text{whenever } 2k-1 = i + t(p-1),$$

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where h' is decomposable in d_j's, $j < i + t(p-1)$. For a $p \in D(k)$ applying f^* to

$$St_p^{2(p-1)} d_{2k-p} \equiv (2k-1)d_{2k-1} + h'$$

gives

$$St_p^{2(p-1)} f^*(d_{2k-p}) \equiv (2k-1)f^*(d_{2k-1}) + f^*(h') \bmod p.$$

Since $a_{2i-1} = 0$, $i < k$, the indecompositable component of the equality is

$$(2k-1)a_{2k-1}d_{2k-1} \equiv 0 \bmod p.$$

Now $a_{2k-1} \equiv 0 \bmod p$ follows from that p is prime to $2k-1$. □

For a self-map f of a finite complex X, its Lefschetz number is defined by

$$L(f) = 1 + \sum (-1)^r \mathrm{Tr}\{f^*\colon H^r(X; Q) \to H^r(X; Q)\},$$

If $X = CS_n$ the formula can be simplified, since $H^{\mathrm{odd}}(CS_n) = 0$, as

$$L(f) = 1 + \sum \mathrm{Tr}\{f^*\colon H^r(X) \to H^r(X)\}.$$

LEMMA 7.2. *Suppose that* $f^*(d_1) = 0$. *Then we have*

(1) $L(f) = 1$ *when* $n = 2, 3, 5$ *and,*
(2) $L(f) \equiv 1 \bmod p$ *for every* $p \in D(k)$ *when* $n > 5$.

Proof. By Lemma 2.2 we have

$$H^*(CS_2) \cong Z[d_1]/d_1^2; \quad H^*(CS_3) \cong Z[d_1]/d_1^4.$$

Thus $f^*(d_1) = 0$ implies that $L(f) = 1$ when $n = 2$ or 3.

Consider the case $n = 5$. With $f^*(d_1) = 0, f^*(d_i) = 0$ for $i = 2, 4$ by the relations R_1 and R_2. Assuming

$$f^*(d_3) = ad_3 + bd_1d_2, \quad a, b \in Z,$$

and applying f^* to $R_3 : d_3^2 - 2d_2d_4 = 0$ yields $(ad_3 + bd_1d_2)^2 = 0$.

Using R_i, $i = 1, 2, 3$, to rewrite this in terms of the basis d_2d_4, $d_1d_2d_3$ for $H^6(CS_6; Z)$ we obtain

$$(2a^2 - b^2)d_2d_4 + 2b(a+b)d_1d_2d_3 = 0.$$

$L(f) = 1$ now follows from $a = b = 0$. This completes the proof of (1).

For a prime p the Z_p-cohomology algebra of CS_n is

$$H^*(CS_n; Z_p) = Z_p[d_1, d_3, \ldots, d_{2k-1}]/L,$$

where L is the ideal generated by l_r's mod-p. Let $Z_p[d_1, \ldots, d_{2k-1}]^{2t}$ be the Z_p vector space spanned by $d_1^{r_1}d_3^{r_2} \ldots d_{2k-1}^{r_k}$, $\sum(2i-1)r_i = t$, and put

$$L^{2t} = L \cap Z_p[d_1, \ldots, d_{2k-1}]^{2t}.$$

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Then we have the exact sequence:

$$0 \to L^{2t} \to Z_p[d_1, \dots, d_{2k-1}]^{2t} \to H^{2t}(CS_n; Z_p) \to 0.$$

Since f^*, as an endomorphism of $Z_p[d_1, d_3, \dots, d_{2k-1}]$, preserves the ideal, L^{2t} is an invariant subspace of f^*. i.e. f^* induces an exact ladder:

$$\begin{array}{ccccccccc} 0 & \to & L^{2t} & \to & Z_p[d_1, \dots, d_{2k-1}]^{2t} & \to & H^{2t}(CS_n; Z_p) & \to & 0 \\ & & \downarrow & & f^* \downarrow & & \downarrow & & \\ 0 & \to & L^{2t} & \to & Z_p[d_1, \dots, d_{2k-1}]^{2t} & \to & H^{2t}(CS_n; Z_p) & \to & 0 \end{array}.$$

It follows that, for each $t > 0$,

$$\mathrm{Tr}(f^* \text{ on } H^{2t}(CS_n; Z_p)) = \mathrm{Tr}(f^* \text{ on } Z_p[d_1, \dots, d_{2k-1}]^{2t}) - \mathrm{Tr}(f^* \text{ on } L^{2t}).$$

Assume now that $n > 5$, $p \in D(k)$ and that $f^*(d_1) = 0$. Then $a_{2i-1} \equiv 0 \bmod p$, $i \leqslant k$, by Lemma 7.1. Consequently

$$\mathrm{Tr}(f^* \text{ on } Z_p[d_1, \dots, d_{2k-1}]^{2t}) = 0 \text{ and } \mathrm{Tr}(f^* \text{ on } L^{2t}) = 0$$

for all $t > 0$. These verifies

$$L(f) \equiv 1 + \sum_{t>0} \mathrm{Tr}(f^* \text{ on } H^{2t}(CS_n; Z_p)) \equiv 1 \bmod p. \qquad \square$$

Proof of Theorem 4. Let f be a self-map of CS_n with $L(f) = 0$. If $f^*(d_1) = ad_1$, $a \neq 0$, then $L(f) = \Pi_{1 \leqslant i \leqslant n-1}(1 + a^i)$ by Theorem 2 (the Poincare polynomial of CS_n is $\Pi_{1 \leqslant i \leqslant n-1}(1 + t^{2i})$ by Lemma 2.1). Now $L(f) = 0$ implies $a = -1$, and $f(d_i) = (-1)^i d_i$ follows from Theorem 2.

If $f^*(d_1) = 0$, there must be $n = 4$ by Lemma 7.2, and $f^*(d_2) = 0$ by R_1. With $L(f) = 0$ we can assume that

$$f^*(d_3) = -d_3 + bd_1d_2, \quad b \in Z.$$

Applying f^* to $R_3 : d_3^2 = 0$, rewriting everything in the resulting equation as multiples of the generator $d_1d_2d_3 \in H^{12}(CS_4) = Z$ by using R_1, R_2, R_3, we get $2b(b-1)$ $d_1d_2d_3 = 0$, i.e. either $f^*(d_3) = -d_3$ or $f^*(d_3) = -d_3 + d_1d_2$. These finish the proof. $\square$

Consider the free algebra

$$\Phi(CS_n) = Z[x_1, x_3, \dots, x_{2k-1}] \otimes \Lambda_Z(y_{[\frac{n+1}{2}]}, y_{[\frac{n+1}{2}]+1}, \dots, y_{n-1}),$$

the tensor product of the polynomial algebra in x_i's with the exterior algebra in y_r's. It is graded by $\deg(x_i) = 2i$ and $\deg(y_r) = 4r - 1$. The differential δ: $\Phi(CS_n) \to \Phi(CS_n)$ of degree 1 given by

$$\delta(x_i) = 0 \quad \text{and} \quad \delta(y_r) = l_r(x_1, x_3, \dots, x_{2k-1})$$

furnishes $\Phi(CS_n)$ with the structure of a differential graded commutative algebra over Z. Indeed Lemma 2.3 implies that

LEMMA 7.3 (cf. [4, Proposition 3]). *The homomorphism*

$$g\colon \Phi(CS_n) \to H^*(CS_n), \quad \textit{given by } g(x_{2i-1}) = d_{2i-1};\ g(y_r) = 0$$

is the minimal model (over Z) for $H^*(CS_n)$.

Proof of Theorem 5. Let f be a self-homotopy equivalence of CS_n. Then

$$f(d_1) = \pm d_1, \quad \text{and} \quad f(d_i) = (\pm 1)^i d_i \quad \text{for all } i \leqslant n-1$$

by Theorem 2. The relations $(4.1)_r$ (resp. $(4.2)_r$) becomes

$$f^*(l_r) = l_r \quad \text{for } \left[\frac{n+1}{2}\right] \leqslant r \leqslant n-1$$

In views of Lemma 7.3, a minimal model

$$\Phi(f)\colon \Phi(CS_n) \to \Phi(CS_n)$$

for f can be chosen to be $\Phi(f)(x_{2i-1}) = (\pm 1)^i x_{2i-1}$ and

$$\Phi(f)(y_r) = y_r.$$

By the rational homotopy theory [8] the forms $y_r \otimes 1$'s $\in \Phi(CS_n) \otimes Q$ constitute a basis for $\mathrm{Hom}(\pi_{\mathrm{odd}}(CS_n), Q)$ and the induced chain endomorphism $\Phi(f) \otimes 1$ of $\Phi(CS_n) \otimes Q$, module decomposables, agrees with the dual action of f_* on $\pi_*(CS_n)$. Thus the proof is done by (7.1). □

8. Examples

This section serves as a supplement to Theorem 3. We present self-maps f of CS_n, for even n, so that $f^*(d_i) = 0$ when $i \neq 2k-1$, but $f^{*N}(d_{2k-1}) \neq 0$ for all $N > 0$.

Let $e_1, \ldots, e_{4k}$ be the standard basis for the Euclidean space R^{4k} and let S^{4k-2} be the unit sphere in the subspace spanned by e_i, $i < 4k$. The map

$$p\colon CS_{2k} \to S^{4k-2}, \quad p(J) = Je_{4k-1} \in S^{4k-2},$$

is a fiber bundle projection whose fiber inclusion over $e_{4k-1} \in S^{4k-2}$ is

$$l\colon CS_{2k-1} \to CS_{2k}, \quad l(J') = J' \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In fact the class d_{2k-1} is cospherical in the sense that

(1) $\pi^*(e) = d_{2k-1}$, where $e \in H^{2k-2}(S^{4k-2}) = Z$ is a generator (cf. [4]).

On the other hand the homotopy exact sequence of p gives the exact sequence of vector spaces over Q

$$\cdots \to \pi_{4k-2}(CS_{2k-1}) \otimes Q \to \pi_{4k-2}(CS_{2k}) \otimes Q \xrightarrow{p_*} \\ \to \pi_{4k-2}(S^{4k-2}) \otimes Q \to \pi_{4k-3}(CS_{2k-1}) \otimes Q \to \cdots$$

From the minimal model for $H^*(CS_{2k}; Q)$ (Lemma 7.3) we find

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$\pi_{4k-2}(CS_{2k-1}) \otimes Q = \pi_{4k-3}(CS_{2k-1}) \otimes Q = 0.$

This implies that

(2) there exists a map α: $S^{4k-2} \to CS_{2k}$ so that $\deg(p \circ \alpha) \neq 0$. Thus if we let $f_\alpha = \alpha \circ p$, for a α satisfying 2), then f_α^* satisfies

(3) $f_\alpha^*(d_i) = 0$ for all $i \neq 2k-1$ but $f_\alpha^{*N}(d_{2k-1}) = \deg(p \circ \alpha)^N d_{2k-1}$. Finally it is worth to point out that

(4) the class $f_\alpha^*(d_{2k-1}) \in H^{4k-2}(CS_{2k})$ is always divisible by $\frac{1}{2}(4k-3)!$ since f_α factors through the sphere S^{4k-2} and since $2d_{2k-1}$ is the $(2k-1)$th Chern class of the bundle γ_{2k} [2].

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