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Moduli spaces of rational graphically stable curves

BY ANDY FRY
Department of Mathematical Sciences,
Lewis and Clark College, Portland, OR 97219, *U.S.A.*
e-mail: andyfry@lclark.edu

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Abstract

We use a graph to define a new stability condition for algebraic moduli spaces of rational curves. We characterise when the tropical compactification of the moduli space agrees with the theory of geometric tropicalisation. The characterisation statement occurs only when the graph is complete multipartite.

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1. *Introduction*

A strong trend in modern algebraic geometry is the study of *moduli (parameter) spaces*. Broadly, a moduli space parameterises geometric objects. An important and well-studied moduli space is $\mathcal{M}_{0,n}$, the moduli space of smooth rational curves with n marked points. The space $\mathcal{M}_{0,n}$ is not compact, which is undesirable for algebraic geometers because of the many applications that require such a condition. A 'nice' compactification of $\mathcal{M}_{0,n}$ brings along with it a modular interpretation, that is, a compact space containing $\mathcal{M}_{0,n}$ as a dense open subset has a boundary (equal to the complement of $\mathcal{M}_{0,n}$) that parametrises n-marked algebraic curves that may not be smooth. The most notable compactification, $\overline{\mathcal{M}}_{0,n}$, is due to Grothendieck [**9**], then constructed as an iterated blow-up by Knudsen [**13**]. The boundary of their compactification is comprised of nodal curves with finite automorphism group called *stable curves*. It is interesting to know what alternate compactifications exist and how the boundary combinatorics differs in each case. Another important family of compactifications, $\overline{\mathcal{M}}_{0,w}$, alters the original stability condition by assigning a weight to each marked point. The moduli spaces of weighted stable curves were established by Hassett in the context of the log minimal model program [**11**].

Tropical mathematics offers tools to investigate the structure of the boundary of compact moduli spaces by relating complex algebraic varieties to piecewise linear objects. A strength of tropical geometry is that it allows us to look at a linear skeleton of a potentially complicated variety, reducing algebro-geometric questions to those of combinatorics. For instance, the tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$ is a cone complex which parameterises leaf-labelled metric trees. The combinatorial relation between algebraic moduli spaces and tropical moduli spaces is that the cones of $\mathcal{M}_{0,n}^{\text{trop}}$ are in bijection with the boundary strata of $\overline{\mathcal{M}}_{0,n}$.

Recently [5], a new family of stability conditions were defined for tropical moduli spaces of rational marked curves determined by the combinatorics of a graph Γ, called graphic stability. This paper investigates how graphic stability is applied to the algebraic moduli spaces and how the algebraic and tropical moduli spaces relate to each other. Previously, the relationship between algebraic and tropical moduli spaces in the weighted stability setting was investigated by Cavalieri-Hampe-Markwig-Ranganathan [2].

Algebraically, we define a compactification of $\mathcal{M}_{0,n}$ using graphic stability called the *moduli space of rational graphically stable pointed curves*, denoted $\overline{\mathcal{M}}_{0,\Gamma}$. Taking the interior, $\mathcal{M}_{0,\Gamma}$, to be smooth Γ-stable curves, these new moduli spaces have many characteristics that we would expect from a modular compactification of $\mathcal{M}_{0,n}$; namely their boundaries are divisors with simple normal crossings. We also construct an embedding of $\mathcal{M}_{0,\Gamma}$ into a torus using the Plücker embedding of the Grassmannian.

For a smooth subvariety of a torus with a simple normal crossings compactification, the theory of geometric tropicalisation relates the combinatorics of the boundary to a balanced fan in a real vector space. Using this theory we show that the tropicalisation of $\mathcal{M}_{0,\Gamma}$ is identified with a projection of the tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$.

This tropicalisation doesn't necessarily line up with the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\text{trop}}$. The obstruction is a lack of injectivity in the tropicalisation map. Specifically, the divisorial valuation map $\pi_\Gamma : \Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}) \to N_{\mathbb{R}}$ may not be injective, this fact is highlighted in equation (3·5). The main result of this work is a classification result stating precisely when the tropical compactification of $\mathcal{M}_{0,\Gamma}$ agrees with the theory of geometric tropicalisation for rational graphically stable curves.

THEOREM 3·3. *For Γ complete multipartite, there is a torus embedding*

$$\mathcal{M}_{0,\Gamma} \hookrightarrow T^{\binom{n}{2}-n-N} = T_\Gamma$$

whose tropicalisation trop$(\mathcal{M}_{0,\Gamma})$ has underlying cone complex $\mathcal{M}_{0,\Gamma}^{trop}$. Furthermore, the tropical compactification of $\mathcal{M}_{0,\Gamma}$ is $\overline{\mathcal{M}}_{0,\Gamma}$, i.e, the closure of $\mathcal{M}_{0,\Gamma}$ in the toric variety $X(\mathcal{M}_{0,\Gamma}^{trop})$ is $\overline{\mathcal{M}}_{0,\Gamma}$.

The motivation for this paper comes from the theory of tropical compactifications, geometric tropicalisation, and log geometry. From work of Tevelev [17] and Gibney-Maclagan [7] it has been shown that there is an embedding of $\mathcal{M}_{0,n}$ into the torus of a toric variety $X(\Sigma)$ where the tropicalisation of $\mathcal{M}_{0,n}$ is a balanced fan $\Sigma \cong \mathcal{M}_{0,n}^{\text{trop}}$. This embedding is special in the sense that the closure of $\mathcal{M}_{0,n}$ in $X(\Sigma)$ is $\overline{\mathcal{M}}_{0,n}$. Cavalieri et al. [2] show a similar embedding can be constructed for weighted moduli spaces when the weights are heavy/light. Although graphical stability doesn't completly generalise weighted stability, each heavy/light space in [2] can be viewed as a graphically stable space for some complete multipartite graph. In this sense, [2, theorem 3·9] is a corollary to theorem 3·3 of this paper.

Graphical stability can be viewed as a special case of the simplicial stability described by Blankers and Bozlee [1] as follows. An *independent set* of a graph is a set of vertices in a graph, no two of which are adjacent. The *independence complex* of a graph is a simplicial complex formed by the sets of vertices in the independent sets of the graph. Let Γ be a graph on $n-1$ vertices, and let Γ' be the graph defined by adding an nth vertex to Γ and edges connecting the new vertex to all other vertices. Then the Blankers–Bozlee simplicial

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compactification of $\mathcal{M}_{g,n}$ given by the incidence complex of Γ' agrees with the graphical stability compactification of $\mathcal{M}_{g,n}$ associated to Γ.

The paper is organised as follows. Chapter 2 discusses preliminary definitions in the algebraic (Section 2·1) and tropical (Section 2·2) settings which are necessary for this manuscript. Section 2·3 describes the process of geometric tropicalisation and briefly covers this process applied to $\mathcal{M}_{0,n}$.

Chapter 3 is composed of original work. Section 3·1 contains a proof that $\overline{\mathcal{M}}_{0,\Gamma}$ is not only a modular compactification of $\mathcal{M}_{0,n}$, but indeed a simple normal crossings compactification of the locus of smooth Γ-stable curves, $\mathcal{M}_{0,\Gamma}$. To invoke geometric tropicalisation, we also need a torus embedding of $\mathcal{M}_{0,\Gamma}$. Section 3·2 begins by identifying the interior of the moduli space with the quotient of an open set of the Grassmannian, thus creating the necessary torus embedding. We notice that the divisorial valuation map, which furnishes the combinatorics of the boundary with a fan structure, does not in general have the desired underlying cone complex, $\mathcal{M}^{\mathrm{trop}}_{0,\Gamma}$. Indeed, we achieve this compatibility only when Γ is complete multipartite. After the main theorem, we conclude with an example where the graph is not complete multipartite. In this case, the toric variety does not have enough boundary strata to contain the modular compactification.

2. *Preliminaries*

2·1. *Algebraic moduli spaces*

The moduli space $\mathcal{M}_{0,n}$ parameterises isomorphism classes of smooth, genus 0 curves with n marked points. A point of $\mathcal{M}_{0,n}$ is an isomorphism class of n ordered, distinct marked points on $\mathbb{P}^1$ which we denote $(p_1,\ldots,p_n)$. Two points $(\mathbb{P}^1,p_1,\ldots,p_n),(\mathbb{P}^1,q_1,\ldots,q_n)\in\mathcal{M}_{0,n}$ are equal if there is $\Phi\in\mathrm{Aut}(\mathbb{P}^1)$ such that $\Phi(p_i)=(q_i)$, for all i. Using cross ratios, we may assign any n-tuple $(p_1,\ldots,p_n)$ to $(0,1,\infty,\Phi_{CR}(p_4),\ldots,\Phi_{CR}(p_n))$ where Φ_{CR} is the unique automorphism of $\mathbb{P}^1$ sending p_1, p_2, and p_3 to 0, 1, and ∞. The first two nontrivial cases occur when $n=3$ and $n=4$. As varieties, $\mathcal{M}_{0,3}$ is a point, as we send (p_1,p_2,p_3) to $(0,1,\infty)$ and $\mathcal{M}_{0,4}=\mathbb{P}^1\setminus\{0,1,\infty\}$ because the fourth point is free to vary as long as it doesn't coincide with the other 3 markings. In general, this shows that $\mathcal{M}_{0,n}$ is an $n-3$ dimensional space and

$$\mathcal{M}_{0,n}=\overbrace{\mathcal{M}_{0,4}\times\cdots\times\mathcal{M}_{0,4}}^{n-3\text{ times}}\setminus\{\text{all diagonals}\}.$$

From the $n=4$ example, we can see that $\mathcal{M}_{0,n}$ is not compact in general. The most notable compactification, $\overline{\mathcal{M}}_{0,n}$, is due to Deligne and Mumford which allows nodal curves with finite automorphism group; such curves are called *stable* curves [**4**, **13**].

Definition 2·1. A rational marked curve $(C,p_1,\ldots,p_n)$ is *stable* if:

(1) C is a connected curve of arithmetic genus 0, whose only singularities are nodes;

(2) $(p_1,\ldots,p_n)$ are distinct points of $C\setminus\mathrm{Sing}(C)$;

(3) The only automorphism of C that preserves the marked points is the identity.

Stable nodal curves arise as the limit of a family of smooth curves where a number of points collide, e.g., $p_1\mapsto p_2$. In Figure 1, we see an example of a nodal curve in $\overline{\mathcal{M}}_{0,4}$ where the marked points p_3, and p_4 have collided. This curve also arises if p_1 and p_2 collide. The *dual graph* or *combinatorial type* of a stable curve in $\overline{\mathcal{M}}_{0,n}$, is defined by assigning a

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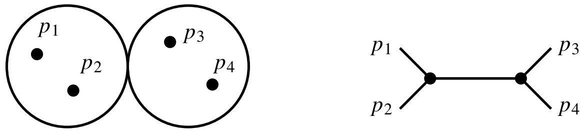

Fig. 1. A marked algebraic curve and its dual graph.

vertex to each component, an edge to each node, and a half-edge to each marked point, as shown in Figure 1. An alternative definition of stability can be posed in terms of dual graphs.

Definition 2·2. A rational marked curve $(C, p_1, \ldots, p_n)$ is *stable* if its dual graph is a tree where each vertex has valence greater than 2.

We define the *boundary of* $\overline{\mathcal{M}}_{0,n}$ to be $\partial\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$; it consists of all points corresponding to nodal stable curves. We call the closure of a codimension one stratum a *boundary divisor*. The boundary is stratified by nodal curves of a given topological type with an assignment of marks to each component. In other words, $\partial\overline{\mathcal{M}}_{0,n}$ is stratified by dual graphs of stable nodal pointed curves. Dual graphs of boundary divisors partition the set of markings into two sets $I \sqcup I^c$. We adopt the convention that the marking $1 \in I^c$; therefore, a boundary divisor $D := D_I$ is uniquely identified by its *index set* I.

2·2. *Tropical moduli spaces*

We begin by introducing necessary background terminology on tropical moduli spaces. For a more thorough survey of tropical moduli spaces, see [**14**]. Consider the space of genus 0, n-marked abstract tropical curves $\mathcal{M}_{0,n}^{\mathrm{trop}}$. Points of $\mathcal{C} \in \mathcal{M}_{0,n}^{\mathrm{trop}}$ are in bijection with metrised trees with *bounded edges* having finite length and n unbounded labelled edges called *ends*. By forgetting the lengths of the bounded edges of $\mathcal{C}$ we get a tree with labelled ends called the *combinatorial type* of $\mathcal{C}$. The space $\mathcal{M}_{0,n}^{\mathrm{trop}}$ naturally has the structure of a cone complex where curves of a fixed combinatorial type with d bounded edges are parameterised by $\mathbb{R}^d_{>0}$. We obtain $\mathcal{M}_{0,n}^{\mathrm{trop}}$ by gluing several copies of $\mathbb{R}^{n-3}_{\geq 0}$ via appropriate face morphisms, one for each trivalent combinatorial type.

The space $\mathcal{M}_{0,n}^{\mathrm{trop}}$ may be embedded into a real vector space as a balanced, weighted, pure-dimensional polyhedral fan as in [**6**]. We briefly recall this construction. A *weighted fan* (X, ω) is a fan X in $\mathbb{R}^n$ where each top-dimensional cone σ has a positive integer weight associated to it, denoted by $\omega(\sigma)$. A weighted fan is *balanced* if for all cones τ of codimension one, the weighted sum of primitive normal vectors of the top-dimensional cones $\sigma_i \supset \tau$ is 0, i.e.,

$$\sum_{\sigma_i \supset \tau} \omega(\sigma_i) \cdot u_{\sigma_i/\tau} = 0 \in V/V_\tau,$$

where $u_{\sigma_i/\tau}$ is the primitive normal vector, V is the ambient real vector space, and V_τ is the smallest vector space containing the cone τ. See [**6**, construction 2.3] for a construction of the primitive normal vectors $u_{\sigma_i/\tau}$.

For a curve $\mathcal{C}$, define $\mathrm{dist}(i, j)$ as the sum of lengths of all bounded edges between the ends marked by i and j. Then the vector

$$d(\mathcal{C}) = (\mathrm{dist}(i, j))_{i<j} \in \mathbb{R}^{\binom{n}{2}}/\Phi(\mathbb{R}^n) = Q_n \tag{2·1}$$

identifies $\mathcal{C}$ uniquely, where $\Phi : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$ by $x \mapsto (x_i + x_j)_{i<j}$.

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In [5], an alternate stability condition using a combinatorial graph is introduced for rational pointed tropical curves. Let Γ be a simple graph on $n-1$ vertices, labelled 2, ..., n with at least one edge. Denote $E(\Gamma)$ as its edge set. The graph Γ controls which pairs of markings are allowed to collide. Edges in Γ correspond to pairs of points that cannot collide, while missing edges, with respect to the complete graph $K_{n-1} \supseteq \Gamma$, correspond to pairs of points that can collide.

Definition 2·3. The *root vertex* of a stable tropical curve $\mathcal{C}$ is the vertex containing the end with marking 1. A stable tropical curve $\mathcal{C}$ with n ends is Γ-*stable* if, at each non-root vertex v of $\mathcal{C}$ with exactly one bounded edge, there exists an edge $e_{ij} \in E(\Gamma)$ where i and j are ends adjacent to v. Define $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ to be the parameter space of all rational n-marked Γ-stable abstract tropical curves.

Note that there are two types of graphs in this definition. The graph $\mathcal{C}$ corresponds to a stable tropical curve, which we will reference using the terminology *bounded edges, ends, unmarked vertices,* and *markings* labelled $1, \ldots, n$. Whereas the graph Γ is a combinatorial graph with *edges* and *vertices* labelled $2, \ldots, n$.

As a cone complex, $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ is a subcomplex of $\mathcal{M}_{0,n}^{\text{trop}}$. Using graphic stability, there exists a projection map from the vector space $Q_n = \mathbb{R}^{\binom{n}{2}-n}$ to $\mathbb{R}^{\binom{n}{2}-n-N}$ that forgets the coordinates corresponding to the N edges removed from K_{n-1} to obtain Γ, see [5, equations 7, 8]. Although there are two projections defined, lemma 3·18 of [5] identifies them via a linear transformation, so we will use pr_Γ to refer to both projection maps. We will see in lemma 3·3 that pr_Γ is the tropicalisation of a regular map between algebraic tori.

2·3. *Geometric tropicalisation for $\mathcal{M}_{0,n}$*

Two theories, developed simultaneously, arise when dealing with tropicalisations of subvarieties of tori: *tropical compactification* and *geometric tropicalisation*. The former, introduced by Tevelev [17], describes a situation where the tropical variety determines a good choice of compactification. Specifically, the tropical compactification of $U \subset \mathbb{T}^r$ is its closure $\overline{U}$ in a toric variety $X(\Sigma)$ with $|\Sigma| = \text{trop}(U)$. The latter, introduced by Hacking, Keel and Tevelev [10] and further developed by Cueto [3], explores the converse statement, how a nice compactification determines its tropicalisation.

We recall some useful definitions for geometric tropicalisation. We note that geometric tropicalisation can be completed with more relaxed conditions, such as replacing a smooth compactification with a normal, $\mathbb{Q}$-factorial compactification and replacing simple normal crossing by combinatorial normal crossings. For explicit details, see [3].

Let U be a smooth subvariety of a torus $\mathbb{T}^r$ and Y be a smooth compactification containing U as a dense open subvariety. The boundary of Y, $\partial Y = Y \setminus U$, is *divisorial* if it is a union of codimension-1 subvarieties of Y. We say $(Y, \partial Y)$ is a *simple normal crossings (snc) pair* when the boundary of Y behaves locally like an arrangement of coordinate hyperplanes. In other words, ∂Y is an *snc divisor* if a non-empty intersection of k irreducible boundary divisors is codimension k and the intersection is transverse. The *boundary complex of* Y, $\Delta(\partial Y)$, is a simplicial complex whose vertices are in bijection with the irreducible divisors of the boundary divisor ∂Y, and whose k-cells correspond to a non-empty intersection of k boundary divisors. The cells containing a face τ correspond to the boundary strata that lie in the closure of τ's stratum.

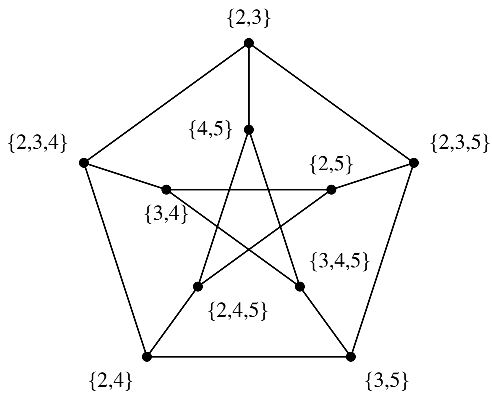

Fig. 2. The boundary complex of $\overline{\mathcal{M}}_{0,5}$ with divisors (vertices) labelled by their index set.

Let $\phi_1, \ldots, \phi_r \in \mathcal{O}^*(U)$. The ϕ_i's define a morphism $\vec{\phi}$ from U to a torus $\mathbb{T}^r$, sending $u \in U$ to $(\phi_1(u), \ldots, \phi_r(u))$. When there are enough invertible functions, this map is an embedding. Given an irreducible boundary divisor $D \subset \partial Y$ we can compute the order of vanishing of each ϕ_i on D, $\mathrm{ord}_D(\phi_i)$, yielding an r-dimensional integer vector $\vec{v}_D = (\mathrm{ord}_D(\phi_1), \ldots, \mathrm{ord}_D(\phi_r))$ living inside the cocharacter lattice of $\mathbb{T}^r$, $N_{\mathbb{T}^r} \subseteq N_{\mathbb{R}} = \mathbb{R}^r$. Let $\pi : \Delta(\partial Y) \to N_{\mathbb{R}}$ be the map defined by sending a vertex v_i to $\vec{v}_{D_i}$ and extending linearly on every simplex. We call π a *divisorial valuation map*. Geometric tropicalisation says precisely that the support of the tropical fan is the cone over this complex and this result is independent of our choice of compactification Y, i.e., $\mathrm{trop}(U) = \mathrm{cone}(\mathrm{Im}(\pi))$. As we will see later, π is not necessarily injective, so $\mathrm{trop}(U)$ may not be the cone over $\Delta(\partial Y)$.

Tevelev [17, theorem 5·5] first computes the tropicalisation of $\mathcal{M}_{0,n}$ via geometric tropicalisation by combining results of [12, 16]. This result is generalised by Gibney and Maclagan [7, theorem 5·7]. They use the fact that $\mathcal{M}_{0,n}$ can be embedded into a torus of dimension $\binom{n}{2} - n$ using the Plücker embedding of the Grassmannian $G(2,n)$ into $\mathbb{P}^{\binom{n}{2}-1}$. For explicit details, see [7, 14]. Comparing the algebraic Plücker embedding to the tropical distance coordinates we realise that the distance coordinates from equation (2·1) can be recovered from the tropicalisation of the Plücker coordinates, for details see [8, section 3·1].

Example 2·4. For $\mathcal{M}_{0,5}$, we have an embedding into $T^{\binom{5}{2}-5} = T^5$. In the boundary of $\overline{\mathcal{M}}_{0,5}$, there are 10 irreducible boundary divisors; they are labelled by their index sets in Figure 2.

We may define $\mathcal{M}_{0,n}^{\mathrm{trop}}$ alternatively as the cone over $\Delta(\partial\overline{\mathcal{M}}_{0,n})$. Geometric tropicalisation states precisely that $\mathrm{cone}(\Delta(\partial\overline{\mathcal{M}}_{0,n})) = \mathrm{trop}(\mathcal{M}_{0,n})$. The following theorem, due to Tevelev and Gibney-Maclagan, states that $\mathcal{M}_{0,n}^{\mathrm{trop}} = \mathrm{trop}(\mathcal{M}_{0,n})$.

THEOREM 2·1 ([7, 17]). *The geometric tropicalisation of $\overline{\mathcal{M}}_{0,n}$ via the embedding*

$$\mathcal{M}_{0,n} \hookrightarrow T^{\binom{n}{2}-n}$$

gives the fan trop($\mathcal{M}_{0,n}$) whose underlying cone complex is identified with $\mathcal{M}_{0,n}^{trop}$. Furthermore, the tropical compactification of $\mathcal{M}_{0,n}$ in the toric variety $X(\mathcal{M}_{0,n}^{trop})$ is $\overline{\mathcal{M}}_{0,n}$.

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It follows from the previous theorem that the divisorial valuation map is injective, and thus induces a bijective map of cone complexes from $\mathcal{M}_{0,n}^{\mathrm{trop}}$ to $\mathrm{trop}(\mathcal{M}_{0,n})$. This is an important fact that we will revisit when discussing Γ-stability.

3. *Tropicalising Moduli Spaces of Rational Graphically Stable Curves*

We define algebraic moduli spaces parameterising rational graphically stable curves, $\overline{\mathcal{M}}_{0,\Gamma}$, and investigate its tropicalisation. The central result classifies *all* graphically stable moduli spaces in which the tropical compactification of $\mathcal{M}_{0,\Gamma}$ agrees with the theory of geometric tropicalisation. Unless otherwise noted we let Γ be as in definition 2·3, a simple graph on $n-1$ vertices, labelled $2, ..., n$ with at least one edge.

3·1. *The moduli space of rational graphically stable curves*

In [**15**], Smyth gives a complete classification of all modular compactifications of $\mathcal{M}_{0,n}$ using combinatorial objects called extremal assignments (theorem 1.9). In this section, we prove that $\overline{\mathcal{M}}_{0,\Gamma}$ is a modular compactification of $\mathcal{M}_{0,n}$ by showing that Γ-stability, as in definition 2·3, is an extremal assignment over $\mathcal{M}_{0,n}$. In addition, we show that the pair $(\overline{\mathcal{M}}_{0,\Gamma}, \partial\overline{\mathcal{M}}_{0,\Gamma})$ is an snc pair. Before we define Γ-stable curves and the moduli space $\overline{\mathcal{M}}_{0,\Gamma}$, we first introduce notation that tracks the markings that may collide.

We say a *subcurve* of a proper algebraic curve C over an algebraically closed field is a reduced closed subscheme of C. Let $(C, p_1, \ldots, p_n)$ be an n-marked curve, let x be a closed point of C, and Z be a subcurve of C. Denote the set of markings at x to be

$$\mathrm{Mar}(x) := \{i \in [n] \mid p_i = x\}$$

and the set of markings a subcurve Z to be

$$\mathrm{Mar}(Z) := \{i \in [n] \mid p_i \in Z\} = \bigcup_{x \in Z} \mathrm{Mar}(x).$$

For a subset $I \subset \{2, \ldots, n\}$ let Γ_I be the induced subgraph of Γ containing the vertices indexed by I and all edges of Γ between those vertices. We denote two special subgraphs of Γ given by the markings at a closed point x and the markings in a subsub Z:

$$\Gamma_x := \Gamma_{\mathrm{Mar}(x)} \text{ and } \Gamma_Z := \Gamma_{\mathrm{Mar}(Z)}.$$

Definition 3·1. The *root component* of a rational stable n-marked curve $(C, p_1, \ldots, p_n)$ is the component containing p_1. A rational stable n-marked curve $(C, p_1, \ldots, p_n)$ is Γ-*stable* if each of the conditions are satisfied:

(1) for each closed smooth point $x \in C$, $E(\Gamma_x) = \emptyset$

(2) for each non-root subcurve $Z \subset \mathcal{C}$ with exactly one node, $E(\Gamma_Z) \neq \emptyset$.

Define $\overline{\mathcal{M}}_{0,\Gamma}$ to be the parameter space of all rational Γ-stable n-marked curves with the interior $\mathcal{M}_{0,\Gamma}$ to be all smooth rational Γ-stable n-marked curves.

Consider the assignment defined by

$$\mathcal{Z}(C) = \{Z \subset C \mid |Z \cap Z^c| = 1,\ p_1 \notin Z,\ E(\Gamma_Z) = \emptyset\} \tag{3·1}$$

If we call a subcurve $Z \subset C$ satisfying $|Z \cap Z^c| = 1$ a *tail*, then the assignment $\mathcal{Z}$ is defined by picking out all tails of $(C, p_1, \ldots, p_n)$ not containing p_1 that have no edges in Γ between

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vertices corresponding to the marked points on the tail. For the purposes of this document, we require Γ to be a simple connected graph on $n-1$ vertices, in which case equation (3·1) is an extremal assignment.

By definition, the only components contracted by $\mathcal{Z}$-stability are exactly those which are contracted by Γ-stability. Every tail is contracted to a point of singularity type (0,1) which is a smooth point. Therefore, $\overline{\mathcal{M}}_{0,\Gamma} = \overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ (as defined in [**15**]) and so $\overline{\mathcal{M}}_{0,\Gamma}$ is a modular compactification of $\mathcal{M}_{0,n}$.

LEMMA 3·2. *The boundary* $\partial\overline{\mathcal{M}}_{0,\Gamma} = \overline{\mathcal{M}}_{0,\Gamma} \setminus \mathcal{M}_{0,\Gamma}$ *is a divisor with simple normal crossings.*

Proof. The boundary $\partial\overline{\mathcal{M}}_{0,\Gamma}$ is divisorial, meaning it is a union of divisors of $\overline{\mathcal{M}}_{0,\Gamma}$. In addition, the boundary strata are parameterised by dual graphs. Each edge of a dual graph corresponds to a node in its associated complex curve, where locally each node is given by an equation $xy = t_i$ when $t_i = 0$. Since a boundary stratum of codimension k is the intersection of k divisors, each divisor acts as a coordinate hyperplane $t_i = 0$. It follows from the deformation theory of nodal curves that the functions $t_1, \ldots, t_k$ are independent. Therefore, $\partial\overline{\mathcal{M}}_{0,\Gamma}$ behaves locally like an arrangement of coordinate hyperplanes.

3·2. *Geometric tropicalisation for* $\mathcal{M}_{0,\Gamma}$

For the remainder of the paper we add the condition that Γ is connected. We also assume without loss of generality that Γ contains the edge e_{23}. The connected assumption is used in the proof of lemma 3·4.

The aim of this section is to walk through the process of geometric tropicalisation for the case of Γ-stability and study the tropical compactification of $\mathcal{M}_{0,\Gamma}$. We begin by investigating the projection of the Plücker embedding of $\mathcal{M}_{0,n}$. Graphic stability defines a projection map that will give a torus embedding using the remaining Plücker coordinates. Next, we examine the divisorial valuation map from the boundary complex of $\overline{\mathcal{M}}_{0,\Gamma}$ into the cocharacter lattice of the torus. Fixing $e_{23} \in E(\Gamma)$ prescribes a set of coordinates on the torus. The tropicalisation is a fan which coincides with the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ if and only if Γ is complete multipartite.

We may set up a torus embedding for $\mathcal{M}_{0,\Gamma}$ in the following way. Recall that the Plücker embedding is given by sending a $2 \times n$ matrix, representing a choice of basis for a subspace V, to its vector of 2×2 minors called the Plücker coordinates. Let $\text{Mat}^{\Gamma}(2, n)$ be the set of $2 \times n$ matrices where the ij^{th} minor, z_{ij}, is nonzero whenever $i = 1$ or $e_{ij} \in E(\Gamma)$. Let $G^{\Gamma}(2, n)$ be the open subspace of $G(2,n)$ given by $\text{Mat}^{\Gamma}(2, n)$; that is, the points of $G^{\Gamma}(2, n)$ are given by the subset of nonvanishing Plücker coordinates z_{ij} whenever $i = 1$ or $e_{ij} \in E(\Gamma)$. Let k be an algebraically closed field and consider the action of the $(n-1)$-dimensional torus $T^{n-1} = (k^*)^n/k^*$ on $\mathbb{P}^{\binom{n}{2}-1}$ given by

$$(t_1, \ldots, t_n) \cdot [z_{ij}]_{1 \le i < j \le n} = [t_i t_j z_{ij}]_{1 \le i < j \le n}. \tag{3·2}$$

The T^{n-1} torus action amounts to a nonzero scaling of the columns of the $2 \times n$ matrices in $G^{\Gamma}(2, n)$ modulo diagonal scaling and T^{n-1} acts freely on $G^{\Gamma}(2, n)$.

An n-tuple of (potentially overlapping) points of $\mathbb{P}^1$, $([x_1 : y_1], \ldots, [x_n : y_n])$, may be encoded into a $2 \times n$ matrix where each point is a column of the matrix. Thus, $([x_1 : y_1], \ldots, [x_n : y_n])$ is sent to $(z_{12} : \cdots : z_{n-1n}) \in \mathbb{P}^{\binom{n}{2}-1}$ where $z_{ij} = x_i y_j - x_j y_i$. The

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$$\begin{array}{ccccccc}
\mathrm{Mat}^{\Gamma}(2,n) & \longrightarrow & G^{\Gamma}(2,n) & \xhookrightarrow{\mathrm{Pl}} & \mathrm{Pl}\left(G^{\Gamma}(2,n)\right) & \subset & \mathbb{P}^{\binom{n}{2}-1} \\
\downarrow & & \downarrow & & \downarrow \mathrm{Pr}_{\Gamma} & & \vdots\ \mathrm{Pr}_{\Gamma} \\
\mathcal{M}_{0,\Gamma} & \xrightarrow{\cong} & G^{\Gamma}(2,n)/T^{n-1} & \hookrightarrow & T^{\binom{n}{2}-1-N}/T^{n-1} & \subset & \mathbb{P}^{\binom{n}{2}-1-N}
\end{array}$$

Fig. 3. Torus embedding of $\mathcal{M}_{0,\Gamma}$ via Plücker map.

coordinates z_{ij} are nonzero precisely when the ith and jth points are distinct; in this way, $\mathcal{M}_{0,\Gamma}$ is equal to the quotient $G^{\Gamma}(2,n)/T^{n-1}$.

Definition 3·3. Let Pr_{Γ} be the rational map from $\mathbb{P}^{\binom{n}{2}-1}$ to $\mathbb{P}^{\binom{n}{2}-1-N}$ dropping all the Plücker coordinates z_{ij} for which e_{ij} is not an edge of Γ. Here N is the number of edges removed from K_{n-1} to obtain Γ, $N = \binom{n-1}{2} - E(\Gamma)$.

After applying Pr_{Γ}, the images of the remaining Plücker coordinates are non-zero, meaning the image of $\mathrm{Pl}\left(G^{\Gamma}(2,n)\right)$ via Pr_{Γ} lives inside a torus in $\mathbb{P}^{\binom{n}{2}-1-N}$. Recall that $\mathcal{M}_{0,n}$ lives inside an $(\binom{n}{2} - n)$-dimensional torus, $T^{\binom{n}{2}-n} \subset \mathbb{P}^{\binom{n}{2}-1}$. The projection map Pr_{Γ} is regular on $T^{\binom{n}{2}-n}$; in fact, we prove in lemma 3·2 that the projection of $T^{\binom{n}{2}-n}$ via Pr_{Γ} is an $\left(\binom{n}{2} - n - N\right)$-dimensional torus containing $\mathcal{M}_{0,\Gamma}$. Note the embedding into the quotient torus is given by Pl, but then we may choose an isomorphism to a torus of correct dimension. The diagram in Figure 3 summarises the above conversation.

LEMMA 3·4. *The open part $\mathcal{M}_{0,\Gamma}$ can be embedded into the torus $Pr_{\Gamma}\left(T^{\binom{n}{2}-n}\right) = T^{\binom{n}{2}-n-N}$ using the Plücker coordinates.*

Proof. Let $(\mathbb{P}^1, (x_1 : y_1), \ldots, (x_n : y_n))$ be a Γ-stable curve in $\mathcal{M}_{0,\Gamma}$. Using the discussion preceding definition 3·3, this marked curve, up to the T^{n-1} action in equation 3·2, corresponds to a point $\vec{x} := (z_{12} : \cdots : z_{n-1n}) \in \mathbb{P}^{\binom{n}{2}-1}$ where $z_{ij} = x_i y_j - x_j y_i$. By Γ-stability, the z_{ij} coordinates are allowed to be zero when $e_{ij} \notin E(\Gamma)$, which are the same coordinates that are forgotten by Pr_{Γ}. Since each coordinate of $\mathrm{Pr}_{\Gamma}(\vec{x}) \in \mathbb{P}^{\binom{n}{2}-1-N}$ is necessarily nonzero, $\mathrm{Pr}_{\Gamma}(\vec{x})$ lies in the big open torus of $\mathbb{P}^{\binom{n}{2}-1-N}$, denoted $T^{\binom{n}{2}-1-N}$. The torus action in equation 3·2 is carried through the projection, therefore

$$(\mathrm{Pr}_{\Gamma} \circ \mathrm{Pl})(\mathcal{M}_{0,\Gamma}) \subset T^{\binom{n}{2}-1-N}/T^{n-1} := T^{\binom{n}{2}-n-N}.$$

Finally, we show that Pr_{Γ} is injective on the image of $\mathcal{M}_{0,\Gamma}$. Fix $\vec{r}, \vec{s} \in \mathrm{Pl}(\mathcal{M}_{0,\Gamma})$ and suppose $\mathrm{Pr}_{\Gamma}(\vec{r}) = \mathrm{Pr}_{\Gamma}(\vec{s})$. Then for some $\lambda \in \mathbb{C}^*$, $r_{ij} = \lambda s_{ij}$ whenever $i = 1$ or $e_{ij} \in E(\Gamma)$. Fix such a λ.

We wish to show $r_{kl} = \lambda s_{kl}$ whenever $e_{kl} \notin E(\Gamma)$. Using the automorphisms of $\mathcal{M}_{0,\Gamma}$, we fix the first two marked points of the Γ-stable curves sent to $\vec{r}$ and $\vec{s}$ to be $(0:1)$ and $(1:0)$. We write the remaining marked points as $(1:t_i)$ and $(1:u_i)$ for $\vec{r}$ and $\vec{s}$, respectively, so that

$$r_{ij} = \begin{cases} -1, & i = 1 < j \\ t_j, & i = 2 < j \\ t_j - t_i & 2 < i < j \end{cases} \qquad s_{ij} = \begin{cases} -1, & i = 1 < j \\ u_j, & i = 2 < j \\ u_j - u_i & 2 < i < j \end{cases}.$$

If $k = 2$, then $r_{2l} = t_l$ and $s_{2l} = u_l$.

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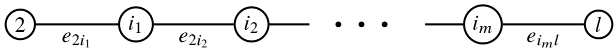

Fig. 4. Path for lemma 3·4.

Consider the path from vertices 2 to l passing through vertices $i_1, \dots, i_m$ (which exists because Γ is a connected graph), as in Figure 4.

This path gives us the following sequence of equations

$$r_{2i_1} = \lambda s_{2i_1},\ r_{i_1 i_2} = \lambda s_{i_1 i_2},\ \dots,\ r_{i_m l} = \lambda s_{i_m l}$$

yielding

$$t_{i_1} = \lambda u_{i_1},\ t_{i_2} - t_{i_1} = \lambda(u_{i_2} - u_{i_1}),\ \dots,\ t_l - t_{i_m} = \lambda(u_l - u_{i_m}).$$

By substituing the first in equation into the second we see that $t_{i_2} = \lambda u_{i_2}$. Continuing the substitution process eventually yields $t_l = \lambda u_l$, and hence $r_{2l} = \lambda s_{2l}$.

On the other hand, if $k \neq 2$, we can repeat the above strategy by picking separate paths from 2 to k and 2 to l to get $t_k = \lambda u_k$ and $t_l = \lambda u_l$, respectively. This completes the proof as

$$r_{kl} = t_l - t_k = \lambda(u_l - u_k) = \lambda s_{kl}.$$

The boundary of $\overline{\mathcal{M}}_{0,\Gamma}$ is divisorial in the same way that $\partial\overline{\mathcal{M}}_{0,n}$ is divisorial, except that there are fewer irreducible divisors. The ratios z_{ij}/z_{23}, for $2 \leq i < j \leq n$, $(i, j) \neq (2, 3)$ and $e_{ij} \in E(\Gamma)$, are rational functions on $\overline{\mathcal{M}}_{0,\Gamma}$ and act as a choice of coordinates on the torus $T^{\binom{n}{2}-n-N}$.

Define the divisorial valuation map $\pi_\Gamma : \Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}) \to N_{\mathbb{R}}$ by assigning the vector $\vec{v}_{D_I} = (\mathrm{ord}_{D_I}(z_{24}/z_{23}), \dots, \mathrm{ord}_{D_I}(z_{n-1n}/z_{23}))$ to a divisor D_I where

$$\mathrm{ord}_{D_I}(z_{ij}/z_{23}) = \begin{cases} 1 & \{2,3\} \not\subset I,\ \{i,j\} \subset I,\ \text{and } e_{ij} \in E(\Gamma); \\ -1 & \{2,3\} \subset I,\ \{i,j\} \not\subset I,\ \text{and } e_{ij} \in E(\Gamma); \\ 0 & \text{else.} \end{cases}$$

This means

$$\pi_\Gamma(D_I) = \vec{v}_{D_I} = \begin{cases} \displaystyle\sum_{\substack{i,j \in I; \\ e_{ij} \in E(\Gamma)}} \vec{e}_{ij} & \{2,3\} \not\subset I; \\ \displaystyle -\sum_{\substack{i,j \notin I; \\ e_{ij} \in E(\Gamma)}} \vec{e}_{ij} & \{2,3\} \subset I. \end{cases} \tag{3·3}$$

The standard basis vectors of $T^{\binom{n}{2}-n-N}$ are given by $\vec{v}_{D_{\{i,j\}}}$, where $e_{ij} \in E(\Gamma) \setminus \{e_{23}\}$. Additionally we have $\vec{v}_{D_{\{2,3\}}} = -\vec{1}$. For a divisor D_I with $|I| \geq 3$,

$$\vec{v}_{D_I} = \sum_{\substack{\{i,j\} \subset I; \\ e_{ij} \in E(\Gamma)}} \vec{v}_{D_{\{i,j\}}}. \tag{3·4}$$

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LEMMA 3·5. *The tropicalisation of the map* Pr_Γ *agrees with the projection* pr_Γ *from* [**5**, *equations* 7, 8].

Proof. A basis of $T^{\binom{n}{2}-n}$ is given by z_{ij}/z_{23} for $2 \le i < j \le n$, $(i, j) \ne (2, 3)$. These coordinates are in bijection with divisors $D_{\{i,j\}}$. The tropicalisation of representatives of such divisors are basis elements of $\mathbb{R}^{\binom{n}{2}-n}$. Both projections Pr_Γ and pr_Γ, forget coordinates that correspond to the edges deleted from K_{n-1} to obtain Γ. The discussion above confirms that the tropicalisation of the basis elements of $T^{\binom{n}{2}-n-N}$ coincide with the basis elements of $\mathbb{R}^{\binom{n}{2}-n-N}$.

PROPOSITION 3·6. *Using the embedding in lemma 3·4, the tropical variety trop*$(\mathcal{M}_{0,\Gamma})$ *is equal to* $pr_\Gamma(\mathcal{M}_{0,n}^{trop})$.

Proof. Geometric tropicalisation requires a simple normal crossings compactification and a torus embedding. These two conditions are satisfied by lemma 3·2 and lemma 3·4. By lemma 3·5 the divisorial valuations of the boundary divisors yield the rays of this fan. Theorem 2·5 from [**3**] states that the weight of each top-dimensional cone $\sigma \subset \mathrm{trop}(\mathcal{M}_{0,\Gamma})$ is equal to the intersection number, with multiplicity, of the divisors corresponding to the rays of σ. A non-empty intersection of $n-3$ hypersurfaces is a single point with multiplicity 1, coinciding with the weights on $\mathrm{pr}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trop}})$.

From [**5**], we know that $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}} = \mathrm{pr}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trop}})$ if and only if Γ is a complete multipartite graph. Tropically, this characterisation comes from studying the injectivity of a restriction morphism on graphic matroids. Algebraically, we study the injectivity of the divisorial valuation maps. Unlike in the $\mathcal{M}_{0,n}$ case (discussed in Section 2·3), the map π_Γ may not be injective. There is a similar relation to equation (3·4) for $\overline{\mathcal{M}}_{0,n}$ that we may use to demonstrate the simplest case of non-injectivity: consider the divisor $D_{\{i,j,k\}}$ in $\overline{\mathcal{M}}_{0,n}$ and its image under the divisorial valuation map

$$\vec{v}_{D_{\{i,j,k\}}} = \vec{v}_{D_{\{i,j\}}} + \vec{v}_{D_{\{i,k\}}} + \vec{v}_{D_{\{j,k\}}}. \tag{3·5}$$

If exactly two of the vectors on the right correspond to Γ-unstable divisors, then π_Γ cannot be injective. This case does not happen when Γ is complete multipartite. Example 3·7 demonstrates the failure of injectivity for π_Γ, while example 3·8 exhibits a case where π_Γ is injective.

Example 3·7. Let $\widetilde{\Gamma}$ be the subgraph of K_4 with edges e_{35} and e_{45} removed; see Figure 5. Then we have $\mathcal{M}_{0,\widetilde{\Gamma}} \hookrightarrow T^{\binom{5}{2}-5-2} = T^3$ with coordinates z_{24}/z_{23}, z_{25}/z_{23}, and z_{34}/z_{23}. In $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$, there are 8 irreducible boundary divisors, labelled in Figure 6(a). Comparing the cone complexes of $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trop}}$ and $\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$, we can see that the cones associated to the boundary strata $D_{\{3,4\}}$, $D_{\{3,4,5\}}$, and $D_{\{3,4\}} \cap D_{\{3,4,5\}}$ in $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trop}}$ are all mapped to the ray given by $D_{\{3,4\}}$ in $\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$. Explicitly, $\pi_{\widetilde{\Gamma}} : \Delta(\partial\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}) \to \mathbb{R}^3$ where the divisors have been mapped to the following primitive vectors:

$$\vec{v}_{D_{\{2,4\}}} = (1, 0, 0) \qquad \vec{v}_{D_{\{2,5\}}} = (0, 1, 0) \qquad \vec{v}_{D_{\{3,4\}}} = (0, 0, 1)$$
$$\vec{v}_{D_{\{2,3\}}} = (-1, -1, -1) \qquad \vec{v}_{D_{\{2,3,4\}}} = (0, -1, 0) \qquad \vec{v}_{D_{\{2,3,5\}}} = (-1, 0, -1)$$
$$\vec{v}_{D_{\{2,4,5\}}} = (1, 1, 0) \qquad \vec{v}_{D_{\{3,4,5\}}} = (0, 0, 1)$$

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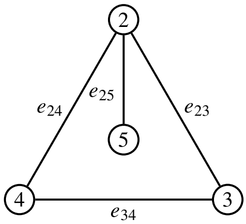
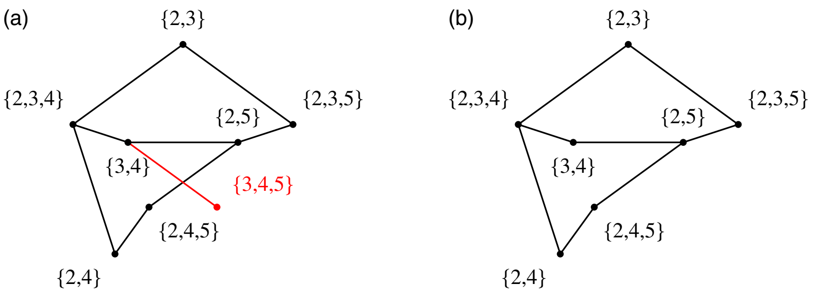


Fig. 5. The graph $\widetilde{\Gamma}$ in example 3·7.



A slice of the cone complex $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ with rays labeled by their divisor index set.

A slice of the cone complex $\text{trop}(\mathcal{M}_{0,\Gamma})$ with rays labeled by their divisor index set.

Fig. 6. Cone complexes for example 3·7.

Example 3·8. Let $\Gamma = K_{2,2}$ be the complete bipartite graph obtained by removing edges e_{25} and e_{34} from K_4, as shown in Figure 7(a). Then we have $\mathcal{M}_{0,K_{2,2}} \hookrightarrow T^{\binom{5}{2}-5-2} = T^3$ with coordinates $z_{24}/z_{23},\ z_{35}/z_{23}$, and z_{45}/z_{23}. In $\overline{\mathcal{M}}_{0,K_{2,2}}$, there are 8 irreducible boundary divisors, labelled in Figure 7(b). Explicitly, $\pi_\Gamma : \Delta(\partial\overline{\mathcal{M}}_{0,K_{2,2}}) \to \mathbb{R}^3$ where the divisors have been mapped to the following primitive vectors:

$$\vec{v}_{D_{\{2,4\}}} = (1,0,0) \qquad \vec{v}_{D_{\{3,5\}}} = (0,1,0) \qquad \vec{v}_{D_{\{4,5\}}} = (0,0,1).$$
$$\vec{v}_{D_{\{2,3\}}} = (-1,-1,-1) \qquad \vec{v}_{D_{\{2,3,4\}}} = (0,-1,-1) \qquad \vec{v}_{D_{\{2,3,5\}}} = (-1,0,-1)$$
$$\vec{v}_{D_{\{2,4,5\}}} = (1,0,1) \qquad \vec{v}_{D_{\{3,4,5\}}} = (0,1,1)$$

Lemma 3·9 gives useful characterisations of a complete multipartite graph also used in [5].

LEMMA 3·9. *Let G be a graph. The following are equivalent:*

(i) G is a complete multipartite graph;

(ii) If e_{ij} is an edge of G, then for any vertex v_k, either e_{ik} or e_{jk} is an edge of G;

(iii) There do not exist 3 vertices whose induced subgraph has exactly 1 edge.

Proof. We can see that all three conditions express that the complement of G is a disjoint union of cliques.

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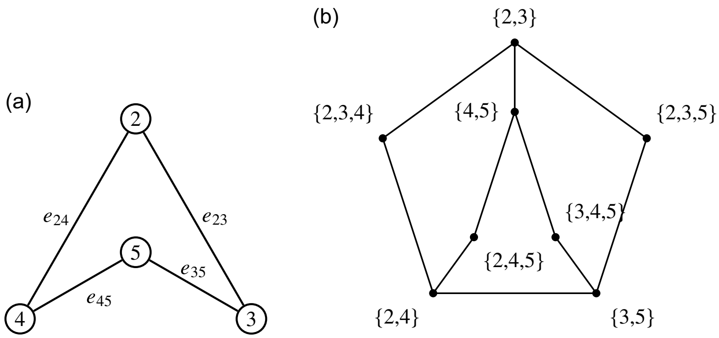

The graph $K_{2,2}$ in Example 3·8

A slice of the cone complex $\mathrm{trop}(\mathcal{M}_{0,K_{2,2}}) = \mathcal{M}^{\mathrm{trop}}_{0,K_{2,2}}$ with rays labeled by their divisor index set.

Fig. 7. Graph and cone complexes for example 3·8.

LEMMA 3·10. *The divisorial valuation map π_Γ is injective if and only if Γ is complete multipartite. In this situation, the cone complex $\mathcal{M}^{trop}_{0,\Gamma}$ is embedded as a balanced fan in a real vector space by π_Γ.*

Proof. We begin by proving the forwards direction by contradiction. Suppose π_Γ is injective and Γ is not complete multipartite. Using lemma 3·9, fix three vertices v_i, v_j, and v_k where $e_{ij} \in E(\Gamma)$ but $e_{ik}, e_{jk} \notin E(\Gamma)$. We have the following contradiction

$$\pi_\Gamma(D_{\{i,j,k\}}) = \vec{v}_{D_{\{i,j,k\}}} = \vec{v}_{D_{\{i,j\}}} = \pi_\Gamma(D_{\{i,j\}}).$$

For the backwards direction, assume Γ is complete multipartite. Let D_I and D_J be two Γ-stable divisors such that $\vec{v}_{D_I} = \vec{v}_{D_J}$. By equation (3·3), $\{2,3\} \subseteq I$ if and only if $\{2,3\} \subseteq J$. If $\{2,3\} \subseteq I, J$, then we have

$$-\vec{1} + \sum_{\substack{\{i,j\}\subset I,\ \{i,j\}\neq\{2,3\};\\ e_{ij}\in E(\Gamma)}} \vec{e}_{ij} = -\vec{1} + \sum_{\substack{\{i,j\}\subset J,\ \{i,j\}\neq\{2,3\};\\ e_{ij}\in E(\Gamma)}} \vec{e}_{ij}.$$

If $\{2,3\} \not\subseteq I, J$, then we have

$$\sum_{\substack{\{i,j\}\subset I;\\ e_{ij}\in E(\Gamma)}} \vec{e}_{ij} = \sum_{\substack{\{i,j\}\subset J;\\ e_{ij}\in E(\Gamma)}} \vec{e}_{ij}.$$

In either case, this implies that the induced subgraphs Γ_I and Γ_J of Γ have the same edge sets, $E(\Gamma_I) = E(\Gamma_J)$. If $I \neq J$, then there exists $i \in I \setminus J$. But $v_i \in \Gamma$ must be isolated in Γ_I, otherwise it would be contained in an edge in $E(\Gamma_I)$ and thus $i \in J$. However, Γ_I is a complete multipartite graph, so it cannot have any isolated vertices. Therefore, $I = J$, concluding the proof that π_Γ is injective.

Additionally, the map π_Γ induces a map of cone complexes from $\mathcal{M}^{\mathrm{trop}}_{0,\Gamma} = \mathrm{cone}(\Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}))$ to $\mathrm{trop}(\mathcal{M}_{0,\Gamma}) = \mathrm{cone}(\mathrm{Im}(\pi_\Gamma))$ which is an isomorphism if and only if Γ is complete multipartite. Furthermore, we know that $\mathcal{M}^{\mathrm{trop}}_{0,\Gamma} = \mathrm{pr}_\Gamma(\mathcal{M}^{\mathrm{trop}}_{0,n})$ is a balanced fan (with constant weight function 1) by [**5**, theorem 29].

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PROPOSITION 3·11. *If Γ is not complete multipartite, there does not exist a balanced embedding of $\mathcal{M}_{0,\Gamma}^{trop}$ into a real vector space by any map.*

Proof. If Γ is not complete multipartite, there exists three markings i,j,k so that $e_{ij} \in E(\Gamma)$, but $e_{ik}, e_{jk} \notin E(\Gamma)$. Consider a codimension-one cone τ of $\mathcal{M}_{0,n}^{\mathrm{trop}}$ where the unique 4-valent vertex of its tropical curve has a single bounded edge and the three markings i,j,k and all of its rays are given by Γ-stable tropical curves.

In $\mathcal{M}_{0,n}^{\mathrm{trop}}$, τ is contained in three facets spanned by the rays of τ and an additional ray coming from one of the tropical curves $D_{\{i,j\}}$, $D_{\{i,k\}}$, or $D_{\{j,k\}}$. However, the tropical curves $D_{\{i,k\}}$ and $D_{\{j,k\}}$ are not Γ-stable, and thus their rays are contracted in $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$. In $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$, the image of τ is a condimension-one cone contained in only a single facet, and thus cannot be balanced in any embedding.

Before the main result of the paper, we prove a lemma that identifies the units of $\mathcal{M}_{0,\Gamma}$ as forgetful morphisms to $\mathcal{M}_{0,4}$ by extending cross ratios on $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,\Gamma}$. Not all forgetful morphisms can be considered as units of $\mathcal{M}_{0,\Gamma}$ because the space may contain points corresponding to curves that are not mapped to $\mathcal{M}_{0,4}$ by every forgetful morphism. For example, if $\Gamma = K_{2,3}$, $\mathcal{M}_{0,\Gamma}$ contains a point corresponding to the curve with three distinct marked points p_1, $p_2 = p_3$, and $p_4 = p_5 = p_6$. If p_2 and p_3 are forgotten, the point of $\mathcal{M}_{0,\Gamma}$ doesn't land in $\mathcal{M}_{0,4}$. Thus we want to only keep the extended forgetful morphisms for which this doesn't happen.

LEMMA 3·12. *The units of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ are generated by cross ratios, i.e. forgetful morphisms to $\mathcal{M}_{0,4}$.*

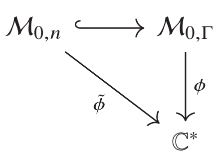
Proof. The space $\mathcal{M}_{0,n}$ can be viewed as the subset of $(\mathbb{C}^* \setminus \{1\})^{n-3}$ minus the hyperplanes $x_i - x_j = 0$. The functions which don't vanish on $\mathcal{M}_{0,n}$ are rational functions that have zeros and poles on the hyperplanes, i.e. Laurent monomials in x_i, $x_i - 1$, and $x_i - x_j$. We claim that we can write any monomial function as a product of cross ratios:

$$\frac{(P_1 - P_2)(P_3 - P_4)}{(P_1 - P_3)(P_2 - P_4)}$$

for marked points P_1, P_2, P_3, P_4.

For x_i, let $P_1 = x_i$, $P_2 = 0$, $P_3 = \infty$, and $P_4 = 1$.
For $x_i - 1$, let $P_1 = x_i$, $P_2 = 1$, $P_3 = \infty$, and $P_4 = 0$.
For $x_i - x_j$, take a product of x_i and $P_1 = x_i$, $P_2 = x_j$, $P_3 = 0$, and $P_4 = \infty$.

Consider the embedding of $\mathcal{M}_{0,n}$ into $\mathcal{M}_{0,\Gamma}$ in the diagram below where ϕ is a unit of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$.



From arguments above, $\tilde{\phi}$ must be a product of cross ratios. Because $\mathcal{M}_{0,n}$ is dense in $\mathcal{M}_{0,\Gamma}$, ϕ must be the extension of $\tilde{\phi}$ which does not vanish, nor acquire poles at the extra interior points in $\mathcal{M}_{0,\Gamma} \setminus \mathcal{M}_{0,n}$. Indeed, ϕ is also a product of cross ratios.

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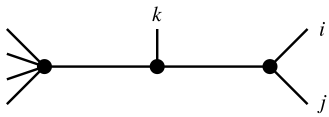

Fig. 8. Dual graph of the stratum S contained in $D_{\{i,j\}}$ and $D_{\{i,j,k\}}$.

Theorem 3·13. *For* Γ *compelte multipartite, there is a torus embedding*

$$\mathcal{M}_{0,\Gamma} \hookrightarrow T^{\binom{n}{2}-n-N} = T_\Gamma$$

whose tropicalisation $\text{trop}(\mathcal{M}_{0,\Gamma})$ *has underlying cone complex* $\mathcal{M}_{0,\Gamma}^{\text{trop}}$*. Furthermore, the tropical compactification of* $\mathcal{M}_{0,\Gamma}$ *is* $\overline{\mathcal{M}}_{0,\Gamma}$*, i.e, the closure of* $\mathcal{M}_{0,\Gamma}$ *in the toric variety* $X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ *is* $\overline{\mathcal{M}}_{0,\Gamma}$*.*

Proof. As in [**2**, theorem 3·9], we wish to show the map $\overline{\mathcal{M}}_{0,\Gamma} \to X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ is an embedding. According to [**10**, lemma 2·6 (4) and theorem 2·10], this occurs when the following two conditions hold. For a locally closed stratum S of $\overline{\mathcal{M}}_{0,\Gamma}$, let $\mathcal{M}_S$ be $\mathcal{O}^*(S)/k^*$ and $\mathcal{M}^S_{\mathcal{M}_{0,\Gamma}}$ be the sublattice of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})/k^*$ generated by units having zero valuation on S.

(1) For each boundary divisor D containing S, there is a unit $u \in \mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ with valuation 1 on D and valuation 0 on other boundary divisors containing S.

(2) S is very affine and the restriction map $\mathcal{M}^S_{\mathcal{M}_{0,\Gamma}} \to \mathcal{M}_S$ is surjective.

We note that condition (1) occurs if and only if Γ is a complete multipartite graph, but condition (2) does not force Γ to be complete multipartite.

For condition (1), recall that the general element of a boundary divisor D_I has exactly one node and may be described by I, the set of marked points on a component. By lemma 3·12, the units in $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ are generated by forgetful morphisms to $\mathcal{M}_{0,4}$ using *cross ratios*. Such a forgetful morphism has valuation 1 on D if the image of the general element of D is nodal and valuation 0 on D if the image of the general element of D is smooth. We show forgetful morphisms with this property exists if and only if Γ is a complete multipartite graph.

We prove the forwards direction by way of contradiction. Assume Γ is not complete multipartite. Using lemma 3·9, fix three vertices v_i, v_j, and v_k where $e_{ij} \in E(\Gamma)$ but $e_{ik}, e_{jk} \notin E(\Gamma)$. Consider the divisors $D_{\{i,j,k\}}$ and $D_{\{i,j\}}$ whose intersection yields the stratum S whose dual graph is shown in Figure 8. Every forgetful morphism that has valuation 1 on $D_{\{i,j,k\}}$ must not forget i and j, otherwise, the image of the general element of $D_{\{i,j,k\}}$ is smooth. However, any such morphism also has valuation 1 on $D_{\{i,j\}}$, a contradiction.

Now suppose Γ is complete multipartite. Fix a stratum S and a divisor D_I containing S, as shown in Figure 9. Our aim is to find four markings, $\{a,b,c,d\}$, such that the general element of D_I remains nodal and the general element of all other divisors containing S become smooth in $\mathcal{M}_{0,4}$ on $\{a,b,c,d\}$. We proceed by fixing $d = 1$ so that a,b,c correspond to vertices in Γ and without loss of generality, let $a, b \in I$ and $c \in I^c \setminus \{1\}$. To ensure the image of the general element of D_I remains nodal, we must pick a,b such that $e_{ab} \in E(\Gamma)$, though, we have no additional restrictions of c because of the "global stability" the marking 1 carries with it.

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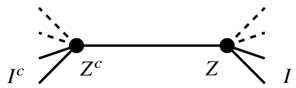

Fig. 9. Dual graphs of the stratum S and divisor D_I from the proof of theorem 3·3, where dashed edges represent potential extra components.

Consider the subcurves Z and Z^c of S and $\overline{Z}$ and $\overline{Z}^c$ of D_I that share a node, as illustrated by their dual graphs in Figure 9. We partition I in the following way. Split S into connected components by separating S at the nodes of Z. Now, S has been deconstructed into several connected components: Z with its markings and a connected component for each node on Z. Let λ_I be the partition of I given by the markings on Z and the components previously attached to Z, excluding the component with Z^c.

For brevity, we highlight that many facts in this paragraph are justified by stability in some way. If Z has only one node, then $\lambda_I = I$ and we need only choose $a, b \in I$ so that $e_{ab} \in E(\Gamma)$. Suppose Z has more than one node. There exists at least one part $A \in \lambda_I$ with $|A| \geq 2$, fix such a part A. Let $B = I \setminus A$, which is necessarily nonempty. Fix an edge $e_{a_1a_2} \in E(\Gamma)$ for markings $a_1, a_2 \in A$. Lemma 3·4 says that for a marking $b \in B$, either e_{a_1b} or e_{a_2b} is in $E(\Gamma)$. Hence, there exists $a, b \in I$ so that $e_{ab} \in E(\Gamma)$ when Z has multiple nodes.

Indeed, in each of the above cases the forgetful morphism which remembers $\{1, a, b, c\}$ has valuation 1 on D_I since $e_{ab} \in E(\Gamma)$. Let D_J be any other divisor containing S. From our choices of a,b,c, we know $|J \cap \{1, a, b, c\}| \neq 2$. This means that the image of the general element D_J under the forgetful morphism which keeps $\{1, a, b, c\}$ will be smooth, and thus D_J has valuation 0.

For condition (2), a stratum S is very affine because it can be viewed as a product of $\mathcal{M}_{0,\Gamma'}$s. Each component of the universal curve over a point S contains at least one node which acts as the 'special' marking 1; the marked points behave under Γ'-stability, since any subgraph of a complete multipartite graph is complete multipartite; and any extra node serves as a marked point whose vertex in Γ' is connected to all other vertices, which keeps Γ' as a complete multipartite graph since a fully connected vertex serves as its own independent set. Finally, since the boundary of $\overline{\mathcal{M}}_{0,\Gamma}$ is a simple normal crossings divisor, as in the case of $\overline{\mathcal{M}}_{0,w}$, the surjectivity of the restriction map follows the same proof outline as in [**2**, theorem 3·9]. The local structure of $\partial\overline{\mathcal{M}}_{0,\Gamma}$ is an intersection of coordinate hyperplanes and restricting the coordinates is surjective.

Many statements remain true when Γ is not complete multipartite. The geometric tropicalisation of $\overline{\mathcal{M}}_{0,\Gamma}$ using the embedding in lemma 3·4 still equals $\mathrm{pr}_\Gamma(\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}) = \mathrm{trop}(\mathcal{M}_{0,\Gamma})$. However, not all cones are mapped injectively. On the algebraic side, we still have a map from $\overline{\mathcal{M}}_{0,\Gamma}$ to the toric variety $X(\mathrm{trop}(\mathcal{M}_{0,\Gamma}))$, but it does not map all boundary strata injectively. Example 3·14 highlights this observation.

Example 3·14. Let $\widetilde{\Gamma}$ be the subgraph of K_4 with edges e_{35} and e_{45} removed as in example 3·6; see Figure 6(a). Then Figure 6(b) depicts the boundary of $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$ and a slice of $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trop}}$ while Figure 2 depicts the boundary of the closure of $\mathcal{M}_{0,\widetilde{\Gamma}}$ in $X(\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ and a slice of $\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$.

There are only 8 2D cones and 7 rays in $\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$ while the cone over $\partial\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$ has 9 2D cones and 8 rays. This means $X(\mathrm{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ isn't large enough to contain $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$. In other

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words, the locus of smooth curves $\mathcal{M}_{0,\widetilde{\Gamma}}$ is missing the limit as the marked points 3, 4, and 5 collide. The modular compactification of $\mathcal{M}_{0,\widetilde{\Gamma}}$ assigns a $\mathbb{P}^1$ to the limit but $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ doesn't have enough coordinates to include a $\mathbb{P}^1$. Rather, this limit gets closed with a single point in $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ (which is the intersection of two smooth curves where the marked points 3 and 4 and 4 and 5, have collided).

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