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A MOORE STRONGLY RIGID SPACE

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ABSTRACT. It is proved that for every Hausdorff space $\mathbb{R}$ and for every Hausdorff (regular or Moore) space X, there exists a Hausdorff (regular or Moore, respectively) space S containing X as a closed subspace and having the following properties:

1a) Every continuous map of S into $\mathbb{R}$ is constant.

b) For every point x of S and every open neighbourhood U of x there exists an open neighbourhood V of x, $V \subseteq U$ such that every continuous map of V into $\mathbb{R}$ is constant.

2) Every continuous map f of S into S ($f \neq$ identity on S) is constant.

In addition it is proved that the Fomin extension of the Moore space S has these properties.

The first example of a strongly rigid space was given by J. de Groot [2]. In [4, Remark 3.5.4] V. Kannan and M. Rajagopalan posed the question whether every Hausdorff space can be embedded in a Hausdorff strongly rigid space. (A space S is called *strongly rigid* if every continuous map $f: S \longrightarrow S, f \neq$ identity on S, is constant).

We solve this problem by proving that for every Hausdorff space $\mathbb{R}$ and for every Hausdorff (or regular) space X there exists a Hausdorff (or regular) space S containing X as a closed subspace and having the following properties: 1) Every continuous map of S into $\mathbb{R}$ is constant. 2) For every point x of S and every open neighbourhood U of x there exists an open neighbourhood V of x, $V \subseteq U$, such that every continuous map of V into $\mathbb{R}$ is constant. (Spaces having these properties are called in [3] $\mathbb{R}$*-monolithic* and *locally* $\mathbb{R}$*-monolithic,* respectively and by their construction are connected and locally connected). 3) The space S is strongly rigid.

The method of construction of these spaces is basically the same as in [3] which needs an auxiliary space T having two points a, b such that $f(a) = f(b)$, for every continuous map f of T into $\mathbb{R}$. Thus, using in place of space T the Moore space constructed in [1, Lemma 2] it follows that for every Hausdorff space $\mathbb{R}$ and for every Moore space X, there exists a Moore space S containing X as a closed subspace and having properties (1), (2), (3). A direct consequence of this, is that the Fomin extension of the Moore space S has properties (1), (2), (3).

The terminology and the notation used here are the same as in [3], which is necessary background for the later results.

Let $\mathbb{R}$ be a Hausdorff space and $\aleph$ be a cardinal number such that $\aleph > \max\{\psi^+(\mathbb{R}), \aleph_1\}$. We construct the space $T_1(\mathbb{R})$, [3, Theorem 1] setting $|T_{4n+2}| =$

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$|T^m_{2q+1}| = \aleph^+$ and considering that for every point α^m_{4t} a basis of open neighbourhoods are the sets of the form $\{\alpha^m_{4t}\} \cup B$, where the set B contains all but $\aleph$ number of elements of the set $T^m_{4t-1} \cup T^m_{4t+1}$. We denote this space by $T(\aleph^+)$.

LEMMA 1. *The space $T(\aleph^+)$ has the following properties:*

(1) It is regular totally disconnected and for every continuous map f of $T(\aleph^+)$ into $\mathbb{R}$, $f(p^-) = f(p^+)$.

(2) If M is a subspace of $T(\aleph^+)$ containing the points p^-, p^+ and having cardinality $< \aleph^+$, then the points p^-, p^+ are separated by disjoint open-and-closed subsets in M.

PROOF. (1). That $T(\aleph^+)$ is regular totally disconnected is easily proved. The proof that, for every continuous map f of $T(\aleph^+)$ into $\mathbb{R}$, $f(p^-) = f(p^+)$ is similar to that of the corresponding property of $T_1(\mathbb{R})$ (in [2]). It should be noticed that the proof in [3] is based on the fact that $|T_{4n+2}| = |T^m_{2q+1}| = \max\{\psi^+(\mathbb{R}), \aleph_1\}$, for then both sets $A^m_{4t+1} = T^m_{4t+1} \backslash f^{-1}\big(f(\alpha^m_{4t})\big)$, $A^m_{4t-1} = T^m_{4t-1} \backslash f^{-1}\big(f(\alpha^m_{4t})\big)$ have cardinality $\leq \max\{\psi(\mathbb{R}), \aleph_0\}$. In our case here, the cardinality of both sets A^m_{4t+1}, A^m_{4t-1} is $\aleph$, that is, the map f is constant on a neighbourhood of the point α^m_{4t}.

(2) Let $M \subseteq T(\aleph^+)$, $|M| < \aleph^+$, $p^-, p^+ \in M$ and let $U(n, p^+)$ be an open neighbourhood of p^+ in $T(\aleph^+)$. Then

$$U(n, p^+) = \bigcup_{k > 4n+2} T_k \cup \bigcup_{k > 4n+2} T^m_k \cup \left\{\alpha^m_k : k > 4n+2,\ m = 1, 2, \ldots\right\} \cup \{p^+\}.$$

But then the points α^m_{4n} of $U(n, p^+)$ (if they belong to M) are isolated in M, because $|M| \leq \aleph$ and every open neighbourhood of α^m_{4n} consists of all but $\aleph$ number of elements of the set $T_{4n+1} \cup T_{4n-1}$. Hence $\overline{U(n, p^+) \cap M} = U(n, p^+) \cap M$ and therefore the points p^-, p^+ are separated by open-and-closed subsets in M.

We now apply Theorem 2, [3], setting $X = T(\aleph^+)$, $T_1(\mathbb{R}) = T(\aleph^+)$ and $a = p^+$ and we construct the space $I\big(T(\aleph^+)\big)$ which in the sequel will be denoted by $C(p^+, \aleph^+)$.

LEMMA 2. *The space $C(p^+, \aleph^+)$ has the following properties:*

(1) It is regular $\mathbb{R}$-monolithic and locally $\mathbb{R}$-monolithic only at the point p^+.

(2) The cardinality of every open set is $\aleph^+$.

(3) $\psi\big(C(p^+, \aleph^+)\big) = \aleph^+$.

(4) There is no non-trivial connected (hence $\mathbb{R}$-monolithic) subspace of $C(p^+, \aleph^+)$ containing the point p^+ and having cardinality $< \aleph^+$.

PROOF. (1) That it is regular $\mathbb{R}$-monolithic and locally $\mathbb{R}$-monolithic at the point p^+, is proved as in [3, Lemma 2 and Theorem 1]. Since the subspace $C(p^+, \aleph^+) \backslash \{p^+\}$ is totally disconnected [3, Theorem 2], it follows that $C(p^+, \aleph^+)$ is locally $\mathbb{R}$-monolithic only at the point p^+.

(2) and (3) are obvious by the construction of $C(p^+, \aleph^+)$ and by the fact that $|T(\aleph^+)| = \aleph^+$.

(4) Let M, $|M| < \aleph^+$ be a non-trivial connected subspace of $C(p^+, \aleph^+)$ containing the point p^+.

By Lemma 1, (2) and the definition of topology on $C(p^+, \aleph^+)$ [3,§4] it follows that for the set $O\big(U(n,p^+), H, G\big)$, (which is an open neighbourhood of p^+ in $C(p^+, \aleph^+)$), it holds that $\overline{O\big(U(n,p^+), H, G\big)} \cap M = O\big(U(n,p^+), H, G\big) \cap M$, which implies that M is not connected hence not $\mathbb{R}$-monolithic since every $\mathbb{R}$-monolithic is obviously connected.

THEOREM. *For every Hausdorff space $\mathbb{R}$ and for every Hausdorff (or regular) space X, there exists a Hausdorff (or regular, respectively) $\mathbb{R}$-monolithic, locally $\mathbb{R}$-monolithic, strongly rigid space S containing X as a closed subspace.*

PROOF. Let $\mathbb{R}$ be a Hausdorff space, X be a Hausdorff (or regular) space and I_0 an index set for which $|I_0| = |X|$. Let A_0 be a set of cardinal numbers such that

(a) $|A_0| = |X|$,

(b) For every $\aleph_{0i} \in A_0$, $i \in I_0$, $\aleph_{0i}^+ \notin A_0$,

(c) For every $i \in I_0$, $\aleph_{0i}^+ > \max\{\psi^+(X), \psi^+(\mathbb{R}), \aleph_1\}$.

We construct for every $\aleph_{0i} \in A_0$, $i \in I_0$, the spaces $T(\aleph_{0i}^+)$ and then the corresponding spaces $C(p_{0i}^+, \aleph_{0i}^+)$. We attach the spaces $\{C(p_{0i}^+, \aleph_{0i}^+)\}_{i \in I_0}$ to the space $X = X_0$ as follows:

First we set

$$C = C(p_{0i}^+, \aleph_{0i}^+) \setminus \{p_{0i}^-, p_{0i}^+\}.$$

Then we fix a point $x_i \in X_0$ and we consider the set

$$\Lambda_0(x_i) = \{x_i\} \times (X_0 \setminus \{x_i\}).$$

For every $\lambda = (x_i, x) \in \Lambda_0(x_i)$ we denote by C^λ the copy of C attached to the points x_i, x.

We set

$$C_0^\lambda(x_i) = \{x_i, x\} \cup C^\lambda, \quad \lambda = (x_i, x)$$

and

$$L_0(x_i) = \bigcup_{\lambda \in \Lambda_0(x_i)} C_0^\lambda(x_i).$$

We consider the set

$$X_1 = X_0 \cup \bigcup_{\substack{\lambda \in \Lambda_0(x_i) \\ x_i \in X_0}} C^\lambda$$

on which we define a topology in exactly the same manner as on the set $I^1(X, \Lambda_0)$ in [3].

The space X_{n+1}, $n = 1, 2, \ldots$, is constructed by induction: first we consider the space $S_n = X_n \setminus X_{n-1}$ and an index set I_n such that $|I_n| = |S_n|$. Then we consider a set A_n of cardinal numbers such that

(a) $|A_n| = |S_n|$,

(b) For every $\aleph_{ni} \in A_n$, $i \in I_n$, $\aleph_{ni}^+ \notin A_n$,

(c) For every $i \in I_n$, $\aleph_{ni}^+ > \psi^+(X_n)$.

We construct for every $\aleph_{ni} \in A_n$, $i \in I_n$ the spaces $T(\aleph_{ni}^+)$ and then the corresponding spaces $C(p_{ni}^+, \aleph_{ni}^+)$. We attach $\{C(p_{ni}^+, \aleph_{ni}^+)\}_{i \in I_n}$ to $S_n = X_n \setminus X_{n-1}$ and we construct the space

$$X_{n+1} = X_n \cup \bigcup_{\substack{\lambda \in \Lambda_n(x_i) \\ x_i \in S_n}} C_n^\lambda$$

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where

$$\Lambda_n(x_i) = \{x_i\} \times (S_n \setminus \{x_i\})$$

and

$$C_n = C(p_{ni}^+, \aleph_{ni}^+) \setminus \{p_{ni}^-, p_{ni}^+\}.$$

For every $\lambda = (x_i, x) \in \Lambda_n(x_i)$ we set

$$C_n^\lambda(x_i) = \{x_i, x\} \cup C_n^\lambda$$

and

$$L_n(x_i) = \bigcup_{\lambda \in \Lambda_n(x_i)} C_n^\lambda(x_i).$$

Thus to the fixed point x_i of S_n, $n = 1, 2, \ldots$, are attached $|S_n| = |\Lambda_n(x_i)|$ copies C_n^λ, $\lambda = (x_i, x)$ as x runs over the set S_n.

It should be observed that if $x_i, x_j \in S_n$ and $x_i \neq x_j$, then for the attached spaces $L_n(x_i)$ and $L_n(x_j)$, it holds that

$$L_n(x_i) \cap L_n(x_j) = S_n \text{ and } \aleph_{ni}^+ \neq \aleph_{nj}^+.$$

Also, by the definition of $C_n^\lambda(x_i)$ it follows that if $\lambda = (x_i, x)$, $\mu = (x_j, y)$, $x, y \in S_n$ then

$$\begin{aligned} C_n^\lambda(x_i) \cap C_n^\mu(x_j) &= \{x_i\}, \text{ if } x_i = x_j, \quad x \neq y, \\ C_n^\lambda(x_i) \cap C_n^\mu(x_j) &= \emptyset, \text{ if } x_i \neq x_j, \quad x \neq y, \\ C_n^\lambda(x_i) \cap C_n^\mu(x_j) &= \{x\}, \text{ if } x_i \neq x_j, \quad x = y. \end{aligned}$$

It should also be observed that since for every $n = 0, 1, 2, \ldots$ and $i \in I_n$, $\max\{\psi^+(\mathbb{R}), \aleph_1\} < \aleph_{ni}^+ < \aleph_{(n+1)i}^+$, it follows that $f(p^-) = f(p^+)$, for every continuous map f of $T(\aleph_{ni}^+)$ into $\mathbb{R}$. Hence for every $n = 0, 1, 2, \ldots, \aleph_{ni} \in A_n$ and $i \in I_n$, the corresponding spaces $C(p_{ni}^+, \aleph_{ni}^+)$ satisfy Lemma 2.

Also it is obvious that for every $n = 0, 1, 2, \ldots, \lambda \in \Lambda_n(x_i)$ and $\aleph_{ni} \in A_n$, the space $C_n^\lambda(x_i)$ is homeomorphic to the space $C(p_{ni}^+, \aleph_{ni}^+)$ and hence it also satisfies Lemma 2.

We consider the set $S = \bigcup_{n=0}^\infty X_n$ on which we define a topology in exactly the same manner as on the set $I(X)$ in [3].

That S is Hausdorff (or regular, if the initial space X_0 is regular) $\mathbb{R}$-monolithic, locally $\mathbb{R}$-monolithic containing X_0 as a closed subspace is proved as in [3, Lemma 2 and Theorem 1].

We prove that S is strongly rigid. Let f be a continuous map of S into S and let $s_i \in S$ such that $f(s_i) \neq s_i$. Let n, m be the minimal integers for which $s_i \in X_n$ and $f(s_i) \in X_m$. The space $C_n^\lambda(s_i)$ is an $\mathbb{R}$-monolithic subspace of X_{n+1} and has cardinality $\aleph_{ni}^+$. Hence the space $f\big(C_n^\lambda(s_i)\big)$ is $\mathbb{R}$-monolithic (because the continuous image of an $\mathbb{R}$-monolithic is obviously $\mathbb{R}$-monolithic) and has cardinality $\leq \aleph_{ni}^+$.

Suppose first $n < m$. There exists a natural number k such that $n+k = m$. Since by (c), (see the construction of the space X_{n+1}) the corresponding cardinals $\aleph_{ni}^+$, $i \in I_n$ satisfy the

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inequality $\aleph_{ni}^{+} > \psi^{+}(X_n)$, it follows that for the construction of X_{n+k}, $k = 2, 3, \ldots, m-n$, the corresponding cardinals $\aleph_{(n+k-1)i}^{+}$, $i \in I_{n+k-1}$, satisfy the inequalities

$$\aleph_{(n+k-1)i}^{+} > \psi^{+}(X_{n+k-1}) > \aleph_{ni}^{+} = \psi\left(C_n^{\lambda}(s_i)\right),$$

(the latter by Lemma 2, (3) and by the fact that $C_n^{\lambda}(s_i)$ is homeomorphic to $C(p_{ni}^{+}, \aleph_{ni}^{+})$. Hence, for every $i \in I_{n+k-1}$, every $C(p_{(n+k-1)i}^{+}, \aleph_{(n+k-1)i}^{+})$ which is attached to a point of $S_{n+k-1} = X_{n+k-1} \setminus X_{n+k-2}$ (in order to construct X_{n+k}) has cardinality $> \aleph_{ni}^{+}$ and none of them contains a non-trivial connected subspace having cardinality $\leq \aleph_{ni}^{+}$ (Lemma 2, (4)). Hence $f\left(C_n^{\lambda}(s_i)\right) = f(s_i)$ which implies that $f\left(L_n(s_i)\right) = f(s_i)$ and finally that $f(X_n) = f(s_i)$.

Now suppose $n \geq m$. By the construction of spaces $X_1, X_2, \ldots, X_m$, it follows that $f\left(C_n^{\lambda}(s_i)\right) \subseteq X_{n+1}$, because by (c) again, the connected subspaces of $S \setminus X_{n+1}$ have cardinality $> \aleph_{ni}^{+}$. Consider the space $T(\aleph_{ni}^{+})$ which was used for the construction of $C(p_{ni}^{+}, \aleph_{ni}^{+})$. Then for the points of $T(\aleph_{ni}^{+})$ having the form α_{4t}^{m}, $t = 0, 1, \ldots,\ m = 1, 2, \ldots$, it holds that the points $f(\alpha_{4t}^{m})$ belong to an $\mathbb{R}$-monolithic subspace $C_m^{\mu}(s_k)$ of X_{n+1} having cardinality $< \aleph_{ni}^{+}$ (because $\left|f\left(C_n^{\lambda}(s_i)\right)\right| \leq \aleph_{ni}^{+}$ and no $\mathbb{R}$-monolithic subspace of X_{n+1} has cardinality exactly $\aleph_{ni}^{+}$ besides $C_n^{\lambda}(s_i)$). Therefore, for the pseudocharacter of $f(\alpha_{4t}^{m})$ in $C_m^{\mu}(s_k)$ it holds that $\psi\left(C_m^{\mu}(s_k), f(\alpha_{4t}^{m})\right) < \aleph_{ni}^{+}$. But then, by the construction of $C(p_{ni}^{+}, \aleph_{ni}^{+})$ it follows that $f\left(C_n^{\lambda}(s_i)\right) = f(s_i)$, which implies that $f\left(L_n(s_i)\right) = f(s_i)$ and finally that $f(X_n) = f(s_i)$.

Thus in both cases $f(X_n) = f(s_i)$. Consequently, if s is an arbitrary point of $S \setminus X_n$ and k is the minimal integer for which $s \in X_k$ then by the above it follows that $f(X_k) = f(s)$ and since $X_n \subseteq X_k$ we have $f(s_i) = f(s)$ and therefore $f(S) = f(s_i)$, i.e., the space S is strongly rigid.

COROLLARY 1. *For every Hausdorff space $\mathbb{R}$ and for every Moore space X there exists an $\mathbb{R}$-monolithic, locally $\mathbb{R}$-monolithic, strongly rigid Moore space S containing X as a closed subspace.*

PROOF. In [1, Lemma 2] it is proved that for every Hausdorff space $\mathbb{R}$ (denoted there by Y) there exists a Moore space $T_1(\mathbb{R})$ (denoted by S) having two points $-\infty$, $+\infty$ such that $f(-\infty) = f(+\infty)$, for every continuous map f of $T_1(\mathbb{R})$ into $\mathbb{R}$. By its construction $T_1(\mathbb{R})$ is totally disconnected. Applying again Theorem 2 [3] (as we did before for the construction of $C(p^{+}, \aleph^{+})$) we construct the space $I\left(T_1(\mathbb{R})\right)$ setting in place of the space X in Theorem 2 [3], the above space $T_1(\mathbb{R})$ and in place of a the point $+\infty$. Denote $I\left(T_1(\mathbb{R})\right)$ by $C(+\infty, 2^{\aleph})$, where $\aleph$ is a cardinal number such that $|\mathbb{R}| < \aleph$ and $\aleph^{\aleph_0} = 2^{\aleph}$ (see [1]).

That $C(+\infty, 2^{\aleph})$ is Moore is proved as in [3, Theorem 3]. That it is $\mathbb{R}$-monolithic and locally $\mathbb{R}$-monolithic only at the point $+\infty$ is proved as property (1) in Lemma 2. That the cardinality of every open set is $2^{\aleph}$ (i.e., property (2) of Lemma 2) is implied by the construction of $C(+\infty, 2^{\aleph})$ and because $|T_1(\mathbb{R})| = 2^{\aleph}$. Property (4) is implied by the construction of space $C(+\infty, 2^{\aleph})$. For property (3), obviously $\psi\left(C(+\infty, 2^{\aleph})\right) = \aleph_0$ (because every Moore space is first countable).

We now follow the proof of the Theorem above making the appropriate modifications. That is, for the construction of the space X_1 we consider an index set I_0, $|I_0| = |X|$ and a set A_0 of cardinal numbers such that

(a) $|A_0| = |X|$,

(b) For every $\aleph_{0i} \in A_0$, $i \in I_0$, $\aleph_{01}^{\aleph_0} = 2^{\aleph_{0i}}$,

(c) For every $i \in I_0$, $\aleph_{0i} > \max\{|\mathbb{R}|, |X|\}$.

Thus the spaces to be attached to $X = X_0$ are $\{C(+\infty_{0i}, 2^{\aleph_{0i}})\}_{i \in I_0}$.

For the construction of space $X_{n+1}, n = 1, 2, \ldots$ we consider an index set I_n, $|I_n| = |S_n|$ and a set A_n of cardinal numbers such that

(a) $|A_n| = |S_n|$,

(b) For every $\aleph_{ni} \in A_n$, $i \in I_n$, $\aleph_{ni}^{\aleph_0} = 2^{\aleph_{ni}}$

(c) For every $i \in I_n$, $\aleph_{ni} > |X_n|$,

and thus the spaces to be attached to $S_n = X_n \setminus X_{n-1}$ are $\{C(+\infty_{ni}, 2^{\aleph_{ni}})\}_{i \in I_n}$.

The final space S is defined as in the Theorem above and the proof that it is Moore is again the same as in [3, Theorem 3]. The other properties of S are proved as in the Theorem.

COROLLARY 2. *If S is the Moore space constructed in Corollary 1, then the Fomin extension σS of S is $\mathbb{R}$-monolithic, locally $\mathbb{R}$-monolithic, strongly rigid.*

PROOF. That σS is $\mathbb{R}$-monolithic, locally $\mathbb{R}$-monolithic is obvious since S is dense in σS.

We prove that σS is strongly rigid. Let $f: \sigma S \longrightarrow \sigma S$ be continuous and $f \neq$ identity on S. Since S is first countable and the sequential closure of S in σS is S [5, Theorem 5.12], it follows that if $s \in S$ and $f(s) \in \sigma S \setminus S$, then $f(S) \subseteq \sigma S \setminus S$. Hence f is constant, because $\sigma S \setminus S$ is totally disconnected [5, Lemma 5.3(b)]. Therefore $f(S) \subseteq S$ and consequently f is constant on S and hence on σS.

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