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FINITE STABILITY DOMAINS FOR DIFFERENCE EQUATIONS

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Dedicated to George Szekeres on his 65th birthday, with affection and respect

Abstract

A technique for the estimation of domains of local stability for difference equations is discussed. A Liapunov function is used in the estimation. Sharper results are possible if there is only one type of nonlinearity, when open Liapunov surfaces are possible. An example of the technique is given.

1

Liapunov's direct method is a powerful tool for the stability analysis of both continuous and discrete systems. Significant problems arise in its application when the Liapunov function is only decreasing on orbits of the system in an open region about an equilibrium point. Then in general, stability is not global but a finite stability domain exists. Typically, Liapunov functions have been used to estimate such regions of attraction for differential equations (Noldus, Galle and Josson (1973); Weissenberger (1968); Willems (1969)), but this approach is difficult to apply to discrete systems (Diamond (1975); Hurt (1967)).

This note describes a method of calculating finite domains of asymptotic stability, around the origin, for discrete autonomous systems

(1) $$x_{n+1} = f(x_n),\ n = 0, 1, 2, \cdots, x_0 = x.$$

Here x is a k-vector and f a function from R^k to itself.

Let V, W be nonnegative real valued functions defined on R^k and suppose that in a domain G containing the origin

(2) $$V(x_{n+1}) - V(x_n) \leqq -W(x_n) \leqq 0,$$

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and that in G, $W(x)=0$ iff $x=0$. The function V will be termed a Liapun(function on G for the system (1). Set

$$V_{\min} = \min\{V(x): x \in bdy\, G\},$$

and define

$$J_m = \{x: V(x_m) < V_{\min}\}, \qquad m = 0, 1, 2, \cdots.$$

THEOREM 1. *The regions J_m are domains of asymptotic stability for the syste described by equation* (1).

PROOF. If $x \in J_m$, by (2) $V(x_{m+n}) \leqq V(x_m) < \dot{V}_{\min}$, $n = 1, 2, \cdots$, and : $x_n \in J_m$. It follows that x_n approaches the set $\{x \in G: W(x) = 0\}$ as n approach infinity (Hurt (1967)). So x_n can approach at most the origin and possibly son part of the boundary of J_m. Since $V(x_n)$ is a nonincreasing function of n, tl latter is impossible.

NOTE. The condition on W could be weakened and this would give fini regions of attraction for the set $\{x \in \bar{G}: W(x) = 0\}$, rather than just the origin.

2

Estimates of the region of attraction can be sharpened for systems with single type of nonlinearity:

$$(3) \qquad \begin{aligned} x_{n+1} &= Ax_n + bg(s), \\ s &= c'x_n, \end{aligned}$$

and where the Liapunov inequality holds for $p < c'x < q$, where $p < 0$ and $q >$ As before, x, b, c are k-vectors, c' the transpose of c, A is a $k \times k$ matrix and a scalar function. The system (3) is a discrete analogue of a feedback loop wi amplification $g(s)$.

Consider the hyperplanes

$$E_p: c'(A - I)x + c'bg(p) = 0, \quad E_q: c'(A - I)x + c'bg(q) = 0.$$

Put

$$V_E = \min\{V(x): x \in E_p \cup E_q\},$$

and define

$$V_M = \max(V_E, V_{\min}).$$

Obviously $V_M \geqq V_{\min}$, where $V_{\min}$ is here the minimum value attained by $V($: on the hyperplanes $c'x = p$, $c'x = q$. Suppose that

$$c'bg(p) < c'bg(q).$$

The conditions and arguments of the following theorem are very much the same if this inequality is reversed.

THEOREM 2. *If*

$$c'b(g(s)-g(p)) \geqq 0, \textit{ and } c'b(g(s)-g(q)) \leqq 0, \textit{ for } p<s<q,$$

then

$$H_m = \{x: V(x_m) < V_M, p < c'x < q\}$$

is a finite domain of asymptotic stability for the system (3).

PROOF. The case where E_p is parallel to $c'x = p$ and the other two planes is covered by theorem 1, with $V_M = V_{\min}$, as is also the case when $V_E < V_{\min}$. So in what follows it is assumed that such a degenerate case does not occur. By construction, every point of H_0 lies between the two hyperplanes E_p, E_q. Since $x \in H_0$ lies above E_p,

$$c'(A-I)x + c'bg(p) > 0.$$

Thus

$$c'x_1 = c'Ax + c'bg(c'x) > c'x + c'b(g(c'x)-g(p)) > c'x > p.$$

Similarly $c'x_1 < q$ and so the orbits of the system cannot pass through the hyperplanes $c'x = p$, $c'x = q$, without first passing through E_p and E_q. But the surface $V(x) = V_M$ is, at most, tangent to one or both of E_p, E_q. The Liapunov property ensures that the orbits cannot pass through this surface whilst $p < c'x < q$, and so must remain between E_p and E_q. It follows that $x_n \in H_0$ for all n and the conclusion follows, as in Theorem 1, by observing that if x_n converged to a point y, $c'y = p$, then $c'y_1 > p$. Similar reasoning applies to the other H_m.

NOTES. (1) The conditions of the theorem would be satisfied by any function which was increasing on $p < s < q$, provided the Liapunov property held there.

(2) The region H_0 is bounded by a closed Liapunov surface only if $V_M = V_{\min}$.

3

As an example, consider

$$x_{n+1} = y_n, \quad y_{n+1} = ax_n + y_n^2,$$

and the Liapunov function

$$V(x, y) = a^2x^2 + y^2.$$

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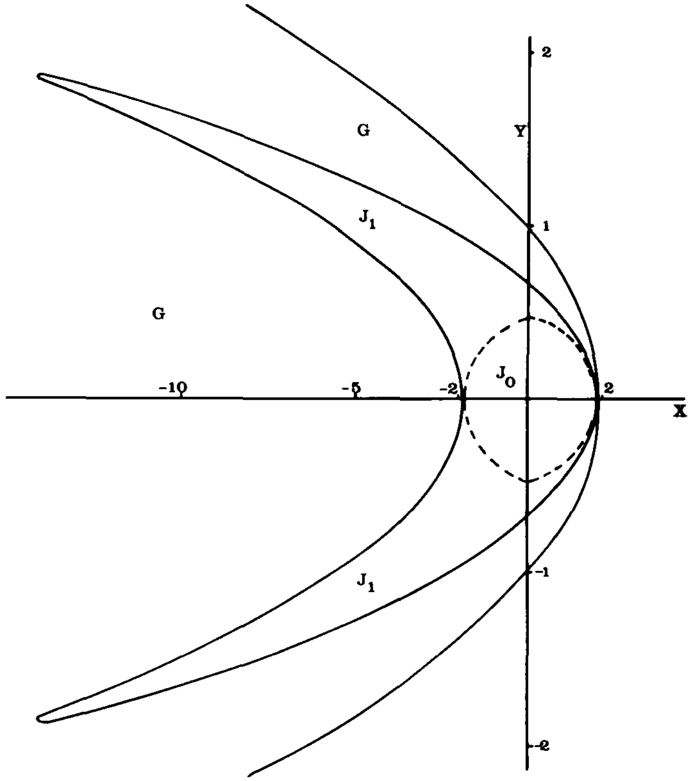

Local stability regions for the example. G is the domain in which the Liapunov inequality (2) holds, and J_0, J_1 are finite stability domains, as in Theorem 1.

Then

$$V(x_1, y_1) - V(x, y) = y^2(a^2 - 1 + 2ax + y^2)$$

and $G = \{(x, y)\colon a^2 - 1 + 2ax + y^2 < 0\}$. It is easy to show that $V_{\min} = (1 - a^2)^2/4$, whence $J_0 = \{(x, y)\colon a^2x^2 + y^2 < (1 - a^2)^2/4\}$ and

$$J_1 = \{(x, y)\colon a^2x^2 + (a^2 + 2ax)y^2 + y^4 < (1 - a^2)^2/4\}.$$

The regions G, J_0 and J_1 are shown in the figure for the value $a = 1/4$. Note that J_1 is a significant improvement upon J_0 and that both are larger than any disc in G centred at the origin.

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