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THE LEAST COMMON MULTIPLE OF CONSECUTIVE ARITHMETIC PROGRESSION TERMS

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Abstract Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. We define the arithmetic function $g_{k,a,b}$ for any positive integer n by

$$g_{k,a,b}(n) := \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\operatorname{lcm}(b+na, b+(n+1)a, \dots, b+(n+k)a)}.$$

If we let $a = 1$ and $b = 0$, then $g_{k,a,b}$ becomes the arithmetic function that was previously introduced by Farhi. Farhi proved that $g_{k,1,0}$ is periodic and that $k!$ is a period. Hong and Yang improved Farhi's period $k!$ to $\operatorname{lcm}(1, 2, \dots, k)$ and conjectured that $(\operatorname{lcm}(1, 2, \dots, k, k+1))/(k+1)$ divides the smallest period of $g_{k,1,0}$. Recently, Farhi and Kane proved this conjecture and determined the smallest period of $g_{k,1,0}$. For the general integers $a \geqslant 1$ and $b \geqslant 0$, it is natural to ask the following interesting question: is $g_{k,a,b}$ periodic? If so, what is the smallest period of $g_{k,a,b}$? We first show that the arithmetic function $g_{k,a,b}$ is periodic. Subsequently, we provide detailed p-adic analysis of the periodic function $g_{k,a,b}$. Finally, we determine the smallest period of $g_{k,a,b}$. Our result extends the Farhi–Kane Theorem from the set of positive integers to general arithmetic progressions.

Keywords: arithmetic progression; least common multiple; p-adic valuation; arithmetic function; smallest period

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1. Introduction

Many beautiful and important theorems about the arithmetic progression in number theory are known: Dirichlet's Theorem [**1**, **11**] and the Green–Tao Theorem [**9**] being the two most famous examples. For some other results, see, for example, [**4**, **12**, **15**, **21**, **22**]. Meanwhile, the topic of the least common multiple of any given sequence of positive integers has received a lot of attention from many authors: see, for example, [**2**, **3**, **5**–**7**, **10**, **11**, **13**, **14**, **16**, **19**, **20**]. For detailed background information about the least common multiple of finite arithmetic progressions, we refer readers to [**17**].

Farhi [**6**, **7**] investigated the least common multiple of a finite number of consecutive integers. Let $k \geqslant 0$ be an integer. It was proved in [**6**] and [**7**] that $\operatorname{lcm}(n, n+1, \dots, n+k)$ is divisible by $n\binom{n+k}{k}$ and also divides

$$n\binom{n+k}{k}\operatorname{lcm}\left(\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right).$$

Farhi [**6**,**7**] showed that the last equality holds if $k!|(n+k+1)$. Farhi also introduced the arithmetic function g_k, which is defined for any positive integer n by

$$g_k(n) := \frac{n(n+1)\cdots(n+k)}{\mathrm{lcm}(n,n+1,\dots,n+k)}.$$

Farhi then proved that the sequence $\{g_k\}_{k=0}^{\infty}$ satisfies the recursive relation $g_k(n) = \gcd(k!,(n+k)g_{k-1}(n))$ for all positive integers n, where $\gcd(a,b)$ means the greatest common divisor of integers a and b. Using this relation, we can easily show (by induction on k) that for any non-negative integer k, the function g_k is periodic of period $k!$. This is a result due to Farhi [**7**]. Define P_k to be the smallest period of the function g_k. Farhi's result then says that $P_k|k!$. Define $L_0 := 1$ and, for any integer $k \geqslant 1$, define $L_k := \mathrm{lcm}(1,2,\dots,k)$. Hong and Yang [**17**] showed that $g_k(1)|g_k(n)$ for any non-negative integer k and any positive integer n. Consequently, using this result, they showed that $P_k|L_k$ for all positive integers k. This improves Farhi's period. In [**17**], Hong and Yang raised a conjecture stating that $L_{k+1}/(k+1)$ divides P_k for all non-negative integers k. From this conjecture, one can read that $k|P_k$ and $P_k = L_k$ if $k+1$ is a prime. Very recently, Farhi and Kane [**8**] found a proof of the Hong–Yang conjecture. Furthermore, Farhi and Kane determined the exact value of P_k, which solved the open problem posed by Farhi in [**7**].

Throughout this paper, let $\mathbb{Q}$ and $\mathbb{N}$ denote the field of rational numbers and the set of positive integers, respectively. Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $k,b \in \mathbb{N}_0$ and let $a \in \mathbb{N}$. We define the arithmetic function $g_{k,a,b} : \mathbb{N} \to \mathbb{N}$ by

$$g_{k,a,b}(n) = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\mathrm{lcm}(b+na,b+(n+1)a,\dots,b+(n+k)a)}.$$

Note that $g_{k,1,0} = g_k$. It is natural to ask the following interesting question.

Problem 1.1. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. Is $g_{k,a,b}$ periodic and, if so, what is the smallest period of $g_{k,a,b}$?*

Assume that $g_{k,a,b}$ is periodic and that $P_{k,a,b}$ is the smallest period of $g_{k,a,b}$. We can then use $P_{k,a,b}$ to give a formula for $\mathrm{lcm}(b+na,b+(n+1)a,\dots,b+(n+k)a)$ as follows: for any positive integer n, we have

$$\mathrm{lcm}(b+na,b+(n+1)a,\dots,b+(n+k)a) = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{g_{k,a,b}(\langle n\rangle_{P_{k,a,b}})},$$

where $\langle n\rangle_{P_{k,a,b}}$ denotes the least non-negative residue of n modulo $P_{k,a,b}$. Therefore, it is important to determine the exact value of $P_{k,a,b}$.

In this paper, we investigate the least common multiple of consecutive terms in arithmetic progressions. As usual, for any prime number p, we let v_p be the normalized p-adic valuation of $\mathbb{Q}$, i.e. $v_p(a) = s$ if $p^s \parallel a$. For any real number x, by $\lfloor x\rfloor$ we denote the largest integer no more than x. Let $e_{p,k} := \lfloor \log_p k\rfloor = \max_{1\leqslant i\leqslant k}\{v_p(i)\}$ be the largest exponent of a power of p that is at most k. We can now give the main result of this paper.

Theorem 1.2. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. The arithmetic function $g_{k,a,b}$ is then periodic, and if $\gcd(a,b) = 1$, then its smallest period equals $Q_{k,a}$, where*

$$Q_{k,a} := \frac{L_k}{\delta_{k,a} \prod_{\text{prime } q \mid \gcd(a, L_k)} q^{e_{q,k}}}, \tag{1.1}$$

and

$$\delta_{k,a} := \begin{cases} p^{e_{p,k}} & \text{if } p \nmid a \text{ and } v_p(k+1) \geqslant e_{p,k} \text{ for some prime } p \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

For $\gcd(a,b) > 1$, the smallest period of $g_{k,a,b}$ is equal to $Q_{k,a'}$ with $a' = a/(\gcd(a,b))$.

Thus we answer Problem 1.1 completely. Our result extends the Farhi–Kane Theorem from the set of positive integers to general arithmetic progressions.

The paper is organized as follows. In § 2, by using a well-known result of Hua [**18**] we show that the arithmetic function $g_{k,a,b}$ is periodic (see Theorem 2.5). Then, in § 3, we provide detailed p-adic analysis of the periodic function $g_{k,a,b}$ and determine the smallest period of $g_{k,a,b}$. In the last section, we prove Theorem 1.2 and give an example to illustrate its validity.

2. The periodicity of $g_{k,a,b}$

Hong and Yang [**17**] proved that L_k is a period of g_k. In this section, we introduce a new method to show that for any integers $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$, the arithmetic function $g_{k,a,b}$ is periodic, and in particular L_k is also a period of $g_{k,a,b}$. First we need a well-known result of Hua. One can easily deduce this result by using the principle of inclusion–exclusion (see, for instance, [**18**, p. 11]).

Lemma 2.1 (Hua [18]). *Let $a_1, a_2, \dots, a_n$ be any n positive integers. We then have*

$$\operatorname{lcm}(a_1, a_2, \dots, a_n) = a_1 a_2 \cdots a_n \prod_{r=2}^{n} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(a_{i_1}, \dots, a_{i_r}))^{(-1)^{r-1}}.$$

Lemma 2.2. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any $2n$ positive integers. Let $3 \leqslant t \leqslant n$ be a given integer. If $\gcd(a_{i_1}, \dots, a_{i_t}) = \gcd(b_{i_1}, \dots, b_{i_t})$ for any $1 \leqslant i_1 < \cdots < i_t \leqslant n$, we then have*

$$\frac{a_1 a_2 \cdots a_n}{\operatorname{lcm}(a_1, a_2, \dots, a_n)} \prod_{r=2}^{t-1} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(a_{i_1}, \dots, a_{i_r}))^{(-1)^{r-1}}$$
$$= \frac{b_1 b_2 \cdots b_n}{\operatorname{lcm}(b_1, b_2, \dots, b_n)} \prod_{r=2}^{t-1} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (\gcd(b_{i_1}, \dots, b_{i_r}))^{(-1)^{r-1}}.$$

Proof. If $\gcd(a_{i_1}, \dots, a_{i_t}) = \gcd(b_{i_1}, \dots, b_{i_t})$ for any $1 \leqslant i_1 < \cdots < i_t \leqslant n$, then we have $\gcd(a_{i_1}, \dots, a_{i_k}) = \gcd(b_{i_1}, \dots, b_{i_k})$ for any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and any $n \geqslant k \geqslant t$. Thus, by using Lemma 2.1, we get the result of Lemma 2.2. □

In particular, we have the following result.

Lemma 2.3. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any $2n$ positive integers. If, for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$, we have $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$, then*

$$\frac{1}{\prod_{1\leqslant i<j\leqslant n}\gcd(a_i,a_j)}\frac{a_1a_2\cdots a_n}{\operatorname{lcm}(a_1,a_2,\dots,a_n)} = \frac{1}{\prod_{1\leqslant i<j\leqslant n}\gcd(b_i,b_j)}\frac{b_1b_2\cdots b_n}{\operatorname{lcm}(b_1,b_2,\dots,b_n)}.$$

Proof. Since $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$ for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$, we have $\gcd(a_{i_1}, \dots, a_{i_k}) = \gcd(a_{i_1}, \dots, a_{i_k})$ for any $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $k \geqslant 3$. By using Lemma 2.1, we get the conclusion of Lemma 2.3. □

Notice that if $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leqslant i < j \leqslant n$, then $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = \gcd(b_{i_1}, b_{i_2}, b_{i_3})$ for any $1 \leqslant i_1 < i_2 < i_3 \leqslant n$. It follows immediately from Lemma 2.3 that the following is true.

Corollary 2.4. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any $2n$ positive integers. If $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leqslant i < j \leqslant n$, we then have*

$$\frac{a_1a_2\cdots a_n}{\operatorname{lcm}(a_1,a_2,\dots,a_n)} = \frac{b_1b_2\cdots b_n}{\operatorname{lcm}(b_1,b_2,\dots,b_n)}.$$

We can now give the main result of this section. This also gives an alternative proof of the Hong–Yang period of the periodic function g_k [**17**].

Theorem 2.5. *Let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers. The arithmetic function $g_{k,a,b}$ is then periodic, and L_k is a period of $g_{k,a,b}$.*

Proof. Let n be any positive integer. For any $0 \leqslant i < j \leqslant k$, we have

$$\begin{aligned}\gcd(b+(n+i+L_k)a, b+(n+j+L_k)a) &= \gcd(b+(n+i+L_k)a, (j-i)a)\\ &= \gcd(b+(n+i)a, (j-i)a)\\ &= \gcd(b+(n+i)a, b+(n+j)a).\end{aligned}$$

Thus, by Corollary 2.4, we obtain

$$\begin{aligned}&\frac{(b+(n+L_k)a)(b+(n+1+L_k)a)\cdots(b+(n+k+L_k)a)}{\operatorname{lcm}(b+(n+L_k)a, b+(n+1+L_k)a,\dots,b+(n+k+L_k)a)}\\ &\qquad = \frac{(b+na)(b+(n+1)a)\cdots(b+(n+k)a)}{\operatorname{lcm}(b+na, b+(n+1)a,\dots,b+(n+k)a)}.\end{aligned}$$

In other words, for any positive integer n, we have $g_{k,a,b}(n + L_k) = g_{k,a,b}(n)$, as desired. □

Evidently, Theorem 2.5 gives an affirmative answer to the first part of Problem 1.1.

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3. p-adic analysis of $g_{k,a,b}$

Throughout this section we always let $k \geqslant 0$, $a \geqslant 1$ and $b \geqslant 0$ be integers such that $\gcd(a, b) = 1$. From the main result of the previous section (Theorem 2.5), we know that the arithmetic function $g_{k,a,b}$ is periodic. Let $P_{k,a,b}$ denote the smallest period of $g_{k,a,b}$. By Theorem 2.5 we then know that $P_{k,a,b}$ is a divisor of L_k. But the exact value of $P_{k,a,b}$ is still unknown. In this section, we will determine the exact value of $P_{k,a,b}$. We need some more notation. Let

$$S_{k,a,b}(n) := \{b + na, b + (n + 1)a, \dots, b + (n + k)a\}$$

be any $k+1$ consecutive terms in the arithmetic progression $\{b + ma\}_{m\in\mathbb{N}_0}$. For a given prime number p, define $g_{p,k,a,b}(n) := v_p(g_{k,a,b}(n))$. Since $g_{k,a,b}$ is a periodic function, $g_{p,k,a,b}$ is also a periodic function for each prime p and $P_{k,a,b}$ is a period of $g_{p,k,a,b}$. Let $P_{p,k,a,b}$ be the smallest period of $g_{p,k,a,b}$. We have the following result.

Lemma 3.1. *We have* $P_{k,a,b} = \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$.

Proof. Since, for any $n \in \mathbb{N}$, we have that $v_p(g_{k,a,b}(n + P_{k,a,b})) = v_p(g_{k,a,b}(n))$, i.e. $P_{p,k,a,b}|P_{k,a,b}$ for each prime p. Hence we have $\mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})|P_{k,a,b}$.

Conversely, for any $n \in \mathbb{N}$, we have that

$$v_p(g_{k,a,b}(n + \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b}))) = v_p(g_{k,a,b}(n))$$

for each prime p. Thus, we have

$$g_{k,a,b}(n + \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})) = g_{k,a,b}(n)$$

for any $n \in \mathbb{N}$: that is, we have $P_{k,a,b}|\,\mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$. Therefore, we have $P_{k,a,b} = \mathrm{lcm}_{p\,\mathrm{prime}}(P_{p,k,a,b})$, as required. ☐

Hence we only need to compute $P_{p,k,a,b}$ for each prime p to get the exact value of $P_{k,a,b}$. The following result is due to Farhi [**6**]. An alternative proof of it was given by Hong and Feng [**13**].

Lemma 3.2. *Let $\{u_i\}_{i\in\mathbb{N}_0}$ be a strictly increasing arithmetic progression of non-zero integers and let k be any given non-negative integer. The integer $\mathrm{lcm}(u_0, u_1, \dots, u_k)$ is then a multiple of*

$$\frac{u_0u_1\cdots u_k}{k!(\gcd(u_0, u_1))^k}.$$

Lemma 3.3. *For any positive integer n, we have $g_{k,a,b}(n)|k!$.*

Proof. Let $u_i = b + a(n + i)$ for $0 \leqslant i \leqslant k$. Then $\gcd(u_0, u_1) = 1$, since a and b are coprime. So by Lemma 3.2 we know that there is an integer A such that

$$\mathrm{lcm}(b + na, b + (n + 1)a, \dots, b + (n + k)a) = A\frac{(b + an)(b + a(n + 1))\cdots(b + a(n + k))}{k!}.$$

It then follows that $k! = Ag_{k,a,b}(n)$. ☐

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It follows from Lemma 3.3 that $g_{p,k,a,b}(n) = v_p(g_{k,a,b}(n)) = 0$ for each prime $p > k$ and any positive integer n. Hence we have $P_{p,k,a,b} = 1$ for each prime $p > k$. So, by Lemma 3.1, in order to determine the exact value of $P_{k,a,b}$, it suffices to compute the exact value of $P_{p,k,a,b}$ for all the primes p such that $1 < p \leqslant k$. First we consider the case in which $p|a$ and $1 < p \leqslant k$. Since $\gcd(a,b) = 1$, we have $\gcd(p,b) = 1$, and thus $\gcd(p, b+(n+i)a) = 1$ for any integer $n \in \mathbb{N}$ and if $0 \leqslant i \leqslant k$. Hence $\gcd(p, g_{k,a,b}(n)) = 1$ for any integer $n \geqslant 1$, i.e. we have $g_{p,k,a,b}(n) = 0$ for any integer $n \geqslant 1$ if $p|a$. Thus $P_{p,k,a,b} = 1$ if $p|a$. We put these facts into the following lemma.

Lemma 3.4. *Let p be a prime such that either $p > k$ or $p|a$. We then have $P_{p,k,a,b} = 1$.*

In what follows we treat the remaining case in which $p \nmid a$ and $1 < p \leqslant k$. Clearly, we have

$$\begin{aligned} g_{p,k,a,b}(n) &= \sum_{m \in S_{k,a,b}(n)} v_p(m) - \max_{m \in S_{k,a,b}(n)} v_p(m) \\ &= \sum_{e \geqslant 1} \sum_{m \in S_{k,a,b}(n)} (1 \text{ if } p^e|m) - \sum_{e \geqslant 1} (1 \text{ if } p^e \text{ divides some } m \in S_{k,a,b}(n)) \\ &= \sum_{e \geqslant 1} \#\{m \in S_{k,a,b}(n) : p^e|m\} - \sum_{e \geqslant 1} (1 \text{ if } p^e \text{ divides some } m \in S_{k,a,b}(n)) \\ &= \sum_{e \geqslant 1} \max(0, \#\{m \in S_{k,a,b}(n) : p^e|m\} - 1). \end{aligned} \tag{3.1}$$

We then have the following lemmas.

Lemma 3.5. *If $p \nmid a$ and $e > e_{p,k}$, then there is at most one element of $S_{k,a,b}(n)$ which is divisible by p^e.*

Proof. Suppose that there exist two integers i and j such that $p^e|b + (n+i)a$ and $p^e|b + (n+j)a$, where $0 \leqslant i < j \leqslant k$. We then have $p^e|(j-i)a$. Since $\gcd(p,a) = 1$, we get $p^e|(j-i)$. From it we deduce that $v_p(j-i) \geqslant e > e_{p,k}$. This is a contradiction. □

Lemma 3.6. *Let e be a positive integer. If $p \nmid a$, then any p^e consecutive terms in the arithmetic progression $\{b + ma\}_{m \in \mathbb{N}_0}$ are pairwise incongruent modulo p^e. Furthermore, if $e \leqslant e_{p,k}$, then there is at least one element of $S_{k,a,b}(n)$ that is divisible by p^e.*

Proof. Suppose that there exist two integers i and j such that $b+(m+i)a \equiv b+(m+j)a \pmod{p^e}$, where $m \geqslant 0$ and $0 \leqslant i < j \leqslant p^e - 1$. Then $p^e|(j-i)a$. Since $\gcd(p,a) = 1$, we have $p^e|(j-i)$. This is impossible. Thus the first part is true.

Now let $e \leqslant e_{p,k}$. Then $1 \leqslant p^e \leqslant k$. Hence $S_{k,a,b}(n)$ holds p^e consecutive terms and one of these is divisible by p^e by the above discussion. Therefore the second part holds. □

By Lemma 3.5, we know that all the terms on the right-hand side of (3.1) are 0 if $e > e_{p,k}$. By Lemma 3.6, there is at least one element divisible by p^e in the set $S_{k,a,b}(n)$ if $e \leqslant e_{p,k}$. Therefore, by (3.1) we obtain

$$g_{p,k,a,b}(n) = \sum_{e=1}^{e_{p,k}} f_e(n), \tag{3.2}$$

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where $f_e(n) := \#\{m \in S_{k,a,b}(n) : p^e|m\} - 1$. Since $b + (n + i + p^e)a \equiv b + (n + i)a \pmod{p^e}$ for any $i \in \{0, 1, \ldots, k\}$, we have $f_e(n + p^e) = f_e(n)$. Therefore, p^e is a period of $f_e(n)$. Hence $f_e(n + p^{e_{p,k}}) = f_e(n)$ is true for each $e \in \{1, \ldots, e_{p,k}\}$. This implies that $g_{p,k,a,b}(n + p^{e_{p,k}}) = g_{p,k,a,b}(n)$. Consequently, $p^{e_{p,k}}$ is a period of $g_{p,k,a,b}(n)$. Thus $P_{p,k,a,b}|p^{e_{p,k}}$. It follows immediately that the $P_{p,k,a,b}$ are relatively prime for different prime numbers p. But Lemmas 3.1 and 3.4 tell us that $P_{k,a,b} = \mathrm{lcm}_{p \text{ prime}, p \leqslant k, p \nmid a}(P_{p,k,a,b})$. Therefore, we get the following result.

Lemma 3.7. *We have*

$$P_{k,a,b} = \prod_{p \text{ prime}, p \nmid a, p \leqslant k} P_{p,k,a,b},$$

where $P_{p,k,a,b}$ satisfies that $P_{p,k,a,b}|p^{e_{p,k}}$.

According to Lemma 3.7, it suffices to compute the p-adic valuation of $P_{p,k,a,b}$ for the prime numbers p satisfying $p \nmid a$ and $p \in (1, k]$. Now let us determine the p-adic valuation of $P_{k,a,b}$ for these prime numbers p.

Proposition 3.8. *Let $a \geqslant 1$ and $b \geqslant 0$ be integers such that $\gcd(a, b) = 1$. Let $k \geqslant 2$ be an integer and let $p \in (1, k]$ be a prime number such that $p \nmid a$.*

(i) *If $v_p(k + 1) < e_{p,k}$, then $v_p(P_{k,a,b}) = e_{p,k}$.*

(ii) *If $v_p(k + 1) \geqslant e_{p,k}$, then $v_p(P_{k,a,b}) = 0$.*

Proof. (i) Since $p^{e_{p,k}}$ is a period of $g_{p,k,a,b}$, it suffices to prove that $p^{e_{p,k}-1}$ is not the period of $g_{p,k,a,b}$, from which it follows that $p^{e_{p,k}}$ is the smallest period of $g_{p,k,a,b}$. By (3.2), we have

$$g_{p,k,a,b}(n) = \sum_{e=1}^{e_{p,k}} f_e(n) = \sum_{e=1}^{e_{p,k}-1} f_e(n) + f_{e_{p,k}}(n).$$

Since $p^{e_{p,k}-1}$ is a period of $\sum_{e=1}^{e_{p,k}-1} f_e(n)$, it is sufficient to prove that $p^{e_{p,k}-1}$ is not the period of $f_{e_{p,k}}(n)$. We claim that there exists a positive integer n_0 such that $f_{e_{p,k}}(n_0 + p^{e_{p,k}-1}) = f_{e_{p,k}}(n_0) - 1$.

By $v_p(k+1) < e_{p,k}$, we deduce that $p^{e_{p,k}} \nmid (k + 1)$ and $p^{e_{p,k}} \leqslant k$. We can suppose that $k + 1 \equiv l \pmod{p^{e_{p,k}}}$ for some $1 \leqslant l \leqslant p^{e_{p,k}} - 1$. We divide the proof of part (i) into the following two cases.

Case 1. $1 \leqslant l \leqslant p^{e_{p,k}} - p^{e_{p,k}-1}$. Since $p \nmid a$, we can always find a suitable n_0 such that $b + n_0 a \equiv 0 \pmod{p^{e_{p,k}}}$. Consider the following two sets:

$$S_{k,a,b}(n_0) = \{b + n_0 a, \ldots, b + (n_0 + p^{e_{p,k}-1} - 1)a, b + (n_0 + p^{e_{p,k}-1})a, \ldots, b + (n_0 + k)a\}$$

and

$$S_{k,a,b}(n_0 + p^{e_{p,k}-1}) = \{b + (n_0 + p^{e_{p,k}-1})a, \ldots, b + (n_0 + k)a, \\ b + (n_0 + k + 1)a, \ldots, b + (n_0 + k + p^{e_{p,k}-1})a\}.$$

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We now have that $\{b+(n_0+p^{e_{p,k}-1})a,\dots,b+(n_0+k)a\}$ is the intersection of $S_{k,a,b}(n_0)$ and $S_{k,a,b}(n_0+p^{e_{p,k}-1})$. So to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$, it suffices to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. By Lemma 3.6, any $p^{e_{p,k}}$ consecutive terms in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ are pairwise incongruent modulo $p^{e_{p,k}}$. Thus the terms divisible by $p^{e_{p,k}}$ in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ must be of the form $b+(n_0+tp^{e_{p,k}})a$, $t\in\mathbb{Z}$. Since $k+1\equiv l \pmod{p^{e_{p,k}}}$ and $1\leqslant l\leqslant p^{e_{p,k}}-p^{e_{p,k}-1}$, we have $k+j\equiv l+j-1\not\equiv 0 \pmod{p^{e_{p,k}}}$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. Hence $p^{e_{p,k}}\nmid(b+(n_0+k+j)a)$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. Thus none of the elements in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$ are divisible by $p^{e_{p,k}}$. On the other hand, since $b+an_0\equiv 0 \pmod{p^{e_{p,k}}}$, it follows from Lemma 3.6 that there is exactly one term in the set $\{b+n_0a, b+(n_0+1)a,\dots, b+(n_0+p^{e_{p,k}-1}-1)a\}$ that is divisible by $p^{e_{p,k}}$. Therefore, the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$ is equal to the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ minus 1. Namely, $f_{e_{p,k}}(n_0+p^{e_{p,k}-1})=f_{e_{p,k}}(n_0)-1$ as required. The claim is proved in this case.

Case 2. $p^{e_{p,k}}-p^{e_{p,k}-1}<l\leqslant p^{e_{p,k}}-1$. Since $p\nmid a$, it is easy to see that there is a positive integer n_0 such that $b+(n_0+p^{e_{p,k}-1}-1)a\equiv 0 \pmod{p^{e_{p,k}}}$. As in the discussion of Case 1, to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$, it suffices to compare the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$ with the number of terms divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. From $b+(n_0+p^{e_{p,k}-1}-1)a\equiv 0 \pmod{p^{e_{p,k}}}$ one can deduce that the terms divisible by $p^{e_{p,k}}$ in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ must be of the form $b+(n_0+p^{e_{p,k}-1}-1+tp^{e_{p,k}})a$ with $t\in\mathbb{Z}$. Since $k+1\equiv l \pmod{p^{e_{p,k}}}$ for some $p^{e_{p,k}}-p^{e_{p,k}-1}<l\leqslant p^{e_{p,k}}-1$, we have $p^{e_{p,k}}-p^{e_{p,k}-1}+1\leqslant l+j-1\leqslant p^{e_{p,k}}+p^{e_{p,k}-1}-2$ and so $k+j\equiv l+j-1\not\equiv p^{e_{p,k}-1}-1 \pmod{p^{e_{p,k}}}$ for all $1\leqslant j\leqslant p^{e_{p,k}-1}$. It follows that for all $1\leqslant j\leqslant p^{e_{p,k}-1}$, we have $p^{e_{p,k}}\nmid(b+(n_0+k+j)a)$. That is, there does not exist an integer divisible by $p^{e_{p,k}}$ in the set $\{b+(n_0+k+1)a,\dots,b+(n_0+k+p^{e_{p,k}-1})a\}$. But the term $b+(n_0+p^{e_{p,k}-1}-1)a$ is the only term divisible by $p^{e_{p,k}}$ in the set $\{b+n_0a, b+(n_0+1)a,\dots,b+(n_0+p^{e_{p,k}-1}-1)a\}$. Thus the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0+p^{e_{p,k}-1})$ equals the number of terms divisible by $p^{e_{p,k}}$ in the set $S_{k,a,b}(n_0)$ minus 1. Hence the desired result $f_{e_{p,k}}(n_0+p^{e_{p,k}-1})=f_{e_{p,k}}(n_0)-1$ follows immediately. The proof of the claim is complete.

From the claim we deduce immediately that $p^{e_{p,k}-1}$ is not a period of $g_{p,k,a,b}$. Thus $p^{e_{p,k}}$ is the smallest period of $g_{p,k,a,b}$. It follows that $v_p(P_{k,a,b})=e_{p,k}$ as desired.

(ii) By Lemma 3.7, we know that to prove part (ii) it is sufficient to prove that $v_p(P_{q,k,a,b})=0$ for each prime q with $q\leqslant k$ and $q\nmid a$. For any prime q different from p, since $P_{q,k,a,b}\mid q^{e_{q,k}}$, we then have $v_p(P_{q,k,a,b})=0$. In what follows we deal with the remaining case $q=p$.

From $v_p(k+1)\geqslant e_{p,k}$, we deduce that $p^{e_{p,k}}\mid(k+1)$ and $p^{e}\mid(k+1)$ for each $e\in\{1,\dots,e_{p,k}\}$. By Lemma 3.6, any p^e consecutive terms in the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$ are pairwise incongruent modulo p^e since $p \nmid a$. Hence for each $e \in \{1,\dots,e_{p,k}\}$, there are exactly $(k+1)/p^e$ terms divisible by p^e in any $k+1$ consecutive terms of the arithmetic progression $\{b+ma\}_{m\in\mathbb{N}_0}$. So we have that $f_e(n) = ((k+1)/p^e) - 1$ for each $e \in \{1,\dots,e_{p,k}\}$. In other words, for every $n \in \mathbb{N}$, we have $f_e(n+1) = f_e(n)$. It then follows from (3.2) that for every $n \in \mathbb{N}$, we have $g_{p,k,a,b}(n+1) = g_{p,k,a,b}(n)$. Thus $P_{p,k,a,b} = 1$ and $v_p(P_{k,a,b}) = 0$. Therefore, part (ii) is proved. □

4. Proof of Theorem 1.2

In this section, we first prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.5, we know that $g_{k,a,b}$ is periodic. Denote by $P_{k,a,b}$ its smallest period. First, let $\gcd(a,b) = 1$. Then, by Lemma 3.7, for any prime p such that $p|a$, we have $v_p(P_{k,a,b}) = 0$. For any prime p satisfying $p \nmid a$ and $p \leqslant k$, we have, by Lemma 3.7, $P_{p,k,a,b} = p^{v_p(P_{p,k,a,b})} = p^{v_p(P_{k,a,b})}$. So, by Proposition 3.8 we infer that

$$P_{k,a,b} = \prod_{p \text{ prime},\, p\leqslant k} p^{e_p(k,a)},$$

where

$$e_p(k,a) := \begin{cases} 0 & \text{if } v_p(k+1) \geqslant e_{p,k} \text{ or } p|a, \\ e_{p,k} & \text{otherwise.} \end{cases}$$

Using the integer L_k, we obtain immediately that $P_{k,a,b} = Q_{k,a}$ as required, where $Q_{k,a}$ is defined as in (1.1).

Now let $\gcd(a,b) > 1$. If $\gcd(a,b) = d$ and $a = da'$ and $b = db'$, then $\gcd(a',b') = 1$ and we can easily check that $g_{k,a,b}(n) = d^k g_{k,a',b'}(n)$ for any $n \in \mathbb{N}$. From this one can easily derive that the periodic functions $g_{k,a,b}$ and $g_{k,a',b'}$ have the same smallest period, i.e. $P_{k,a,b} = P_{k,a',b'}$. But the result for the case $\gcd(a,b) = 1$ applied to the function $g_{k,a',b'}$ gives us that $P_{k,a',b'} = Q_{k,a'}$, with $Q_{k,a'}$ defined as in (1.1). The desired result $P_{k,a,b} = Q_{k,a'}$ therefore follows immediately. This completes the proof of Theorem 1.2. □

It was proved by Farhi and Kane [**8**] that there is at most one prime $p \leqslant k$ such that $v_p(k+1) \geqslant e_{p,k}$. We noticed that such a prime p was given in Proposition 3.3 of [**8**] without the condition $p \leqslant k$, but such a restriction condition is clearly necessary because otherwise Proposition 3.3 of [**8**] would not be true. For example, letting p be any prime number greater than $k+1$ gives us $v_p(k+1) = 0 = e_{p,k}$. Comparing the smallest period $P_{k,a,b}$ of the function $g_{k,a,b}$ with the smallest period P_k of the function $g_k = g_{k,1,0}$, we arrive at the relation between $P_{k,a,b}$ and P_k as follows:

$$P_{k,a,b} = \frac{P_k}{\prod_{\text{prime } p |\gcd(a',P_k)} p^{e_{p,k}}},$$

where $a' = a/(\gcd(a,b))$. From this one can read that $P_{k,a,b} = P_k$ if $a|b$.

Finally, we give an application of Theorem 1.2 as the conclusion of this paper.

Example 4.1. Let us consider the least common multiple of any $k+1$ consecutive positive odd numbers. To study this problem, we define an arithmetic function h_k by

$$h_k(n) := \frac{(2n+1)(2n+3)\cdots(2n+2k+1)}{\operatorname{lcm}(2n+1, 2n+3, \dots, 2n+2k+1)} \quad (n \in \mathbb{N}).$$

By Theorem 1.2, we know that h_k is periodic and, for any integer $k \geqslant 2$, the exact period R_k of h_k is given by

$$R_k = \frac{L_k}{2^{e_{2,k}} D_k},$$

where

$$D_k = \begin{cases} p^{e_{p,k}} & \text{if } v_p(k+1) \geqslant e_{p,k} \text{ for some odd prime } p \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

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