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INDUCED HOMOTOPY EQUIVALENCES ON MAPPING SPACES AND DUALITY

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Introduction. In this paper we present three results involving function constructions, together with an example which shows the usefulness of our considerations. The first result, Theorem 1, states roughly that homotopy is self related through function space constructions. Section 1 is devoted to the precise statement and proof of this theorem.

In order to clarify the results of § 2 we draw an analogy. Recall that under certain mild restrictions on the spaces involved, a map $p: E \to B$ is a fibration, and a map $i: A \to X$ is a cofibration, if and only if the induced maps $p_*: E^Z \to B^Z$, and $i^*: Z^X \to Z^A$, are fibrations for all spaces Z. The two theorems of § 2 give, in a more general category, analogous results for the notions of limit and colimit. Finally, in § 3 we suggest the importance of this type of result by using it, together with Theorem 1 and the above analogy, to deduce a result from its dual.

Let T be a Cartesian closed topological category in the sense of [**5**]. Thus T could be the category of Steenrod's compactly generated spaces [**8**], or of Spanier's quasi-topological spaces [**7**]. We choose T to be Cartesian closed to ensure the existence of products, a point (or terminal object), and that the exponential map $X^{Y \times Z} \to (X^Z)^Y$ is a homeomorphism for all spaces X, Y, Z in T. In § 3 we will also require T to be finitely complete and finitely cocomplete.

1. Induced homotopy equivalences.

THEOREM 1. *Let $f: A \to B$ be a map in T; then the following are equivalent:*
(i) *f is a homotopy equivalence;*
(ii) *The induced map $f_*: A^Z \to B^Z$ is a homotopy equivalence for each Z in T;*
(iii) *The induced map $f^*: Z^B \to Z^A$ is a homotopy equivalence for each Z in T.*

Proof. The results and proofs that condition (i) implies conditions (ii) and (iii) are well known and can be found, for example, in [**1**, Corollary 3.9]. The implication (ii) implies (i) is seen to be trivial by putting Z equal to a point.

To show that (iii) implies (i) let $g: A^A \to A^B$ be a homotopy inverse of $f^*: A^B \to A^A$ and let $h = g(1_A)$. Since $f^*g \simeq 1$, it follows that there is a path $K: I \to A^A$ from hf to 1_A, and hence a homotopy $hf \simeq 1$. We cannot deduce that h induces g. However, we can deduce that $f^*h^* \simeq 1: Z^B \times I \to Z^B$ for all spaces Z (see [**1**, Proposition 3.8]). Since both f^* and 1 are homotopy equivalences for each Z, it follows that h^* is a homotopy equivalence for each Z.

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We repeat the argument applied to h^* to deduce that h has a left homotopy inverse q. Thus $fh \simeq qhfh \simeq qh \simeq 1$.

It is interesting to note that one can prove the implication (ii) implies (i) by an argument which is "dual" to the above proof, in the sense that this argument uses right inverses.

2. Limits, colimits, and function structures. Let **C** denote any Cartesian closed category in the sense of [**5**], and let Hom(X, Y) denote the set of morphisms from X to Y in **C**. We use the conventions of Mitchell [**6**] and refer to colimits and limits as opposed to direct, or inverse limits.

The following two theorems show how the dual notions of limit and colimit are related through function structures.

THEOREM 2. *A family of morphisms* $F = \{f_i: L \to D_i\}$ *is a limit for a diagram* D *in* C *if and only if the family* $F^Z = \{f_{i*}: L^Z \to D_i^Z\}$ *is a limit for the diagram* D^Z *in* **C** *for each object* Z *of* **C**.

THEOREM 2*. *A family of morphisms* $G = \{g_i: D_i \to L\}$ *is a colimit for a diagram* D *in* **C** *if and only if the family* $Z^G = \{g_i^*: Z^L \to Z^{D_i}\}$ *is a limit for the diagram* Z^D *in* **C** *for each object* Z *of* **C**.

Proof of Theorem 2*. Mitchell [**6**, Propositions 5.1 and 5.1* of Chapter 2] proved: (i) that a family of morphisms $\{L \to D_i\}$ is a limit for a diagram D in **C** if and only if for each object A of **C**, the family

$$\{\mathrm{Hom}(A, L) \to \mathrm{Hom}(A, D_i)\}$$

is a limit for the diagram Hom(A, D) in the category of sets, (ii) that a family $\{D_i' \to L'\}$ is a colimit for a diagram D' if and only if for each object A of **C**, the family $\{\mathrm{Hom}(L', A) \to \mathrm{Hom}(D_i', A)\}$ is a limit for Hom(D', A).

Suppose that G is a colimit for D, then for all Z and A in **C**, Hom(G, Z^A) is a limit for Hom(D, Z^A). Now Hom(G, Z^A) is naturally isomorphic to Hom(A, Z^G) and so Hom(A, Z^G) is a limit for Hom(A, Z^D) for all A and Z belonging to **C**. It follows that Z^G is a limit for Z^D for all Z in **C**. Conversely, suppose that Z^G is a limit for Z^D for all Z in **C**, then for A equal to a point object, $*$, $\mathrm{Hom}(*, Z^G) \cong \mathrm{Hom}(G, Z)$ is a limit for $\mathrm{Hom}(*, Z^D) \cong \mathrm{Hom}(D, Z)$ for all Z in **C**.

It should be noted that the above argument holds in a much more general category than **C**. It holds, for example, in a symmetric monoidal category in the sense of [**5**]. In fact, we require only that for each triple A, B, Z of objects of the category, there is a natural isomorphism $\mathrm{Hom}(A, Z^B) = \mathrm{Hom}(B, Z^A)$. It is not even essential to have a point object in the category, since one can prove the converse implication of the argument by means of an indirect proof.

3. Fibred and cofibred homotopy equivalences. This section gives an application of §§ 1, 2. Recall that a map $p: E \to B$ is called a weak fibration if it has the covering homotopy property for all homotopies $Z \times I \to B$

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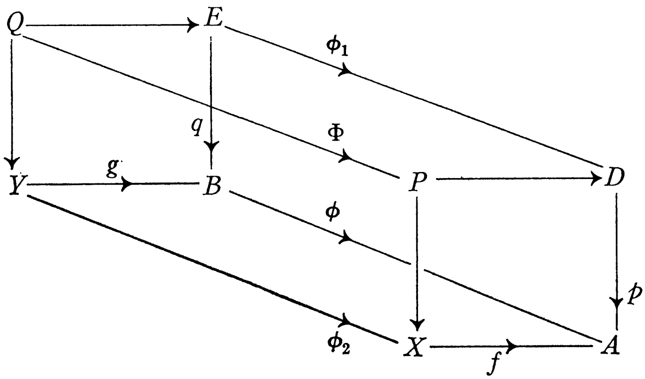
which are stationary on $Z \times [0, \frac{1}{2}]$. This property has been shown by Dold [**4**] and Weinzweig [**9**] to be convenient for studying fibre homotopy equivalences.

Consider the following commutative diagram:



in which Q and P are the pullbacks of g, q and f, p, respectively.

THEOREM 3. *If p and q are weak fibrations and ϕ, ϕ_1, ϕ_2 are homotopy equivalences, then so also is Φ.*

The proof of this theorem appears in [**3**]. We next show how we can use this theorem to prove its dual.

Recall that a map i: $A \rightarrow X$ is called a weak cofibration if it has the homotopy extension property with respect to all homotopies $A \times I \rightarrow Y$, which are stationary on $A \times [0, \frac{1}{2}]$. The proof of the following lemma is left to the reader.

LEMMA 4. *A map i: $A \rightarrow X$ in T is a weak cofibration if and only if the induced map i^*: $Z^X \rightarrow Z^A$ is a weak fibration.*

Consider the following commutative diagram:

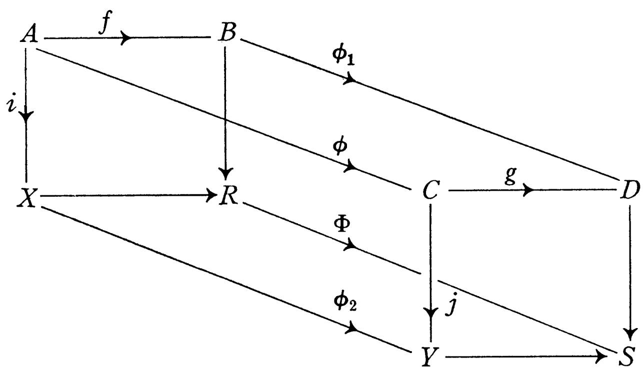


FIGURE 3*

R and S are the pushouts of i, f and j, g, respectively.

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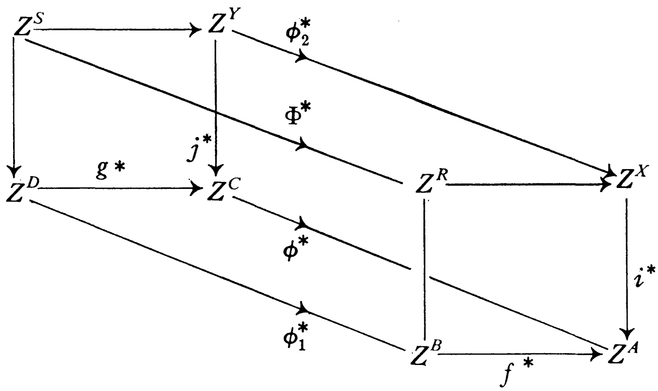
THEOREM 3*. *If i and j are weak cofibrations and ϕ, ϕ_1, ϕ_2 are homotopy equivalences, then so also is Φ.*

Proof. For each object Z of T, Figure 3* induces a commutative diagram



in which Z^S and Z^R are pullbacks of j^*, g^* and i^*, f^*, respectively, by Theorem 2*; i^* and j^* are weak fibrations by Lemma 4; and ϕ^*, ϕ_1^*, and ϕ_2^* are homotopy equivalences by Theorem 1. It follows from Theorem 3 that Φ^* is a homotopy equivalence for all Z in T, and hence, by Theorem 1, Φ is a homotopy equivalence.

An alternative proof of Theorem 3* can be found in [**2**], in the case where i and j are cofibrations.

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