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# REMARKS ON BOUNDED SOLUTIONS OF LINEAR SYSTEMS

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In the case of continuous time systems with bounded operators (coefficients) the following result, of Perron type is well known: "The linear differential system $\dot{x} = Ax + f(t)$ has, for every function $f$ continuous and bounded on $\mathbb{R}$, a unique bounded solution on $\mathbb{R}$, if and only if the spectrum of the operator $A$ has no points on the imaginary axis".

In this paper we give a discrete version of this result. The case of continuous time systems with $A$ an unbounded, infinitesimal generator of a $C_0$ group, is considered in the last section.

## 1. PRELIMINARIES

Let $X$ be a complex Banach space. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from $X$ into itself. If $A \in \mathcal{B}(X)$ and $I$ is the unity of $\mathcal{B}(X)$ then

$$\rho(A) = \{\lambda \in \mathbb{C} : \exists(\lambda I - A)^{-1} = R(\lambda, A) \in \mathcal{B}(X)\}$$

and

$$\sigma(A) = \{\lambda \in \mathbb{C} \ : \ \lambda \notin \rho(A)\}$$

will denote respectively, the resolvent set and the spectrum of the operator A.

Let now $A : X \to X$ a bounded linear operator. For every continuous function $f : \mathbb{R} \to X$ we consider the differential equation

$$(A, f) \qquad \dot{x} = Ax + f(t).$$

Connections between admissibility, asymptotical behaviours of solutions of the system $(A, f)$ and some spectral properties of the operator $A$, have been studied by many authors [**1**, **2**, **3**, **4**].

It is known (see for example, [**1**, **2**]) that $(A, f)$ has a unique bounded solution on $\mathbb{R}$, for every $f$ continuous and bounded on $\mathbb{R}$, if and only if

$$(1.1) \qquad i\mathbb{R} = \{\lambda \in \mathbb{C} : Re\lambda = 0\} \subset \rho(A).$$

In this paper, using a discrete Green's function (constructed in Section 2), a similar result is given for linear discrete time systems.

Section 3 is concerned with the case when $A$ is unbounded. In this case we do not know if (1.1) is a sufficient condition to assure the validity of the previous result.

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## 2. Bounded solutions of linear discrete time systems

Let us consider $A \in \mathcal{B}(X)$ and

$$f = (f_n)_{n \in Z} = (\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots)$$

a $\mathbb{Z}$-indexed sequence of elements from $X$.

If $0 \in \rho(A)$ then it is easy to see that the linear discrete time system

$$(A, f)_d \qquad x_{n+1} = Ax_n + f_n$$

has solution $(x_n)_{n \in Z}$ given by

$$x_n = \begin{cases} A^n x_0 + \sum_{j=1}^{n-1} A^{n-j-1} f_j, & \text{for } n \geqslant 1 \\ A^n x_0 + \sum_{j=n}^{-1} A^{n-j-1} f_j, & \text{for } n < -1 \end{cases}$$

uniquely determined by the value $x_0 \in X$. In particular, if $f_n = 0$ for every $n \in \mathbb{Z}$ then $(x_n = A^n x_0)_{n \in Z}$ is the unique solution for the discrete time system

$$(A)_d \qquad x_{n+1} = Ax_n, \qquad x_0 \in X.$$

Throughout this section $A$ will be a bounded linear operator with the property $0 \in \rho(A)$.

Also we denote

$$\ell^\infty(\mathbb{Z}, X) = \{x = (x_n)_{n \in Z} : \|x\| = \sup_n \|x_n\| < \infty\}$$

and

$$D_1 = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$
$$C_1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

**Proposition 2.1.** *If $\sigma(A) \cap C_1 = \emptyset$ and $(x_n = A^n x_0)_{n \in Z}$ is a solution of the system $(A)_d$ with $x_0 \neq 0$ then $(x_n)_{n \in Z} \notin \ell^\infty(\mathbb{Z}, X)$.*

Proof: Let us suppose that

$$x = (x_n)_{n \in Z} = (A^n x_0)_{n \in Z} \in \ell^\infty(\mathbb{Z}, X),$$

that is

$$\|x\|_\infty = \sup_{n \in Z} \|x_n\| < \infty.$$

(i) If $\sigma(A) \subset D_1$ then we find a positive number $a < 1$ such that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| < a < 1\}$$

and hence (see for example, [5]) we can find $N > 0$ such that

$$\|A^n\| < Na^n, \quad \text{for every} \quad n > 0,$$

from where, in particular, we obtain that:

$$\sup_{n>o} \|A^n x\| < \infty, \qquad \text{for every } x \in X.$$

Since for $m < 0$, $x_m = A^m x_0$ , we have

$$\|x_0\| = \left\|A^{-m} x_m\right\| \leqslant \left\|A^{-m}\right\| \cdot \|x_m\| \leqslant Na^{-m} \cdot \|x\|_\infty, \quad \forall\, m < 0$$

from where we obtain that $x_0 = 0$.

(ii) If

$$\sigma(A) \subset \operatorname{Ext}(D_1) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$$

then $\sigma\left(A^{-1}\right) \subset D_1$ and

$$\sup_{n\in Z} \|A^n x_0\| = \sup_{n\in Z} \left\|A^{-n} x_0\right\| = \sup_{n\in Z} \left\|\left(A^{-1}\right)^n x_0\right\|.$$

Hence, using case (i),

$$\left(A^n x_0\right)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$$

if and only if $x_0 = 0$. Moreover it follows that

$$\sup_{n>o} \left\|\left(A^{-1}\right)^n x\right\| = \sup_{n<o} \|A^n x\| < \infty, \qquad \text{for every } x \in X.$$

(iii) Finally, if

$$\sigma(A) = \sigma_1 \cup \sigma_2,\ \sigma_1 \neq \emptyset,\ \sigma_2 \neq \emptyset \text{ and } \sigma_1 \subset D_1,\ \sigma_2 \subset \operatorname{Ext}(D_1)$$

then, by the Dunford functional calculus, we can define the coresponding spectral projections

$$P_1 = \frac{1}{2\pi i} \int_{\Gamma_1} R(\lambda, A)\, d\lambda, \quad P_2 = \frac{1}{2\pi i} \int_{\Gamma_2} R(\lambda, A)\, d\lambda$$

where, for $j = 1, 2$, $\Gamma_j$ is the boundary of a Cauchy domain $D_j$ which contains $\sigma_j$ and $\overline{D}_1 \cap \overline{D}_2 = \emptyset$.

It is well known [**2, 6**] that $P_1 + P_2 = I$, $P_1P_2 = 0$ and if $X_j = P_jX$, $j = 1,2$ then both the subspaces $X_1$ and $X_2$ are invariant under the operator $A$. Moreover if $A_j$ is the restriction of $A$ to the subspace $X_j$ then $\sigma(A_j) = \sigma_j$, $j = 1,2$.

Let now $(x_n)_{n\in Z} = (A^n x_0)_{n\in Z}$ be arbitrary solution of the system $(A)_d$. Then for every $n \in \mathbb{Z}$ we can write:

$$x_n = A^n(P_1 + P_2)x_0 = A^nP_1x_0 + A^nP_2x_0 = A_1^nP_1x_0 + A_2^nP_2x_0.$$

If $P_1x_0 = 0$ (or $P_2x_0 = 0$) then $x_n = A_2^nP_2x_0$ (respectively $x_n = A_1^nP_1x_0$) and by (ii) ( respectively by (i)) the boundedness of $(x_n)_{n\in Z}$ implies $P_2x_0 = 0$ (respectively $P_1x_0 = 0$), that is, $x_0 = 0$.

If $P_1x_0 \neq 0$ and $P_2x_0 \neq 0$ then $\sup\limits_{n>0} \|A_1^nP_1x_0\| < \infty$ while $(A_2^nP_2x_0)_{n\geqslant 0}$ is unbounded and hence, $(x_n = A_1^nP_1x_0 + A_2^nP_2x_0)_{n\in Z}$ is unbounded. □

**Proposition 2.2.** *Let $A$ be a bounded linear operator from $X$ into itself, with the property $0 \notin \sigma(A)$. The linear discrete time system $(A,f)_d$ has for each*

$$f = (f_n)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$$

*a unique solution*

$$x = (x_n)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$$

*if and only if $\sigma(A) \cap C_1 = \emptyset$.*

Proof: Let us suppose that for every $f \in \ell^\infty(\mathbb{Z}, X)$ the system $(A,f)_d$ has a unique bounded solution $(x_n)_{n\in Z}$. In particular, for every fixed $y \in X$ and for each $a \in C_1$ the system

$$x_{n+1} = Ax_n - a^{n-1}y \tag{2.1}$$

has a unique bounded solution.

By the substitution $x_n = a^{n-1}z_n$ the system (2.1) becames

$$z_{n+1} = a^{-1}Az_n - a^{-1}y. \tag{2.2}$$

It follows from here that $(x_n)_{n\in Z}$ is a solution for (2.1) if and only if

$$(z_n)_{n\in Z} = \left(a^{1-n}x_n\right)_{n\in Z}$$

is a solution of (2.2); moreover $(x_n)_{n\in Z}$ is bounded if and only if $(z_n)_{n\in Z}$ is bounded. Since the system (2.1) has, for every $y \in X$ and $a \in C_1$, a unique bounded solution, it follows that the system (2.2) has a unique bounded solution too, for every $y \in X$ and $a \in C_1$.

For $a \in C_1$ let us denote by

$$z_y = (z_n)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$$

the unique bounded solution of the discrete time system (2.2).

Now we remark that

$$z_{n+k+1} = a^{-1}Az_{n+k} - a^{-1}y$$

and therefore, for every $k \in \mathbb{Z}$, $(z_{n+k})_{n\in Z}$ is also a bounded solution of (2.2). If we use the uniqueness property, it follows that $z_{n+k} = z_n$, for every $n \in \mathbb{Z}$, and $k \in \mathbb{Z}$; In particular we obtain $z_k = z_0$, for all $k \in \mathbb{Z}$. Hence we deduce that the solution $z_y = (z_n)_{n\in Z}$ is given by a constant sequence, that is $z_n = z_0$ for every $n \in \mathbb{Z}$. By virtue of (2.2) we can write

$$z_0 = a^{-1}Az_0 + a^{-1}y$$

or equivalently

$$(aI - A)z_0 = y$$

and hence, for every $y \in X$ we can find a unique element $z_0$ (which depend on $y$) such that $(aI - A)z_0 = y$. □

From the previous arguments that $a \in \rho(A)$ and taking into acount that $a \in C_1$ was arbitrary, we obtain $\sigma(A) \cap C_1 = \emptyset$.

Conversely, let us now suppose that $\sigma(A) \cap C_1 = \emptyset$. By virtue of Proposition 2.1 we know that the equation $(A)_d$ has no bounded solution on $\mathbb{Z}$, except the case $x_0 = 0$. If for a sequence $(f_n)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$, the equation

$$(A,f)_d \qquad x_{n+1} = Ax_n + f_n$$

has two bounded solutions

$$(x'_n)_{n\in Z} \text{ and } (x''_n)_{n\in Z}$$

then $(x'_n - x''_n)_{n\in Z}$ is a bounded solution of equation $(A)_d$ and hence $x'_n = x''_n$ for every $n \in \mathbb{Z}$.

It remain to show that for each bounded sequence $f = (f_n)_{n\in Z}$, the equation $(A,f)_d$ has a bounded solution.

Let us take $f = (f_n)_{n\in Z} \in \ell^\infty(\mathbb{Z}, X)$. By writing

$$\sigma_1 = \sigma(A) \cap D_1, \quad \sigma_2 = \sigma(A) \cap \operatorname{Ext}(D_1),$$

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we have that $\sigma_1 \cup \sigma_2 = \sigma(A)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Therefore we can associate the spectral projections $P_1, P_2$ and the linear operators $A_1, A_2$ (as in Proposition 2.1); it is possible that $\sigma_1 = \emptyset$ (or $\sigma_2 = \emptyset$) and in this case $P_1 = 0$ (or $P_2 = 0$).

The function $G : \mathbb{Z} \to \mathcal{B}(X)$ defined by

$$G(n) = \begin{cases} A_1^n P_1 & \text{if} \quad n \geqslant 0 \\ -A_2^n P_2 & \text{if} \quad n < 0 \end{cases}$$

will be called the discrete Greens function associated with the operator $A$.

Now for each

$$f = (f_n)_{n \in Z} \in \ell^\infty(\mathbb{Z}),$$

we define:

$$x_n = \sum_{j=-\infty}^{\infty} G(n-j-1) f_j = \sum_{j=-\infty}^{n-1} G(n-j-1) f_j$$
$$- \sum_{j=n}^{\infty} G(n-j-1) f_j = \sum_{j=-\infty}^{n-1} A_1^{n-j-1} P_1 f_j - \sum_{j=n}^{\infty} A_2^{n-j-1} P_2 f_j$$

and so we obtain a sequence $(x_n)_{n \in Z}$.

Since

$$\begin{aligned} Ax_n + f_n &= A(P_1 + P_2) x_n + (P_1 + P_2) f_n \\ &= (A_1 P_1 + A_2 P_2) \left( \sum_{j=-\infty}^{n-1} A_1^{n-j-1} P_1 f_j - \sum_{j=-\infty}^{n-1} A_2^{n-j-1} P_2 f_j \right) \\ &+ (P_1 + P_2) f_n = P_1 f_n + \sum_{j=n}^{\infty} A_1^{n-j} P_1 f_j - \sum_{j=n}^{\infty} A_2^{n-j} P_2 f_j + P_2 f_n \\ &= \sum_{j=-\infty}^{n} A_1^{n-j} P_1 f_j - \sum_{j=n+1}^{\infty} A_2^{n-j} P_2 f_j = x_{n+1} \end{aligned}$$

we conclude that $(x_n)_{n \in Z}$ is a solution of the equation $(A, f)_d$.

On the other hand

$$\begin{aligned} \|x_0\| &\leqslant \left\| \sum_{j=-\infty}^{n-1} A_1^{n-j-1} P_1 f_j \right\| + \left\| \sum_{j=n}^{\infty} A_2^{n-j-1} P_2 f_j \right\| \\ &\leqslant \sum_{j=0}^{\infty} \left\| A_1^j P_1 f_{n-j-1} \right\| + \sum_{j=1}^{\infty} \left\| (A_2^{-1})^j P_2 f_{n-j-1} \right\| \\ &\leqslant \left( \left\| A_1^j \right\| \cdot \|P_1\| + \sum_{j=1}^{\infty} \left\| (A_2^{-1})^j \right\| \cdot \|P_2\| \right) \cdot \|f\|_\infty . \end{aligned}$$

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Since

$$\sigma(A_1) \subset \text{ int } D_1$$

and

$$\sigma\left(A_2^{-1}\right) \subset \left\{\frac{1}{\lambda} : |\lambda| > 1\right\} \subset \text{ int } D_1$$

we can find two numbers $N_1, N_2 > 0$ and $a \in (0,1)$ such that

$$\left\|A_1^j\right\| \leqslant N_1 \cdot a^j, \qquad \left\|\left(A_2^{-1}\right)^j\right\| \leqslant N_2 \cdot a^j.$$

Hence

$$\|x_n\| \leqslant max(N_1, N_2) \cdot (1+a) \cdot (1-a)^{-1} \cdot \|f\|_\infty$$

for every $n \in \mathbb{Z}$. Hence $(x_n)_{n\in\mathbb{Z}}$ is a bounded solution of the equation $(A, f)_d$.

## 3. Bounded solutions of linear systems with unbounded operators

Throughout this section, $A$ will be a closed linear operator which generates a $C_0$-group $\{T(t)\}_{t\in\mathbb{R}}$ of bounded linear operators on a Banach space $X$.

**Propisition 3.1.** *If for every function $f$, continuous and bounded on $\mathbb{R}$, the system (1.1) has a unique mild solution*

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$$

*bounded on $\mathbb{R}$, then*

$$\sigma(A) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) = 0\} = \sigma(A) \cap i\mathbb{R} = \emptyset.$$

Proof: For every $\beta \in \mathbb{R}$ and $y \in X$ we can consider the equations:

$$\begin{aligned} &\left(A; -e^{i\beta t}y\right) && \dot{x} = Ax - e^{i\beta t}y \\ &(A - i\beta I; -y) && \dot{z} = (A - i\beta I)z - y. \end{aligned}$$

It is well known that $A - i\beta I$ generates the following $C_0$-group of bounded linear operators

$$\{S(t)\}_{t\in\mathbb{R}} = \{e^{-i\beta t} \cdot T(t)\}_{t\in\mathbb{R}}$$

and therefore, every solution $x(\cdot)$ of equation $\left(A; -e^{i\beta t}y\right)$ is defined on $\mathbb{R}$ by

$$x(t) = T(t)x(0) - \int_0^t T(t-s)e^{i\beta s}y\, ds \tag{3.1}$$

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Also, every solution of equation $(A - i\beta I, -y)$ is defined on $\mathbb{R}$ by

$$z(t) = S(t)z(0) - \int_0^t S(t-s)y\,ds$$

or

$$z(t) = T(t)e^{-i\beta t}z(0) - \int_0^t T(t-s)e^{-i\beta(t-s)}y\,ds. \tag{3.2}$$

In particular, if $x(0) = z(0) = x^0 \in X$, then

$$e^{i\beta t}z(t) = T(t)x(0) - \int_0^t T(t-s)e^{i\beta s}y\,ds$$

and therefore we have a one to one correspondence between the solutions of equations $\left(A; -e^{i\beta t}y\right)$ and $(A - i\beta I : -y)$, given by the equality:

$$x(t) = e^{i\beta t}z(t),$$

for every $t \in \mathbb{R}$.

Moreover, from the previous equality it follows that this correspondence preserves the bounded solutions and therefore, by virtue of the hypothesis, for every $\beta \in \mathbb{R}$ and $y \in X$, the equation $(A - i\beta I; -y)$ has a unique solution on $\mathbb{R}$.

Now, we fix the number $\beta \in \mathbb{R}$ and denote by $z_y(\cdot)$ the unique bounded solution on $\mathbb{R}$ of equation $(A - i\beta I; -y)$. If $z_y(0) = x^0 \in X$, then for every $\tau \in \mathbb{R}$, we have

$$\begin{aligned} z_y(t+\tau) &= S(t+\tau)x^0 - \int_0^{t+\tau} S(t+\tau-s)y\,ds \\ &= S(t)S(\tau)x^0 - \int_{-\tau}^t S(t-u)y\,du \\ &= S(t)\left[S(\tau)x^0 - \int_{-\tau}^0 S(-u)y\,du\right] - \int_0^t S(t-u)y\,du. \end{aligned}$$

It follows from here that the function

$$t \to z_y(t+\tau)$$

is a bounded solution on $\mathbb{R}$ for the equation $(A - i\beta I; -y)$ and thus, by hypothesis,

$$z_y(t+\tau) = z_y(t), \qquad \forall t, \tau \in \mathbb{R}$$

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and hence

$$z_y(t) = z_y(0) = x^0, \qquad \forall t \in \mathbb{R}.$$

Now we obtain that

$$x^0 = S(t)x^0 - \int_0^t S(t-s)y\,ds, \qquad \forall t \in \mathbb{R}$$

or equivalently

$$\frac{1}{t}\left[S(t)x^0 - x^0\right] + \frac{1}{t}\int_0^t S(t-s)y\,ds \quad \forall\, t \neq 0$$

Hence

$$\lim_{t\to 0} \frac{1}{t} S(t)x^0 - x^0 = y$$

so that

$$x^0 \in \mathcal{D}(A - i\beta I) = \mathcal{D}(A)$$

and

$$(A - i\beta I)x^0 = y.$$

From the above arguments it follows that for every $y \in X$, we can find a unique element $x^0 \in \mathcal{D}(A)$ with the property $(A - i\beta I)x^0 = y$ and hence $A - i\beta I : \mathcal{D}(A) \to X$ is a bijective map. This $(A - i\beta I)^{-1}$, which is closed and defined on the whole of $X$, belongs to the space $\mathcal{B}(X)$, that is, $i\beta \notin \sigma(A)$. ∎

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