independent variables, and, in fact, it was this combination that I first considered, in seeking to extend Legendre's mode of solving (27). I am convinced, however, that in practice it is simpler to use the methods successively, as exemplified in the two preceding examples, just as in the ordinary method of changing variables we usually arrive at the ultimate transformation through many intermediate steps.

It need hardly be added that instead of this complete reciprocation, Routh's method of partial reciprocation or modification may be used in many cases with advantage.

The Plane Triangle ABC: Intimoscribed Circles, etc.

By R. E. Anderson, M.A.

§ I. *On an infinite series of Triad Circles derived from the inscribed circle. Determination of a direct relation between* r *and the three radii of the* n[th] *triad.*

Each of the first triad touches two sides of ABC and the inscribed circle: generally, each circle of the m^{th} triad touches two sides of ABC and also touches one of the circles of the $(m-1)^{th}$ triad.

Fig. 27.

In the diagram only one member of each successive triad is indicated—viz., the circles forming a diminishing series between the in-circle and B.

Let $\quad ON = OY = r$

$$\left.\begin{array}{l} O'N' = r_2' \\ O''N'' = r_2'' \\ O'''N''' = r_3''' \\ \text{etc., etc.} \end{array}\right\} \quad \therefore \quad \left\{\begin{array}{llll} r_2' & r_3' & r_1' & \text{are radii of first triad} \\ r_2'' & r_3'' & r_1'' & \ldots\ldots \quad \text{second} \\ r_2''' & r_3''' & r_1''' & \ldots\ldots \quad \text{third} \\ & \text{etc.} & & \ldots\ldots \quad \text{etc.} \end{array}\right.$$

$$\text{and} \quad \rho_2 \quad \rho_3 \quad \rho_1 \quad \ldots\ldots \quad n^{th} \text{ triad,}$$

where in every case the suffix 2 has reference to B, 3 to C, and 1 to A—a rule which also holds for the subsequent sections.

Now $\qquad ON = O'N' + OO'\sin\frac{1}{2}B,$

$$\therefore \qquad r = r_2' + (r + r_2')\sin\tfrac{1}{2}B$$

$$r_2' = r_2'' + (r_2' + r_2'')\sin\tfrac{1}{2}B$$

etc., for ever

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and $$\frac{r_2'}{r} = \frac{1-\sin\frac{1}{2}B}{1+\sin\frac{1}{2}B} = \tan^2\tfrac{1}{4}(\pi - B)$$
$$= t_2^2 \text{ suppose.}$$

Thus $$t_2 = \sqrt{\frac{r_2'}{r}} = \sqrt{\frac{r_2''}{r_2'}} = \sqrt{\frac{r_2'''}{r_2''}} = \text{etc.},$$

or $$t_2 = \sqrt[2]{\frac{r_2'}{r}} = \sqrt[4]{\frac{r_2''}{r}} = \sqrt[6]{\frac{r_2'''}{r}} = \text{etc.}$$

∴ $$t_3 = \sqrt[2]{\frac{r_3'}{r}} = \sqrt[4]{\frac{r_3''}{r}} = \sqrt[6]{\frac{r_3'''}{r}} = \text{etc.}$$

$$t_1 = \sqrt[2]{\frac{r_1'}{r}} = \sqrt[4]{\frac{r_1''}{r}} = \sqrt[6]{\frac{r_1'''}{r}} = \text{etc.}$$

where $t_2 = \tan\frac{1}{4}(\pi - B)$, $t_3 = \tan\frac{1}{4}(\pi - C)$ and $t_1 = \tan\frac{1}{4}(\pi - A)$ or $\tan\frac{1}{4}(A - \pi)$.

But $$\tfrac{1}{4}(\pi - B) + \tfrac{1}{4}(\pi - C) + \tfrac{1}{4}(\pi - A) = \frac{\pi}{2}$$

and ∴ $$t_2t_3 + t_3t_1 + t_1t_2 = 1.$$

Thus the angular functions are easily eliminated between those three sets of equations so as to establish a direct relation between r and the radii of any triad whatever. Take, for example, the third triad.

$$t_2 = \sqrt[6]{\frac{r_2'''}{r}} \quad t_3 = \sqrt[6]{\frac{r_3'''}{r}} \quad t_1 = \sqrt[6]{\frac{r_1'''}{r}}$$

∴ $$t_2t_3 + t_3t_1 + t_1t_2 = 1 = \sqrt[6]{\frac{r_2'''r_3'''}{r^2}} + \sqrt[6]{\frac{r_3'''r_1'''}{r^2}} + \sqrt[6]{\frac{r_1'''r_2'''}{r^2}}$$

Otherwise $$\sqrt[3]{r} = \sqrt[6]{r_2'''r_3'''} + \sqrt[6]{r_3'''r_1'''} + \sqrt[6]{r_1'''r'''}$$

Hence these results :—

First triad,	$r =$	$\sqrt[2]{r_2'\ r_3'} + \sqrt[2]{r_3'\ r_1'} + \sqrt[2]{r_1'\ r_1'}$
Second ,,	$\sqrt[2]{r} =$	$\sqrt[4]{r_2''\ r_3''} + \sqrt[4]{r_3''\ r_1''} + \sqrt[4]{r_1''\ r_2''}$
Third, ,,	$\sqrt[3]{r} =$	$\sqrt[6]{r_2'''r_3'''} + \sqrt[6]{r_3'''r_1'''} + \sqrt[6]{r_1'''r_2'''}$
etc.		etc.
n^{th} triad,	$\sqrt[n]{r} =$	$\sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho_2}$

§ II. *On the series of Triad Circles similarly derived from each of the exscribed circles. Determination of the equation between* r_1 *(or* r_2*, etc.) and the three radii of the* n^{th} *triad.*

Fig. 28.

Let O and Q_1 in the diagram be centres of the inscribed and exscribed circles, then first triad will touch the ex-circle at X_1, X_2

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and X_3. As before, one member only of the successive triads is shown, viz., the series between X_2 and B.

Let $Q_1M = Q_1X_2 = r_1$

$$\left.\begin{array}{l} Q'\,M' = a_2' \\ Q''M'' = a_2'' \\ Q'''M''' = a_2''' \\ \quad \text{etc.} \end{array}\right\} \quad \therefore \quad \left\{\begin{array}{llll} a_1' & a_2' & a_3' & \text{are radii of first triad} \\ a_1'' & a_2'' & a_3'' & \ldots\ldots \text{ second} \\ a_1''' & a_2''' & a_3''' & \ldots\ldots \text{ third} \\ & \text{etc.} & & \text{etc.} \end{array}\right.$$

$$\text{and} \quad a_1 \ a_2 \ a_3 \quad \ldots\ldots \ n^{th} \text{ triad.}$$

Now $$Q_1M = Q'M' + (Q_1M + Q'M')\sin\tfrac{1}{2}(\pi - B)$$

$$\therefore \quad r_1 = a_2' + (r_1 + a_2')\cos\tfrac{1}{2}B$$

$$\therefore \quad \frac{a_2'}{r_1} = \frac{1 - \cos\frac{1}{2}B}{1 + \cos\frac{1}{2}B} = \frac{\sin^2\frac{1}{4}B}{\cos^2\frac{1}{4}B} = \tan^2\tfrac{1}{4}B$$

$$= t_2^2 \text{ suppose.}$$

Thus $$t_2 = \sqrt{\frac{a_2'}{r_1}} = \sqrt{\frac{a_2''}{a_2'}} = \sqrt{\frac{a_2'''}{a_2''}} = \text{etc.}$$

or

$$\left.\begin{array}{l} t_2 = \sqrt[2]{\dfrac{a_2'}{r_1}} = \sqrt[4]{\dfrac{a_2''}{r_1}} = \sqrt[6]{\dfrac{a_2'''}{r_1}} = \text{etc.} \\ \therefore \quad t_3 = \sqrt[2]{\dfrac{a_3'}{r_1}} = \sqrt[4]{\dfrac{a_3''}{r_1}} = \sqrt[6]{\dfrac{a_3'''}{r_1}} = \text{etc.} \\ t_1 = \sqrt[2]{\dfrac{a_1'}{r_1}} = \sqrt[4]{\dfrac{a_1''}{r_1}} = \sqrt[6]{\dfrac{a_1'''}{r_1}} = \text{etc.} \end{array}\right\} \begin{array}{l} \text{where} \\ \quad t_2 = \tan\frac{1}{4}B \\ \quad t_3 = \tan\frac{1}{4}C \\ \text{and } t_1 = \tan\frac{1}{4}(\pi - A) \\ \text{or} \quad\quad \tan\frac{1}{4}(A - \pi) \end{array}$$

But $\frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}(A - \pi) = 0$, and the expedient applied in (I.) here fails. I therefore use the theorem that, when the sum of three angles vanishes, the sum of their tangents = the product of their tangents; *i.e.*, $t_1 + t_2 + t_3 = t_1t_2t_3$

$$\therefore \quad \sqrt{\frac{a_1'}{r_1}} + \sqrt{\frac{a_2'}{r_1}} + \sqrt{\frac{a_3'}{r_1}} = \sqrt{\frac{a_1'a_2'a_3'}{r_1^3}} = \frac{1}{r_1}\sqrt{\frac{a_1'a_2'a_3'}{r_1}}$$

Thus

First triad, $$\frac{1}{r_1} = \frac{1}{\sqrt{a_2'a_3'}} + \frac{1}{\sqrt{a_3'a_1'}} + \frac{1}{\sqrt{a_1'a_2'}} = \frac{\sqrt{a_1'} + \sqrt{a_2'} + \sqrt{a_3'}}{\sqrt{a_1'a_2'a_3'}}$$

Second triad, $$\frac{1}{\sqrt[2]{r_1}} = \frac{1}{\sqrt[4]{a_2''a_3''}} + \frac{1}{\sqrt[4]{a_3''a_1''}} + \frac{1}{\sqrt[4]{a_1''a_2''}}$$

Third triad, $$\frac{1}{\sqrt[3]{r_1}} = \frac{1}{\sqrt[6]{a_2'''a_3'''}} + \frac{1}{\sqrt[6]{a_3'''a_1'''}} + \frac{1}{\sqrt[6]{a_1'''a_2'''}}$$

etc. etc.

n^{th} triad, $$\frac{1}{\sqrt[n]{r_1}} = \frac{1}{\sqrt[2n]{\alpha_2\alpha_3}} + \frac{1}{\sqrt[2n]{\alpha_3\alpha_1}} + \frac{1}{\sqrt[2n]{\alpha_1\alpha_2}}$$

For the other two exscribed circles we have of course

$$\frac{1}{\sqrt[n]{r_2}} = \frac{1}{\sqrt[2n]{\beta_2\beta_3}} + \frac{1}{\sqrt[2n]{\beta_3\beta_1}} + \frac{1}{\sqrt[2n]{\beta_1\beta_2}}$$

$$\frac{1}{\sqrt[n]{r_3}} = \frac{1}{\sqrt[2n]{\gamma_2\gamma_3}} + \frac{1}{\sqrt[2n]{\gamma_3\gamma_1}} + \frac{1}{\sqrt[2]{\gamma_1\gamma_2}}$$

$\beta_1\beta_2\beta_3$ being the radii of the n^{th} triad derived from r_2 and

$\gamma_1\gamma_2\gamma_3$ ... ... r_3.

Definition.—Each triad of the four infinite series now discussed is an instance of *intimoscribed circles* (v. infra, Section VI.)

§ III. *On establishing a direct relation between the radii of the twelve circles which form the* n$^{\text{th}}$ *triads of the preceding four intimoscribed series.*

From $rs = r_1s_1 = r_2s_2 = r_3s_3 = \Delta$, we derive

$$\sqrt[n]{r}\left(\frac{1}{\sqrt[n]{r_1}} + \frac{1}{\sqrt[n]{r_2}} + \frac{1}{\sqrt[n]{r_3}}\right) = \sqrt[n]{\frac{s_1}{s}} + \sqrt[n]{\frac{s_2}{s}} + \sqrt[n]{\frac{s_3}{s}}, \quad \text{or}$$

$$\frac{1}{\sqrt[n]{r_1}} + \frac{1}{\sqrt[n]{r_2}} + \frac{1}{\sqrt[n]{r_3}} = \frac{\sqrt[n]{\frac{s_1}{s}} + \sqrt[n]{\frac{s_2}{s}} + \sqrt[n]{\frac{s_3}{s}}}{\sqrt[n]{r}}$$

whence substituting from the four general results obtained in (I., (II.) we obtain

$$\left.\begin{array}{l} \dfrac{1}{\sqrt[n]{\alpha_2\alpha_3}} + \dfrac{1}{\sqrt[2n]{\alpha_3\alpha_1}} + \dfrac{1}{\sqrt[2n]{\alpha_1\alpha_2}} \\ + \dfrac{1}{\sqrt[2n]{\beta_2\beta_3}} + \dfrac{1}{\sqrt[2n]{\beta_3\beta_1}} + \dfrac{1}{\sqrt[2n]{\beta_1\beta_2}} \\ + \dfrac{1}{\sqrt[2n]{\gamma_2\gamma_3}} + \dfrac{1}{\sqrt[2n]{\gamma_3\gamma_1}} + \dfrac{1}{\sqrt[2n]{\gamma_1\gamma_2}} \end{array}\right\} = \frac{\sqrt[n]{\frac{s_1}{s}} + \sqrt{\frac{s_2}{s}} + \sqrt{\frac{s_3}{s}}}{\sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho_2}}$$

or $$\frac{F(\alpha) + F(\beta) + F(\gamma)}{\sqrt[n]{s_1} + \sqrt[n]{s_2} + \sqrt[n]{s_3}} = \frac{s^{-\frac{1}{n}}}{\sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho}} = \frac{1}{\sqrt[n]{rs}}$$

where $$F(\alpha) = \frac{1}{\sqrt[2n]{\alpha_2\alpha_3}} + \frac{1}{\sqrt[2n]{\alpha_3\alpha_1}} + \frac{1}{\sqrt[2n]{\alpha_1\alpha_2}}, \quad F(\beta) = \text{ etc.}$$

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Three corollaries here occur :—

First, when the sum of the reciprocals of the n^{th} roots of $r_1\, r_2\, r_3$ is multiplied by the n^{th} root of r the result involves only a, b, c.

Second,
$$\sqrt{ss_1s_2s_3} = \left(\frac{\sqrt[n]{s_1} + \sqrt[n]{s_2} + \sqrt[n]{s_3}}{F(\alpha) + F(\beta) + F(\gamma)}\right)^n = \sqrt{rr_1r_2r_3}$$

In other words, the area of any Δ is now expressed in terms of the nine radii of the n^{th} triads of intimoscribed circles derived from its three exscribed circles.

Third, if $B' =$ sum of all the circular areas between the in-circle and B,

$$B' = \pi r^2 \frac{t_2^4}{1 - t_2^4} = \pi r^2 \frac{\sin^4\frac{1}{4}(\pi - B)}{\cos\frac{1}{2}(\pi - B)} = \pi r^2 \frac{(1 - \sin\frac{1}{2}B)^2}{4\sin\frac{1}{2}B}$$

$$\therefore \qquad 4B' = \pi r^2\left(\operatorname{cosec}\frac{B}{2} - 2 + \sin\frac{B}{2}\right)$$

Thus

$$\frac{4(A' + B' + C')}{\pi r^2} = \tfrac{1}{2}\sqrt{\frac{r}{R}}\left(\sqrt{\frac{a}{s_1}} + \sqrt{\frac{b}{s_2}} + \sqrt{\frac{c}{s_3}}\right) + 2\sqrt{\frac{R}{r}}\left(\sqrt{\frac{s_1}{a}} + \sqrt{\frac{s_2}{b}} + \sqrt{\frac{s_3}{c}}\right) - 6,$$

$$\therefore \qquad \frac{A' + B' + C'}{\pi r^2} = \frac{1}{8R}(u_1 + u_2 + u_3) + \frac{R}{2}\left(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}\right) - \frac{3}{2}$$

where $A' + B' + C' =$ sum of areas of all the intimoscribed circles within ABC, and $u_1\, u_2\, u_3$ are the radii of the superscribed circles (*v.* Section VI.)

Similarly if $A_1' + B_1' + C_1'$, $A_2' + B_2' + C_2'$, $A_3' + B_3' + C_3'$ are respectively the total areas of the intimoscribed triads depending on r_1 r_2 and r_3, we find

$$\frac{A_1' + B_1' + C_1'}{\pi r_1^2} + \frac{A_2' + B_2' + C_2'}{\pi r_2^2} + \frac{A_3' + B_3' + C_3'}{\pi r_3^2} =$$

$$\frac{A' + B' + C'}{\pi r^2} + \frac{1}{4R}(u_1' + u_2' + u_3') + R\left(\frac{1}{u_1'} + \frac{1}{u_2'} + \frac{1}{u_3'}\right) - 3,$$

where $u_1'\, u_2'\, u_3'$ are the radii of the three outer superscribed circles (*v.* Section VI.)

§ IV. *Extension of the principle to any convex* m*-gon which has an inscribed circle (radius* r*). Determination of the direct relation*

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between r *and* $(\rho_1\, \rho_2\, \rho_3\, \rho_4 \ldots \rho_m)$ *the radii of the* n[th] *set of intimoscribed circles.*

FIG. 29.

O M $= r$ O′ M′ $= r_2'$ O″M″ $= r_2''$ etc.	MBN represents an angle of any rectilineal m-gon. If the figure be named ABCDE, etc., then as already shown in (I.)

$$t_2 = \sqrt[2]{\frac{r_2'}{r}} = \sqrt[4]{\frac{r_2''}{r}} = \sqrt[6]{\frac{r_2'''}{r}} = \text{etc.} = \sqrt[2n]{\frac{\rho_2}{r}}$$

$$t_3 = \sqrt[2]{\frac{r_3}{r}} = \sqrt[4]{\frac{r_3'}{r}} = \sqrt[6]{\frac{r_3'''}{r}} = \text{etc.} = \sqrt[2n]{\frac{\rho_3}{r}}$$

$$t_4 = \sqrt[2]{\frac{r_4'}{r}} = \sqrt[4]{\frac{r_4''}{r}} = \text{etc. etc.}$$

where
$$t_2 = \tan\tfrac{1}{4}(\pi - \mathrm{B}) = \tan\theta_2$$
$$t_4 = \tan\tfrac{1}{4}(\pi - \mathrm{D}) = \tan\theta_4$$
$$t_5 = \tan\tfrac{1}{4}(\pi - \mathrm{E}) = \tan\theta_5, \text{ etc. etc.}$$

Now $\theta_1 + \theta_2 + \theta_3 + \ldots \theta_m = \frac{1}{4}$ sum of the "exterior" angles $= \frac{1}{2}\pi$, whatever m may be.

Let, for example, $m = 12$, then

$$\tan(\theta_1 + \theta_2 + \ldots \theta_6) = \cot(\theta_7 + \theta_8 + \ldots \theta_{12})$$

$$\therefore \quad \frac{s_1 - s_3 + s_5}{1 - s_2 + s_4 - s_6} = \frac{1 - s_2' + s_4' - s_6'}{s_1' - s_3' + s_5'}$$

where $s_3 =$ sum of products of the tans of every three of the first six angles, and $s_3' =$ sum of products of tans of every three of the last six angles.

Multiplying up and compressing we get

$$\sigma_2 + \sigma_6 + \sigma_{10} = 1 + \sigma_4 + \sigma_8 + \sigma_{12}\,;$$

where $\sigma_3 =$ sum of products of tans of every three of the twelve angles. $m = 13$ gives the same result.

Generally whether $m = 2u$ or $2u + 1$

$$\sigma_2 + \sigma_6 + \sigma_{10} + \text{etc.} = 1 + \sigma_4 + \sigma_8 + \text{etc.}$$

or
$$1 - \sigma_2 + \sigma_4 - \sigma_6 + \sigma_8 - \text{etc.} = 0,$$

there being always $u + 1$ terms on left side.

By substituting in that result the values of $t_1\, t_2\, t_3 \ldots t_m$ already found, we can at once reach an equation between r and the m radii of any proposed set of the intimoscribed circles.

Thus for the decagon or hendecagon, n^{th} set,

$$\sigma_2+\sigma_6+\sigma_{10}=1+\sigma_4+\sigma_8 \tag{k}$$

$$\sigma_2=\sqrt[2n]{\frac{\rho_1\rho_2}{r^2}}+\sqrt[2n]{\frac{\rho_2\rho_3}{r^2}}+\ \dots\ =\frac{\Sigma(\sqrt[2n]{\rho_1\rho_2})}{\sqrt[n]{r}}=\frac{\Sigma_2'}{\sqrt[n]{r}}$$

$$\sigma_4=\sqrt[2n]{\frac{\rho_1\rho_2\rho_3\rho_4}{r^4}}+\sqrt[2n]{\frac{\rho_1\rho_2\rho_3\rho_5}{r^4}}+\ \dots\ =\frac{\Sigma(\sqrt[2n]{\rho_1\rho_2\rho_3\rho_4})}{\sqrt[n]{r^2}}=\frac{\Sigma_4'}{\sqrt[n]{r^2}}$$

$$\sigma_6=\frac{\Sigma_6'}{\sqrt[n]{r^3}}\qquad \sigma_8=\frac{\Sigma_8'}{\sqrt[n]{r^4}}\text{, and }\sigma_{10}=\frac{\Sigma_{10}'}{\sqrt[n]{r^5}}$$

where

$$\Sigma_2'=\sqrt[2n]{\rho_1\rho_2}\quad+\sqrt[2n]{\rho_1\rho_3}\quad+\ \dots\text{ etc.}$$

$$\Sigma_4'=\sqrt[2n]{\rho_1\rho_2\rho_3\rho_4}+\sqrt[2n]{\rho_1\rho_2\rho_3\rho_5}+\ \dots\text{ etc.}$$

Substituting in (k)

$$\frac{\Sigma_2'}{\sqrt[n]{r}}+\frac{\Sigma_6'}{\sqrt[n]{r^3}}+\frac{\Sigma_{10}'}{\sqrt[n]{r^5}}=1+\frac{\Sigma_4'}{\sqrt[n]{r^2}}+\frac{\Sigma_8'}{\sqrt[n]{r^4}}$$

$\therefore$

$$1-\Sigma_2'p+\Sigma_4'p^2-\Sigma_6'p^3+\Sigma_8'p^4-\Sigma_{10}'p^5=0$$

if

$$p=\frac{1}{\sqrt[n]{r}}\quad\text{or}\quad r=\frac{1}{p^n}\,.$$

The proof just given for the n^{th} set when $m=10$ or 11 will evidently apply to the most general case, m being any positive integer.

Thus for any m-gon, whether $m=2u$ or $2u+1$ we obtain

$$1-\Sigma_2'p+\Sigma_4'p^2-\Sigma_6'p^3+\ \dots\ +(-1)^u\Sigma_{2u}'p^u=0,$$

an implicit function of r in terms of the m radii of the n^{th} set of intimoscribed circles.

§ V. *On the intimoscribed triads which are derived from the interscribed* circles of any triangle. Determination of an independent equation between the radii of any triad.*

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* *v.* definition *infra*, Section VI.

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FIG. 30.

$$\left.\begin{aligned} RS &= n_2 \\ R'S' &= n_2' \\ R''S'' &= n_2'' \\ &\text{etc.} \end{aligned}\right\}$$

$WZ = n_3 \quad KH = n_1.$

In this system r does not enter but n_2' is derived from n_2, n_2'' from n_2', n_2''' from n_2'', etc., according to the law noted in Section I.

$\therefore$

$$t_2 = \sqrt[2]{\frac{n_2'}{n_2}} = \sqrt[4]{\frac{n_2''}{n_2}} = \sqrt[6]{\frac{n_2'''}{n_2}} = \text{etc.}$$

$$t_3 = \sqrt[2]{\frac{n_3'}{n_3}} = \sqrt[4]{\frac{n''}{n_3}} = \text{etc.}$$

$$t_1 = \sqrt[2]{\frac{n_1'}{n_1}} = \sqrt[4]{\frac{n_1''}{n_1}} = \text{etc.}$$

where

$$t_2t_3 + t_3t_1 + t_1t_2 = 1 \qquad (k')$$

Now

$$h_1^2n_1 = h_2^2n_2 = h_3^2n_3 = \frac{h_1h_2h_3}{2}, \text{ a constant,}$$

where

$$h_1 = 1 + \tan\tfrac{1}{4}A \quad h_2 = 1 + \tan\tfrac{1}{4}B \quad h_3 = 1 + \tan\frac{C}{4}$$

and thus

$$h_1 = 2\sqrt{n_2n_3} \quad h_2 = 2\sqrt{n_3n_1} \quad h_3 = 2\sqrt{n_1n_2}.$$

Substituting in (k')

First triad, $1 = \sqrt[2]{\dfrac{n_2'n_3'}{n_2n_3}} + \sqrt[2]{\dfrac{n_3'n_1'}{n_3n_1}} + \sqrt[2]{\dfrac{n_1'n_2'}{n_1n_2}}$

$$\text{or} \quad \frac{\sqrt{n_2'n_3'}}{h_1} + \frac{\sqrt{n_3'n_1'}}{h_2} + \frac{\sqrt{n_1'n_2'}}{h_3} = \tfrac{1}{2}$$

Second triad, $1 = \sqrt[4]{\dfrac{n_2''n_3''}{n_2n_3}} + \sqrt[4]{\dfrac{n_3''n_1''}{n_3n_1}} + \sqrt[4]{\dfrac{n_1''n_2''}{n_1n_2}}$

$$\text{or} \quad \frac{\sqrt[4]{n_2''n_3''}}{\sqrt{h_1}} + \frac{\sqrt[4]{n_3''n_1''}}{\sqrt{h_2}} + \frac{\sqrt[4]{n_1''n_2''}}{\sqrt{h_3}} = \frac{1}{\sqrt{2}}$$

Third triad, $1 = \sqrt[6]{\dfrac{n_2'''n_3'''}{n_2n_3}} + \sqrt[6]{\dfrac{n_3'''n_1'''}{n_3n_1}} + \sqrt[6]{\dfrac{n_1'''n_2'''}{n_1n_2}}$

$$\text{or} \quad \frac{\sqrt[6]{n_2'''n_3'''}}{\sqrt[3]{h_1}} + \frac{\sqrt[6]{n_3'''n_1'''}}{\sqrt[3]{h_2}} + \frac{\sqrt[6]{n_1'''n_2'''}}{\sqrt[3]{h_3}} = \frac{1}{\sqrt[3]{2}}$$

Generally for m^{th} triad radii $\nu_1\ \nu_2\ \nu_3$.

$$\frac{\sqrt[2m]{\nu_2\nu_3}}{\sqrt[m]{h_1}} + \frac{\sqrt[2m]{\nu_3\nu_1}}{\sqrt[m]{h_2}} + \frac{\sqrt[2m]{\nu_1\nu_2}}{\sqrt[m]{h_3}} = \frac{1}{\sqrt[m]{2}}$$

[*Note.*—$h_1\ h_2\ h_3$ can readily be expressed severally in terms of a, b, c.

Thus $$h_1^2 = (1+\tan\tfrac{1}{4}\mathrm{A})^2 = \sec^2\tfrac{1}{4}\mathrm{A} + 2\tan\tfrac{1}{4}\mathrm{A}$$

$$= \frac{1+\sin\frac{1}{2}\mathrm{A}}{\cos^2\frac{1}{4}\mathrm{A}} = \frac{2(1+\sin\frac{1}{2}\mathrm{A})}{1+\cos\frac{1}{2}\mathrm{A}} = \frac{2(\sqrt{bc}+\sqrt{s_2s_3})}{\sqrt{bc}+\sqrt{ss_1}}.]$$

§ VI. *On a classification of the circles belonging to* ABC, *with notes of a few of their properties.*

1. – three leading circles, the circumscribed, medioscribed and inscribed.
2. – three exscribed circles.
3. – three interscribed circles.
4. – six superscribed circles.
5. – twelve insuperscribed circles.
6. – twelve introscribed circles.
7. – fifteen intimoscribed circles.
8. – four tercentroscribed circles.
9. – fifteen supermedianscribed circles.
10. – fifteen intermedianscribed circles.

(3.) *Interscribed Circles* (radii $n_1\ n_2 n_3$). Each touches two sides of ABC, and also touches the other interscribed circles.

$$h_1^2 n_1 = h_2^2 n_2 = h_3^2 n_3 = \tfrac{1}{2} h_1 h_2 h_3, \text{ a constant,} \qquad \text{i.}$$

where $$h_2^2 = \frac{2(\sqrt{ca}+\sqrt{s_3s_1})}{\sqrt{ca}+\sqrt{ss_2}} \qquad \text{(See V.)}$$

$$2\{\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3} \pm \sqrt{(n_1+n_2+n_3)}\}^2 = h_1h_2h_3 \qquad \text{ii.}$$

(4.) *Superscribed Circles.* These have for their diameters the lines joining the four centres O Q_1, Q_2 Q_3. Taking the inner three (radii $u_1\ u_2\ u_3$) we have

$$u_1u_2u_3 = 2r\mathrm{R}^2 = \frac{1}{2r}\left(\frac{abc}{a+b+c}\right)^2 \qquad \text{iii.}$$

$$v_1^2 + u_2^2 + u_3^2 = \frac{R}{s}(ar_1 + br_2 + cr_3) \qquad \text{iv.}$$

$$\frac{a}{u_1^2} + \frac{b}{u_2^2} + \frac{c}{u_3^2} = \frac{(a+b+c)^2}{abc} \qquad \text{v.}$$

$$\frac{u_2^2 u_3^2}{bc} + \frac{u_3^2 u_1^2}{ca} + \frac{u_1^2 u_2^2}{ab} = R^2 \qquad \text{vi.}$$

(5.) *Insuperscribed Circles.* These have for their diameters the lines joining the angular points of ABC with the four centres. Taking the two inner groups, diameters AO, BO, CO (radii v_1' v_2' v_3') and AQ_1 BQ_2 CQ_3 (radii V_1' V_2' V_3') we have

$$\frac{4v_1'v_2'v_3'}{r} = \frac{abc}{a+b+c} = \frac{u_1 u_2 u_3}{R} \qquad \text{vii.}$$

$$\left.\begin{aligned} 4(av_1'^2 + bv_2'^2 + cv_3'^2) &= abc \\ \text{or} \quad \frac{v_1'^2}{bc} + \frac{v_2'^2}{ca} + \frac{v_3}{ab} &= \tfrac{1}{4} \end{aligned}\right\} \qquad \text{viii.}$$

$$\frac{aV_1'^2 + bV_2'^2 + cV_3'^2}{R} = \frac{3abc - a^3 - b^3 - c^3}{8r} \qquad \text{ix.}$$

$$\frac{bc}{V_1'^2} + \frac{ca}{V_2'^2} + \frac{ab}{V_3'^2} = 4,$$

or

$$\frac{1}{aV_1'^2} + \frac{1}{bV_2'^2} + \frac{1}{cV_3'^2} = \frac{4}{abc} \qquad \text{x.}$$

(6.) *Introscribed Circles.* These twelve are respectively inscribed to the quadrilaterals which the circles of (5.) circumscribe. Taking the groups (radii v_1 v_2 v_3 and V_1 V_2 V_3) corresponding to those selected in the preceding paragraph, we have

$$\frac{v_1}{r - v_1} + \frac{v_2}{r - v_2} + \frac{v_3}{r - v_3} = \frac{s}{r} \qquad \text{xi.}$$

$$\frac{1}{V_1} + \frac{1}{V_2} + \frac{1}{V_3} = \frac{3}{s} + \frac{1}{r} \qquad \text{xii.}$$

(7.) *Intimoscribed Circles.* Five groups, each circle touching two sides of ABC and also touching either the inscribed circle or one of the exscribed circles or one of the interscribed circles. Each group is the first triad of an infinite series, all the circles of

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which may be termed "intimoscribed." See Sections I., II., III., and V. for their properties.

(8.) *Tercentroscribed Circles.* Each passes through three of the four centres O Q_1 Q_2 Q_3; and has radius $R' = 2R$; the triangle ABC being orthic to $Q_1Q_2Q_3 (= \Delta')$. Let a', b', c', be the sides of Δ'.

$$\Delta' = \frac{abc}{2r} = R's = \frac{2R}{r}\Delta \qquad \text{xiii.}$$

$$(a'^2bc + b'^2ca + c'^2ab)^{\frac{1}{2}} = \frac{abc}{r} = R'(a+b+c) \qquad \text{xiv.}$$

$$\frac{a'^2 + b'^2 + c'^2}{bc + ca + ab} = \frac{2R'}{r} \qquad \text{xv.}$$

$$a'^2 + b'^2 + c'^2 + 4(u_1^2 + u_2^2 + u_3^2) = 12R'^2 \qquad \text{xvi.}$$

(9, 10.) These two groups are the circles circumscribed about and inscribed in the 15 triangles formed by the medians (m_1 m_2 m_3). Let the in-radii of the smaller triangles round G be as marked; those of GBC, GCA, GAB be r_4 r_5 r_6 respectively; and those of AM_1B, AM_1C, BM_2C, BM_2A, CM_3A, CM_3B be r_4', r_4'', r_5', r_5'', r_6', r_6'' respectively—the circum-radius in each case being known by changing r to R.

Fig. 31.

$$\frac{1}{r_1'} + \frac{1}{r_2'} + \frac{1}{r_3'} = \frac{1}{r_1''} + \frac{1}{r_2''} + \frac{1}{r_3''} = \frac{3}{2}\left(\frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} - \frac{1}{r}\right)$$

$$= \frac{3}{r} + \frac{3(m_1 + m_2 + m_3)}{\Delta} \qquad \text{xvii.}$$

$$R_1'R_2'R_3' = R_1''R_2''R_3'' = \frac{R_4R_5R_6}{128}\left(\frac{abc}{R}\right)^2 = \frac{R}{54}\,m_1^2m_2^2m_3^2 \qquad \text{xviii.}$$

$$\frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} = \frac{3}{r} + \frac{2(m_1 + m_2 + m_3)}{\Delta} \qquad \text{xix.}$$

$$R_4R_5R_6 = 64\left(\frac{R}{3}\right)^3\left(\frac{m_1m_2m_3}{abc}\right)^2 \qquad \text{xx.}$$

$$\frac{1}{r_4'} + \frac{1}{r_5'} + \frac{1}{r_6'} = \frac{1}{r_4''} + \frac{1}{r_5''} + \frac{1}{r_6''}$$

$$= \frac{3}{r} + \frac{m_1 + m_2 + m_3}{\Delta} = \frac{1}{2}\left(\frac{3}{r} + \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6}\right) \qquad \text{xxi.}$$

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$$R_4'R_5'R_6' = R_4''R_5''R_6'' = \left(\frac{R}{2}\right)^2 \frac{m_1 m_2 m_3}{\Delta} \qquad \text{xxii.}$$

$$\frac{R_4'R_5'R_6'}{R_1'R_2'R_3'} = \frac{54\ R^5}{a^2b^2c^2} \qquad \text{xxiii.}$$

$\therefore$
$$\frac{R_4'R_4''}{R_4} \cdot \frac{R_5'R_5''}{R_5} \cdot \frac{R_6'R_6''}{R_6} = \left(\frac{3R}{4}\right)^3,$$

or
$$\frac{RR_4}{R_4'R_4''} \cdot \frac{RR_5}{R_5'R_5''} \cdot \frac{RR_6}{R_6'R_6''} = \frac{64}{27} \qquad \text{xxiv.}$$

§ VII. *On some algebraic results which symmetrically involve* a, b, c *and* l_1, l_2, l_3 (*the bisectors of* A, B, C *drawn to the opposite sides*).

(1),
$$\frac{l_1^2}{ss_1} = \frac{4bc}{(b+c)^2} \quad \therefore \quad \frac{l_1^2}{bc} = \frac{(b+c+a)(b+c-a)}{(b+c)^2} = 1 - \frac{a^2}{(b+c)^2}$$

Thus
$$1 = \frac{l_1^2}{bc} + \frac{a^2}{(b+c)^2} = \frac{l_2^2}{ca} + \frac{b^2}{(c+a)^2} = \frac{l_3^2}{ab} + \frac{c^2}{(a+b)^2}\ .^*$$

(2),
$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{b^2-c^2}{l_2^2 l_3^2} + \left(\frac{1}{c}+\frac{1}{a}\right)\frac{c^2-a^2}{l_3^2 l_1^2} + \left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2-b^2}{l_1^2 l_2^2} = 0.$$

(3),
$$\frac{\frac{1}{b^2}-\frac{1}{c^2}}{(s-b)(s-c)}\left(\frac{b+c}{l_2^2 l_3^2}\right) + \frac{\frac{1}{c^2}-\frac{1}{a^2}}{(s-c)(s-a)}\left(\frac{c+a}{l_3^2 l_1^2}\right) + \frac{\frac{1}{a^2}-\frac{1}{b^2}}{(s-a)(s-b)}\left(\frac{a+b}{l_1^2 l_2^2}\right) = 0.$$

If $2\sigma^2 = a^2 + b^2 + c^2$,

(4),
$$a^2(\sigma^2-a^2)^2 + b^2(\sigma^2-b^2)^2 + c^2(\sigma^2-c^2)^2 + 2(\sigma^2-a^2)(\sigma^2-b^2)(\sigma^2-c^2) = a^2b^2c^2, \text{ and}$$

---

* These values of l_2 and l_3 give a short *direct* proof that the Δ is isosceles when $l_2 = l_3$. Thus $ca\left(1-\frac{b^2}{(c+a)^2}\right) = ab\left(1-\frac{c^2}{(a+b)^2}\right)$

$\therefore$ $$c(a+b)^2(s-b) = b(c+a)^2(s-c)$$

$\therefore$ $$(b-c)(abc+bcs+a^2s) = 0$$

$\therefore$ $$b-c=0, \text{ or } b=c.$$

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(5), $$\frac{R}{4P_1P_2P_3}=\frac{1}{\sigma^2-a^2}+\frac{1}{\sigma^2-b^2}+\frac{1}{\sigma^2-c^2}-\frac{4\Delta^2}{(\sigma^2-a^2)(\sigma^2-b^2)(\sigma^2-c^2)}$$

and $$\left(\frac{\sigma^2}{a^2}-1\right)\left(\frac{\sigma^2}{b^2}-1\right)\left(\frac{\sigma^2}{c^2}-1\right)=\frac{P_u}{R}\cdot\frac{P_2}{R}\cdot\frac{P_3}{R},$$

$P_1P_2P_3$ being the perpendiculars from the circum-centre.

(6),

$$1=\frac{1}{\left(1-\frac{a}{b}\right)\left(1-\frac{a}{c}\right)}+\frac{1}{\left(1-\frac{b}{c}\right)\left(1-\frac{b}{a}\right)}+\frac{1}{\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right)}$$

$$=\frac{1}{\left(\frac{b}{a}-1\right)\left(\frac{c}{a}-1\right)}+\frac{1}{\left(\frac{c}{b}-1\right)\left(\frac{a}{b}-1\right)}+\frac{1}{\left(\frac{a}{c}-1\right)\left(\frac{b}{c}-1\right)}.$$

(7), $$(f+g+h)^{n+1}r^n=f^{n+1}r_1^{\ n}=g^{n+1}r_2^{\ n}+h^{n+1}r_3^{\ n}$$

if only $$\frac{f}{s_1}=\frac{g}{s_2}=\frac{h}{s_3}$$

(8), $$\frac{a+x}{a(a-b)(a-c)}+\frac{b+x}{b(b-c)(b-a)}+\frac{c+x}{c(c-a)(c-b)}=\frac{x}{abc},$$

x being a line of any length; when also

(9), $$\frac{(a-x)^2}{(a-b)(a-c)}+\frac{(b-x)^2}{(b-c)(b-a)}+\frac{(c-x)^2}{(c-a)(c-b)}=1.$$

(10), $$\frac{bc+ca+ab}{2(a+b+c)}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1+Rr\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

or $$(bc+ca+ab)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+4s.$$

(11), $$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3,\quad \text{when}$$

$$\frac{a+bm}{b+cn}=\frac{b+cm}{c+an}=\frac{c+am}{a+bn}.$$

(12), For any scalene Δ

$$\frac{\frac{1}{a}-\frac{1}{b}}{a^2+b^2}+\frac{\frac{1}{b}-\frac{1}{c}}{b^2+c^2}+\frac{\frac{1}{c}-\frac{1}{a}}{c^2+a^2}=0,$$

if $$a^3(b-c)+a^2(b^2-c^2)+a(b^3-c^3)+(b^4-c^4)=0.$$

(13), When s has a vanishing value, not otherwise,

$$\frac{a^2}{(b+2c)(c+2b)}+\frac{b^2}{(c+2a)(a+2c)}+\frac{c^2}{(a+2b)(b+2a)}=1.$$

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