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COMPLETE QUASI-UNIFORM SPACES

BY

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ABSTRACT. An example is given of a topological space that does not admit a strongly complete quasi-uniform structure.

The basic definitions relating to quasi-uniform spaces are given in Murdeshwar and Naimpally [2]. A quasi-uniform space $(X, \mathcal{U})$ is *complete* (*strongly complete*) if every $\mathcal{U}$-Cauchy filter has non-empty adherence (limit). An open cover $\mathcal{C}$ of a topological space is a *directed open cover* provided that if U and V are members of $\mathcal{C}$, then there is a $W \in \mathcal{C}$ such that $U \cup V \subset W$. A topological space (X, t) is *orthocompact* (*countably orthocompact, weakly orthocompact*) if for every (countable, directed) open cover $\mathcal{O}$, there exists an open refinement $\mathcal{R}$ such that $\mathcal{R}$ is a cover and for each $x \in X$, $\cap\{R \mid x \in R \in \mathcal{R}\} \in t$.

In answer to a long-standing open question, we construct a topological space E that does not admit a strongly complete quasi-uniform structure. The space does admit a $\mathcal{C}$-complete, hence a complete quasi-uniform structure. The space is weakly orthocompact but not countably orthocompact, and it is a dense subspace of a space which does admit a strongly complete quasi-uniform structure.

EXAMPLE. Let $Y = \prod_{i=1}^{\infty} X_i$ with the product topology t, where for each $i \in \mathbb{N}$, $X_i = \{0, 1\}$ with topology $t_i = \{\{0\}, X_i, \phi\}$. Let $E = \{x \in Y \mid \text{for some } i,\ x(i) \neq 1\}$ and let u denote the subspace topology. For each $i \in \mathbb{N}$, let x_i be defined by $x_i(i) = 0$ and $x_i(j) = 1$ whenever $j \neq i$; let y_i be defined by $y_i(j) = 1 - x_i(j)$; let $\bar{0}$ be defined by $\bar{0}(i) = 0$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, x_i belongs to a smallest open set $S_i = \{x \in E \mid x(i) = 0\}$, and $\bar{0}$ belongs to every open set of E.

PROPOSITION 1. *The space (E, u) is weakly orthocompact so that the fine transitive quasi-uniform structure is complete.*

Proof. Let $\mathcal{C}$ be a directed open cover of E. Since $\mathcal{S} = \{S_i \mid i \in \mathbb{N}\}$ is a refinement of every open cover, $\mathcal{S}^* = \{\bigcup_{i=1}^{n} S_i \mid n \in \mathbb{N}\}$ is also a refinement of $\mathcal{C}$. As $\mathcal{S}$ is well ordered by set inclusion, it is a Q-cover. Thus the fine transitive quasi-uniform structure on (E, u) is complete by the second corollary to Theorem 2.2 of [1].

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LEMMA. *For each quasi-uniformity $\mathcal{U}$ compatible with u and for each $V \in \mathcal{U}$, there exists $j \in \mathbb{N}$ such that $S_j \neq V(x_j)$.*

Proof. Let $V \in \mathcal{U}$ and choose $W \in \mathcal{U}$ such that $W^2 \subset V$. Suppose that for each $j \in \mathbb{N}$, $S_j = V(x_j)$. For each $j \in \mathbb{N}$, $\bar{0} \in W(x_j)$ so that $W(\bar{0}) \subset V(x_j)$ and $W(\bar{0}) \subset \bigcap_{i=1}^{\infty} S_i = \{\bar{0}\}$, a contradiction.

PROPOSITION 2. *The space E does not admit a strongly complete quasi-uniform structure.*

Proof. Let $\mathcal{F}$ denote the filter $\{F \mid \{y_i \mid i \in \mathbb{N}\} \subset F \subset E\}$. To show that $\mathcal{F}$ does not converge, let $x \in S_i$. Then $y_i \notin S_i$ so that $S_i \notin \mathcal{F}$. Now let $\mathcal{U}$ be any quasi-uniform structure compatible with u; we show that $\mathcal{F}$ is a $\mathcal{U}$-Cauchy filter. Let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ such that $V^2 \subset U$. By the preceding lemma, there exists $j \in \mathbb{N}$ such that $S_j \neq V(x_j)$. Let $x \in V(x_j) - S_j$. Then $x(j) = 1$ and $y_j \in V(x)$ so that $y_j \in U(x_j)$. For all $i \neq j$, $y_i \in S_j \subset U(x_j)$, thus $U(x_j) \in \mathcal{F}$.

COROLLARY. *The space E is not countably orthocompact.*

REMARK. The following questions remain open: Does there exist a (Hausdorff, regular) topological space that admits no (strongly) complete quasi-uniform structure?

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