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THE STRONG RADICAL AND THE LEFT REGULAR REPRESENTATION

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Abstract

Let A be a semisimple modular annihilator Banach algebra and let L_A be the left regular representation of A. We show how the strong radical of A is related to the strong radical of L_A.

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1. Introduction

Let A be a semisimple Banach algebra and let $B(A)$ be the Banach algebra of all bounded linear operators on A. For each $a \in A$, let L_a be the linear map on A given by $L_a(x) = ax$, $x \in A$. Then the mapping $a \to L_a$ is a norm-decreasing algebra isomorphism of A into $B(A)$. Let L_A be the closure of $\{L_a: a \in A\}$ in $B(A)$. We call L_A the *left angular representation of* A. By the strong radical $\mathfrak{S}_A$ of A we mean the intersection of all maximal modular ideals of A (if there are no such ideals we set $\mathfrak{S}_A = A$). The strong radical of modular annihilator algebras was studied by Yood in [11]. Our main concern in this paper is to show how $\mathfrak{S}_A$ and $\mathfrak{S}_{L_A}$ are related for these algebras, and in particular for semisimple right complemented Banach algebras.

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2. Preliminaries

Let A be a Banach algebra. For any subset S of A, $l_A(S)$ and $r_A(S)$ will denote, respectively, the left and right annihilators of S in A, and $\mathrm{cl}_A(S)$ will denote the closure of S in A. The socle of A will be denoted by S_A. By an ideal we will always mean a two-sided ideal unless otherwise specified. We call A modular annihilator if every maximal modular left (right) ideal of A has a nonzero right (left) annihilator. A semisimple Banach algebra with dense socle is modular annihilator [9, Lemma 3.11, page 41]. We call A an annihilator algebra if every proper closed left (right) ideal of A has a nonzero right (left) annihilator.

All Banach algebras considered in this paper are over the complex field.

A minimal idempotent e in a Banach algebra A is called *finite-dimensional* if eA is finite-dimensional. If A is semisimple then this is equivalent to Ae being finite-dimensional [11, Proposition 2.2, page 82]. An idempotent e in A is called *simple* if eAe is a simple algebra and e is called *central* if $ex = xe$ for all $x \in A$ (see [10, pages 320–322]).

Let A be a semisimple Banach algebra. If M is an ideal of A, then $l_A(M) = r_A(M)$ [9, page 37] and we denote the common value by M^a. (We let $M^{aa} = (M^a)^a$). If $S_A^a = (0)$ then every non-zero left (right) ideal of A contains a minimal idempotent [9, page 37].

We will also be interested in the right multiplication operators R_a, where, for each $a \in A$, $R_a(x) = xa$ for all $x \in A$.

Let A be a semisimple Banach algebra. Then L_A is semisimple and the mapping $a \to L_a$ embeds A as a dense left ideal of L_A. (See [7] or [8].) In the rest of the paper we will identify A as a dense left ideal of L_A. (For a more complete treatment of L_A see [8].)

3. Right complemented Banach algebras

Let A be a Banach algebra and L_r be the set of all closed right ideals in A. We say that A is *right complemented* (r.c.) if there exists a mapping p: $R \to R^p$ of L_r into itself (called a right complementor) having the following properties:

(C1) $$R \cap R^p = (0) \qquad (R \in L_r);$$

(C2) $$R + R^p = A \qquad (R \in L_r);$$

(C3) $$(R^p)^p = R \qquad (R \in L_r);$$

(Cr) $$\text{if } R_1 \subseteq R_2 \quad \text{then } R_2^p \subseteq R_1^p \ (R_1, R_2 \in L_r).$$

If A is a semisimple r.c. Banach algebra then A has dense socle [6, Lemma 5, page 655] and therefore A is modular annihilator.

In the rest of this section, let A be a semisimple r.c. Banach algebra with a right complementor p.

LEMMA 3.1. *Let I and J be closed ideals in A such that $I \cap J = (0)$. Then $J \subset I^p$ and $I \subset J^p$.*

PROOF. Since $IJ \subset I \cap J = (0)$, $J \subset r_A(I)$ and $I \subset l_A(J)$. But, by [6, Lemma 1, page 652], $I^p = r_A(I)$ and $J^p = l_A(J)$. Hence $J \subset I^p$ and $I \subset J^p$.

Let $\{I_\lambda: \lambda \in \Lambda\}$ be the family of all distinct minimal closed ideals of A. Since A is the direct topological sum of the I_λ and since for every closed ideal I of A, $I \oplus r_A(I) = A$, it follows from [2, Theorem 3.5, page 232] that $\{I_\lambda: \lambda \in \Lambda\}$ is an unconditional decomposition for A. For each $a \in A$ and $\lambda \in \Lambda$, write $a = a_\lambda + b_\lambda$ with $a_\lambda \in I_\lambda$ and $b_\lambda \in I_\lambda^p$.

THEOREM 3.2. (1) *For each $a \in A$, $a = \Sigma_\lambda a_\lambda$, where convergence is with respect to the net of finite partial sums.*

(2) *There exists a constant $K > 0$ such that, if $\lambda_1, \ldots, \lambda_n$ are distinct elements of Λ and $c_{\lambda_i} \in I_{\lambda_i}$, then*

$$\|c_{\lambda_1} + \cdots + c_{\lambda_m}\| \leqslant K\|c_{\lambda_1} + \cdots + c_{\lambda_n}\|, \qquad 1 \leqslant m \leqslant n.$$

In particular, for each $a \in A$,

$$\|a_{\lambda_1} + \cdots + a_{\lambda_m}\| \leqslant K\|a_{\lambda_1} + \cdots + a_{\lambda_n}\|, \qquad 1 \leqslant m \leqslant n.$$

PROOF. (1). Since $\{I_\lambda: \lambda \in \Lambda\}$ is an unconditional decomposition for A, we have $a = \Sigma_\lambda c_\lambda$, where $c_\lambda \in I_\lambda$ and convergence is with respect to the net of finite partial sums. We show that $c_\lambda = a_\lambda$ for all $\lambda \in \Lambda$. Let $\lambda_0 \in \Lambda$. Then $a - c_{\lambda_0} = \Sigma_{\lambda \neq \lambda_0} c_\lambda$. By Lemma 3.1, $I_\lambda \subset I_{\lambda_0}^p$ for $\lambda \neq \lambda_0$ and so

$$d_{\lambda_0} = a - c_{\lambda_0} \in cl_A\left(\sum_{\lambda \neq \lambda_0} I_\lambda\right) \subset I_{\lambda_0}^p.$$

Thus $a = c_{\lambda_0} + d_{\lambda_0}$ with $c_{\lambda_0} \in I_{\lambda_0}$ and $d_{\lambda_0} \in I_{\lambda_0}^p$. But $a = a_{\lambda_0} + b_{\lambda_0}$ with $a_{\lambda_0} \in I_{\lambda_0}$ and $b_{\lambda_0} \in I_{\lambda_0}^p$. Therefore, by the uniqueness of decomposition we must have $c_{\lambda_0} = a_{\lambda_0}$ and $d_{\lambda_0} = b_{\lambda_0}$. Hence $a = \Sigma_\lambda a_\lambda$.

(2). This follows from [2, Theorem 3.4, page 231] and (1).

COROLLARY 3.3. *Let P_λ be the projection on A with range I_λ and nullspace I_λ^p. Then the family $\{P_\lambda: \lambda \in \Lambda\}$ is bounded.*

Every minimal idempotent of A is also a minimal idempotent of L_A. Since every minimal closed ideal I of A is of the form $I = \mathrm{cl}_A(AeA)$, where e is a minimal idempotent in A, it follows that, for each $\lambda \in \Lambda$, $\mathscr{I}_\lambda = \mathrm{cl}_{L_A}(I_\lambda)$ is a minimal closed ideal of L_A. Moreover $A = \mathrm{cl}_A(\Sigma_\lambda I_\lambda)$ implies that $L_A = \mathrm{cl}_{L_A}(\Sigma_\lambda \mathscr{I}_\lambda)$. Thus $\{\mathscr{I}_\lambda \colon \lambda \in \Lambda\}$ is the family of all distinct minimal closed ideals of L_A. L_A is an annihilator algebra [8].

THEOREM 3.4. *The family* $\{\mathscr{I}_\lambda \colon \lambda \in \Lambda\}$ *is an unconditional decomposition for* L_A.

PROOF. Let $\lambda_1, \dots, \lambda_n$ be distinct elements of Λ and let T_{λ_i} be any element of $\mathscr{I}_{\lambda_i}$ $(i = 1, \dots, n)$. Let $a \in A$. Then

$$(T_{\lambda_1} + \cdots + T_{\lambda_m})(a) = T_{\lambda_1}(a_{\lambda_1}) + \cdots + T_{\lambda_m}(a_{\lambda_m}), \qquad 1 \leqslant m \leqslant n,$$

and $T_{\lambda_i}(a_{\lambda_i}) \in I_{\lambda_i}$ $(i = 1, \dots, n)$. By Theorem 3.2,

$$\|T_{\lambda_1}(a_{\lambda_1}) + \cdots + T_{\lambda_m}(a_{\lambda_m})\| \leqslant K\|T_{\lambda_1}(a_{\lambda_1}) + \cdots + T_{\lambda_n}(a_{\lambda_n})\|, \qquad 1 \leqslant m \leqslant n.$$

Hence

$$\|T_{\lambda_1} + \cdots + T_{\lambda_m}\| \leqslant K\|T_{\lambda_1} + \cdots + T_{\lambda_n}\|, \qquad 1 \leqslant m \leqslant n.$$

Therefore, by [2, Theorem 3.4, page 231], $\{\mathscr{I}_\lambda \colon \lambda \in \Lambda\}$ is an unconditional decomposition for L_A.

PROPOSITION 3.5. *Let B be a semisimple Banach algebra with dense socle. Then the following statements are equivalent*:

(1) *The minimal closed ideals of B form an unconditional decomposition for B.*

(2) $I \oplus l_B(I) = B$ *for all closed ideals I of B.*

PROOF. Let I be a closed ideal in B, and let e be a minimal idempotent in B. Then either $e \in I$ or $e \in l_B(I) = r_B(I)$ [10, page 320]. Therefore, if M is a minimal closed ideal in B then either $M \subset I$ or $M \subset l_B(I)$. We can apply the argument in the proof of [2, Theorem 3.5, page 232] to show that the statements (1) and (2) are equivalent.

COROLLARY 3.6. *For every closed ideal I of L_A,* $I \oplus l_{L_A}(I) = L_A$.

PROOF. This follows from Theorem 3.4 and Proposition 3.5.

4. Maximal modular ideals

Throughout this section let A and B be semisimple Banach algebras such that A is a dense left ideal in B. Then A is an abstract Segal algebra in B [4]. If A is modular annihilator then so is B. In fact, let $\mathscr{M}$ be a maximal modular left ideal in B. By [4, Lemma 1.3, page 298], $\mathscr{M} \cap A$ is a maximal modular left ideal of A and, by [4, Lemma 3.7, page 305], $\mathscr{M} = \mathrm{cl}_B(\mathscr{M} \cap A)$. Since $r_A(\mathscr{M} \cap A) \neq (0)$, it follows that $r_B(\mathscr{M}) \neq (0)$. Thus B is a right modular annihilator algebra and therefore, by [9, Theorem 3.4, page 38], is a modular annihilator algebra. The converse is also true (see [8]).

NOTATION. We recall that if $M\,(\mathscr{M})$ is an ideal of $A\,(B)$ then $M^a\,(\mathscr{M}^a)$ is the common value $l_A(M) = r_A(M)\,(l_B(\mathscr{M}) = r_B(\mathscr{M}))$.

LEMMA 4.1. *If A is modular annihilator, then A and B have the same finite-dimensional minimal idempotents.*

PROOF. Let $E_A(E_B)$ be the set of all finite-dimensional minimal idempotents in $A(B)$. If $e \in E_A$, then $eA = eB$ so that $e \in E_B$. Conversely, suppose $e \in E_B$. Let $K = \mathrm{cl}_B(BeB)$. Then K is a finite-dimensional minimal closed ideal of B. Also $K \cap A \neq (0)$, for otherwise $KB = (0)$ which is impossible because B is semisimple. Therefore $K \cap A$ contains a minimal idempotent, say f in A [9, page 37]. Since K is finite-dimensional and $\mathrm{cl}_A(AfA)$ is dense in K, we obtain $K = \mathrm{cl}_A(AfA)$. Hence $K \subset A$ and $e \in E_A$.

THEOREM 4.2. *If A is modular annihilator then $M \to \mathrm{cl}_B(M) = \mathscr{M}$ is a one-to-one correspondence between the maximal modular ideals M of A and the maximal modular ideals $\mathscr{M}$ of B and $M = \mathscr{M} \cap A$.*

PROOF. Let M be a maximal modular ideal of A. Then $M^a \neq (0)$ so that $M \oplus M^a = A$. By [3, Theorem 6.4, page 574], A/M is a finite-dimensional algebra with identity. Therefore $M^a = uA = Au$, for some idempotent u in A. Since $M^{aa} = (1-u)A = A(1-u)$ and $M \subset M^{aa}$, by the maximality of M, we get $M = (1-u)A = A(1-u)$. Let $\mathscr{M} = (1-u)B = B(1-u)$. Then $\mathscr{M}^a = uB = Bu$ and M^a is dense in $\mathscr{M}^a$. Since M^a is finite-dimensional, $\mathscr{M}^a = M^a$. Therefore $\mathscr{M}^a$ is a simple algebra and the equality $\mathscr{M} \oplus \mathscr{M}^a = B$ implies that $\mathscr{M}$ is a maximal modular ideal of B. Clearly $\mathscr{M} = \mathrm{cl}_B(M)$.

Conversely, let $\mathscr{M}$ be a maximal modular ideal of B. Since B is modular annihilator, by the argument above, there exists an idempotent v in B such that $\mathscr{M} = (1-v)B = B(1-v)$ and $\mathscr{M}^a = vB = Bv$. As $\mathscr{M}^a$ is a finite-dimensional modular annihilator algebra, $\mathscr{M}^a$ is a finite sum of minimal left ideals. Therefore, by Lemma 4.1, $\mathscr{M}^a \subset A$ and so $v \in A$. Let $M = (1-v)A = A(1-v)$. Since $M^a = vA = Av$ is dense in $\mathscr{M}^a$ and $\mathscr{M}^a$ is finite-dimensional, we obtain $M^a = \mathscr{M}^a$. Therefore M^a is a simple algebra and the equality $M^a \oplus M = A$ implies that M is a maximal modular ideal in A. We have $M = \mathscr{M} \cap A$.

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From the proof of Theorem 4.2 we see that if M is a maximal modular ideal of A then $M = (1 - e)A = A(1 - e)$, for some central simple idempotent e in A. Conversely, if e is a central simple idempotent in A, then $M = (1 - e)A = A(1 - e)$ is a maximal modular ideal of A since $I = Ae = eA$ is a simple algebra and $I \oplus M = A$. Hence the following results.

COROLLARY 4.3. *If A is modular annihilator, then A and B have the same central simple idempotents.*

NOTATION. Let $\mathfrak{M}_A$ ($\mathfrak{M}_B$) be the set of all maximal modular ideals in A (B).

COROLLARY 4.4. *If A is modular annihilator then the mapping $(1 - e)A \to (1 - e)B$, as e runs over the central simple idempotents of A (or equivalently of B), is a one-to-one map of $\mathfrak{M}_A$ onto $\mathfrak{M}_B$. Moreover, $eA = eB$ and is finite-dimensional for every central simple idempotent e of A.*

From Theorem 4.2 it follows that if A is modular annihilator then $\mathfrak{S}_A = \mathfrak{S}_B \cap A$. In the next section we will see that we also have $\mathfrak{S}_B = \mathrm{cl}_B(\mathfrak{S}_A)$ for certain modular annihilator Banach algebras A and B.

5. The strong radicals of A and L_A

Let A be a semisimple Banach algebra. In this section we will see how the strong radical $\mathfrak{S}_A$ of A is related to the strong radical $\mathfrak{S}_{L_A}$ of L_A for certain A.

PROPOSITION 5.1. *Let A and B be semisimple Banach algebras such that A is a dense left ideal in B. Assume that A is modular annihilator. If B has the property that $\mathrm{cl}_B(I \cap A) = I$ for every closed ideal I of B, then $\mathfrak{S}_B = \mathrm{cl}_B(\mathfrak{S}_A)$.*

PROOF. Let $Q = \mathrm{cl}_A(\Sigma\{M^a\colon M \in \mathfrak{M}_A\})$ and $\mathcal{Q} = \mathrm{cl}_B(\Sigma\mathcal{M}^a\colon \mathcal{M} \in \mathfrak{M}_B\})$. Then, by Corollary 4.4, $Q^a = \mathfrak{S}_A$ and $\mathcal{Q}^a = \mathfrak{S}_B$; moreover, $\mathcal{Q} = \mathrm{cl}_B(Q)$ so that $\mathcal{Q}^a = l_B(Q) = r_B(Q)$. Now $r_B(Q) \cap A = r_A(Q)$ and, by the condition in the theorem, $\mathrm{cl}_B(r_A(Q)) = r_B(Q)$. Hence

$$\mathfrak{S}_B = \mathcal{Q}^a = r_B(Q) = \mathrm{cl}_B(r_A(Q)) = \mathrm{cl}_B(Q^a) = \mathrm{cl}_B(\mathfrak{S}_A).$$

COROLLARY 5.2. *Let A be a semisimple right complemented Banach algebra. If L_A has the property that $x \in \mathrm{cl}_{L_A}(xL_A)$ for all $x \in L_A$, then $\mathfrak{S}_{L_A} = \mathrm{cl}_{L_A}(\mathfrak{S}_A)$.*

PROOF. By [1, Lemma 3, page 39], A has the property that $x \in \mathrm{cl}_A(xA)$ for all $x \in A$. Therefore, if L_A also has the property then, by [4, Theorem 2.3, page 299], $\mathrm{cl}_{L_A}(I) \cap A = I$ for every closed ideal I of A. The conclusion now follows from Proposition 5.1, since A is modular annihilator.

PROPOSITION 5.3. *Let A be a semisimple annihilator right complemented Banach algebra. Then*

(i) $\mathrm{cl}_{L_A}(I) \cap A = I$, *for every closed right ideal I of A.*
(ii) $\mathrm{cl}_{L_A}(J \cap A) = J$, *for every closed left ideal J of L_A.*

PROOF. (i) Let I be a closed right ideal of A and let p be the given right complementor on A. Let $\{e_\alpha: \alpha \in \Omega\}$ be a maximal family of mutually orthogonal minimal p-projections in I. (We recall that a minimal idempotent e is called a minimal p-projection if $(eA)^p = (1-e)A$. See [6, page 654].) We claim that $I = \mathrm{cl}_A(\Sigma_\alpha e_\alpha A)$. In fact let $J = \mathrm{cl}_A(\Sigma_\alpha e_\alpha A)$; $J \subset I$. If $J \neq I$ then $J^p \cap I \neq (0)$ and therefore contains a minimal p-projection e. Since every $e_\alpha \in J$ and $e \in J^p$, we have $e_\alpha e = e e_\alpha = 0$ for all $\alpha \in \Omega$ [6, page 654]. As $e \in I$, this shows that $\{e_\alpha: \alpha \in \Omega\}$ is not a maximal family of mutually orthogonal minimal p-projections in I; a contradiction. Therefore $I = \mathrm{cl}_A(\Sigma_\alpha e_\alpha A)$. Let $K = \mathrm{cl}_{L_A}(I)$. Then $K = \mathrm{cl}_{L_A}(\Sigma_\alpha e_\alpha A) = \mathrm{cl}_{L_A}(\Sigma_\alpha e_\alpha L_A)$. We have $I = r_A(l_A(I))$ and $K = r_{L_A}(l_{L_A}(K)) = r_{L_A}(l_{L_A}(I))$. Now $r_{L_A}(l_A(I)) \supseteq r_{L_A}(l_{L_A}(I)) = K$ so that $I = r_A(l_A(I)) = r_{L_A}(l_A(I)) \cap A \supseteq K \cap A$. Since $I \subset K \cap A$, we get $I = K \cap A$.

(ii) By [7, Theorem 3.6], A contains a left approximate identity $\{u_\gamma: \gamma \in \Gamma\}$ such that $\{L_{u_\gamma}: \gamma \in \Gamma\}$ is bounded in L_A. Clearly $\{L_{u_\gamma}: \gamma \in \Gamma\}$ is a bounded left approximate identity for L_A. Therefore $a \in \mathrm{cl}_A(Aa)$ and $b \in \mathrm{cl}_{L_A}(L_A b)$ for all $a \in A$ and $b \in B$. Let J be a closed left ideal of L_A and $x \in J$. Since S_A is a dense ideal in L_A [8], $S_A x \subset J \cap A$ and $x \in \mathrm{cl}_B(L_A x) = \mathrm{cl}_B(S_A x) \subseteq \mathrm{cl}_B(J \cap A)$. Hence $J \subseteq \mathrm{cl}_B(J \cap A)$ and as $\mathrm{cl}_B(J \cap A) \subseteq J$, we obtain $J = \mathrm{cl}_B(J \cap A)$.

It is easy to see that properties (i) and (ii) above also hold for closed ideals.

COROLLARY 5.4. *Let A be a semisimple annihilator right complemented Banach algebra with a right complementor p. Then, for every closed ideal $\mathcal{M}$ of L_A,*

$$\mathcal{M}^a = \mathrm{cl}_{L_A}([\mathcal{M} \cap A]^p).$$

PROOF. By Proposition 5.3 (ii), $\mathcal{M} = \mathrm{cl}_{L_A}(\mathcal{M} \cap A)$ and $\mathcal{M}^a = \mathrm{cl}_{L_A}(\mathcal{M}^a \cap A)$. But $\mathcal{M}^a \cap A = l_A(\mathcal{M})$. Hence $\mathcal{M}^a = \mathrm{cl}_{L_A}(l_A(\mathcal{M}))$ and so

$$\mathrm{cl}_{L_A}(l_A(\mathcal{M})) = \mathrm{cl}_{L_A}(l_A(\mathcal{M} \cap A)).$$

By [6, Lemma 1, page 652], $l_A(\mathcal{M} \cap A) = [\mathcal{M} \cap A]^p$. Therefore

$$\mathcal{M}^a = \mathrm{cl}_{L_A}(\mathcal{M}^a \cap A) = \mathrm{cl}_{L_A}([\mathcal{M} \cap A]^p).$$

THEOREM 5.5. *Let A be a semisimple annihilator right complemented Banach algebra. Then* $\mathfrak{S}_{L_A} = \mathrm{cl}_{L_A}(\mathfrak{S}_A)$ *and* $\mathfrak{S}_A = \mathfrak{S}_{L_A} \cap A$.

PROOF. This follows at once from Proposition 5.1 and 5.3.

Let A be a semisimple Banach algebra. Let

$$N_L = \{x \in A\colon L_x \text{ is compact}\}, \qquad N_R = \{x \in A\colon R_x \text{ is compact}\}.$$

Let

$$\mathcal{N}_L = \{z \in L_A\colon L_z \text{ is compact}\}, \qquad \mathcal{N}_R = \{z \in L_A\colon R_z \text{ is compact}\}.$$

Clearly N_L and N_R ($\mathcal{N}_L$ and $\mathcal{N}_R$) are closed ideals in $A(L_A)$.

Let A be a semisimple right complemented Banach algebra. Since A is modular annihilator, by [11, Theorem 3.3, page 83], $\mathfrak{S}_A = N_L^a = N_R^a$. By [6, Lemma 1, page 652], $N_L^a = N_L^p$ and $N_R^a = N_R^p$, where p is the right complementor on A. As $(N_L^a)^a = (N_L^p)^p = N_L$ and $(N_R^a)^a = (N_R^p)^p = N_R$, we obtain $N_L = N_R = \mathfrak{S}_A^a$.

THEOREM 5.6. *Let A be a semisimple right complemented Banach algebra. Then*

(i) $\mathfrak{S}_A^a = N_L = N_R$.

(ii) $\mathfrak{S}_{L_A}^a = \mathcal{N}_L = \mathcal{N}_R$.

If A is also an annihilator algebra, then

(iii) $\mathcal{N}_L = \mathcal{N}_R = \mathrm{cl}_{L_A}(N_L) = \mathrm{cl}_{L_A}(N_R)$.

(iv) $N_L = N_R = \mathcal{N}_L \cap A = \mathcal{N}_R \cap A$.

PROOF. (i) This was proved above.

(ii) Since $\mathcal{N}_L$ is a closed ideal of L_A, by Corollary 3.6, we have $\mathcal{N}_L \oplus \mathcal{N}_L^a = L_A$. Similarly $\mathcal{N}_L^a \oplus \mathcal{N}_L^{aa} = L_A$. As $\mathcal{N}_L \subset \mathcal{N}_L^{aa}$, we obtain $\mathcal{N}_L = \mathcal{N}_L^{aa}$. Likewise $\mathcal{N}_R = \mathcal{N}_R^{aa}$. Now, by [11, Theorem 3.3, page 83], $\mathfrak{S}_{L_A} = \mathcal{N}_L^a = \mathcal{N}_R^a$. Hence $\mathfrak{S}_{L_A}^a = \mathcal{N}_L = \mathcal{N}_R$.

(iii) Suppose A is an annihilator algebra. Then, by Proposition 5.3, Theorem 5.5 and (i), we have

$$N_L^a = \mathrm{cl}_{L_A}(N_L^a) \cap A = \mathrm{cl}_{L_A}(\mathfrak{S}_A) \cap A = \mathfrak{S}_{L_A} \cap A = \mathcal{N}_L^a \cap A.$$

Therefore, in view of Corollary 5.4,

$$\mathcal{N}_L = \mathcal{N}_L^{aa} = \mathrm{cl}_{L_A}\big([\mathcal{N}_L^a \cap A]^p\big) = \mathrm{cl}_{L_A}\big((N_L^a)^p\big),$$

where p is the right complementor on A. By [6, Lemma 1, page 652], $(N_L^a)^p = (N_L^p)^p = N_L$. Hence $\mathcal{N}_L = \mathrm{cl}_{L_A}(N_L)$. Likewise $\mathcal{N}_R = \mathrm{cl}_{L_A}(N_R)$. By (ii), $\mathcal{N}_L = \mathcal{N}_R$.

(iv) This follows from Proposition 5.3 and (iii).

Theorem 5.6 answers some of the questions in [11] (see [11, page 85]).

References

[1] F. E. Alexander, 'On complemented and annihilator algebras', *Proc. Glascow Math. Assoc.* **19** (1969), 38–45.
[2] G. F. Bachelis, 'Homomorphisms of annihilator Banach algebras', *Pacific J. Math.* **25** (1968), 229–247.
[3] B. A. Barnes, 'Modular annihilator algebras', *Canad. J. Math.* **18** (1966), 566–578.
[4] M. Leinert, 'A contribution to Segal algebras', *Manuscripta Math.* **10** (1975), 297–306.
[5] C. E. Rickart, *Banach algebras* (Van Nostrand, 1960).
[6] B. J. Tomiuk, 'Structure theory of complemented Banach algebras', *Canad. J. Math.* **14** (1962), 651–659.
[7] B. J. Tomiuk, 'Isomorphisms of multiplier algebras', *Glascow Math. J.*, to appear.
[8] B. J. Tomiuk and B. Yood, 'Incomplete normed algebra norms on Banach algebras', submitted.
[9] B. Yood, 'Ideals in topological rings', *Canad. J. Math.* **16** (1964), 28–45.
[10] B. Yood, 'Closed prime ideals in topological rings', *Proc. London Math. Soc.* (3) **24** (1972), 307–323.
[11] B. Yood, 'On the strong radical of certain Banach algebras', *Proc. Edinburgh Math. Soc.* **21** (1978), 81–85.

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