Glasgow Math. J. **45** (2003) 401–413. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001344. Printed in the United Kingdom

SEIBERG-WITTEN INVARIANTS AND (ANTI-)SYMPLECTIC INVOLUTIONS

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(Received 16 November, 2000; accepted 18 February, 2003)

Abstract. Let X be a closed, symplectic 4-manifold. Suppose that there is either a symplectic or an anti-symplectic involution $\sigma : X \to X$ with a 2-dimensional compact, oriented submanifold Σ as a fixed point set.

If σ is a symplectic involution then the quotient X/σ with $b_2^+(X/\sigma) \geq 1$ is a symplectic 4-manifold.

If σ is an anti-symplectic involution and Σ has genus greater than 1 representing non-trivial homology class, we prove a vanishing theorem on Seiberg-Witten invariants of the quotient X/σ with $b_2^+(X/\sigma) > 1$.

If Σ is a torus with self-intersection number 0, we get a relation between the Seiberg-Witten invariants on X and those of X/σ with $b_2^+(X), b_2^+(X/\sigma) > 2$ which was obtained in [**21**] when the genus $g(\Sigma) > 1$ and $\Sigma \cdot \Sigma = 0$.

2000 *Mathematics Subject Classification.* 53D05, 57M12, 57M50, 57R17, 57R57, 57S25.

1. Introduction. Let X be a closed, oriented Riemannian 4-manifold and let $L \to X$ be a complex line bundle satisfying $c_1(L) = w_2(TX)$ mod 2. Then there is a principal Spin^c(4)-bundle $\xi \to X$ associated to L. We say ξ is a *Spin^c-structure* associated to L. Let $W^{\pm}(\xi)$ be $(\pm\frac{1}{2})$-twisted spinor bundles associated to ξ. The determinant bundle det $W^{\pm}$ is isomorphic to L.

Let $\mathcal{A}(L)$ be the set of all Riemannian connections on L. The gauge group $\mathcal{G}(L)$ of all bundle automorphisms on L acts on $\mathcal{A}(L) \times \Gamma(W^+(\xi))$ by $g(A, \psi) = (A - 2g^{-1}dg, g\psi)$, for all $g \in \mathcal{G}(L)$ and $(A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+(\xi))$.

For a unitary connection $A \in \mathcal{A}(L)$ and a positive spinor field $\psi \in \Gamma(W^+(\xi))$ the Seiberg-Witten equations are defined by

$$\begin{cases} F_A^+ = q(\psi), \\ D_A\psi = 0, \end{cases}$$

where $D_A : \Gamma(W^+(\xi)) \to \Gamma(W^-(\xi))$ is the Dirac operator associated with A and $q : C^\infty(W^+(\xi)) \to \Omega_X^+(i\mathbb{R})$ is a quadratic map defined by $q(\psi) = \psi \otimes \psi^* - \frac{||\psi||^2}{2}\text{Id}$.

Let $\mathcal{M}(\xi)$ be the moduli space of the gauge equivalence classes of all solutions of the Seiberg-Witten equations. Then $\mathcal{M}(\xi)$ is compact by [**16**].

This work was supported by a Korea Research Foundation Grant (No KRF-2002-072-C00010).

We consider perturbed Seiberg-Witten equations:

$$\begin{cases} F_A^+ + i\delta = q(\psi), \\ D_A\psi = 0, \end{cases}$$

where δ is a real valued, self-dual 2-form on X.

Then the perturbed moduli space $\mathcal{M}_\delta(\xi)$ is a smooth manifold with its dimension $\dim \mathcal{M}_\delta(\xi) = \frac{1}{4}\{c_1(L)^2[X] - (2\chi(X) + 3\mathrm{Sign}(X))\}$. If the metric on X is chosen so that the perturbed Seiberg-Witten equations admit no reducible solutions, then $\mathcal{M}_\delta(\xi)$ is compact. If $\dim \mathcal{M}_\delta(\xi) = 2d$, then the Seiberg-Witten invariant is defined by

$$\int_{\mathcal{M}_\delta(\xi)} c_1(\mathcal{M}_\delta(\xi)_0)^d,$$

the integral of the maximal power of the Chern class of the circle bundle $\mathcal{M}_\delta(\xi)_0 \to \mathcal{M}_\delta(\xi)$, where $\mathcal{M}_\delta(\xi)_0$ is the framed moduli space.

If $\dim \mathcal{M}_\delta(\xi)$ is odd or negative then the Seiberg-Witten invariant is defined to be zero. For details, see [**23**].

In general, there are infinitely many elements $c_1(L) \in H^2(X;\mathbb{Z})$ satisfying $c_1(L) = w_2(TX)$ mod 2. Each such element induces a Spin^c-structure on X. However there are only finitely many elements in $H^2(X;\mathbb{Z})$ such that their Seiberg-Witten invariants are non-zero. Such an element in $H^2(X;\mathbb{Z})$ is called a *basic class*. Hence the set of basic classes is finite. Furthermore X is said to have *simple type* if all basic classes satisfy $c_1(L)^2[X] = 2\chi(X) + 3\sigma(X)$.

Using the Seiberg-Witten invariants, Taubes [**23**] proved the non-trivialness of the Seiberg-Witten invariants on symplectic 4-manifolds with $b_2^+ > 1$. In Section 2, we consider a symplectic involution σ over a closed symplectic 4-manifold X with a symplectic structure ω. We show that if the symplectic involution σ has a 2-dimensional, compact, oriented submanifold as a fixed point set, then the quotient X/σ with $b_2^+(X/\sigma) \geq 1$ is a closed symplectic 4-manifold.

If a closed, oriented Riemannian 4-manifold X has a basic class, it gives a minimal genus bound for the embedded surface, called the *adjunction inequality*.

THEOREM 1.1. (Adjunction Inequality [**16**,**19**]). *Let X be a smooth 4-manifold with $b_2^+(X) > 1$ and a basic class L, and let Σ_X be an embedded connected oriented surface with $\Sigma_X \cdot \Sigma_X \geq 0$ and $[\Sigma_X] \neq 0 \in H_2(X;\mathbb{Z})$. Then we have an inequality*

$$-\chi(\Sigma_X) \geq \Sigma_X \cdot \Sigma_X + |c_1(L)[\Sigma_X]|.$$

Ozsváth and Szabó [**20**] extended Theorem 1.1 for a 4-manifold X of Seiberg-Witten simple type with $b_2^+(X) > 1$ and $g(\Sigma_X) > 0$ and $\Sigma_X \cdot \Sigma_X < 0$.

Related with a symplectic 4-manifold, there is a Akbulut's conjecture [**15**] (Problem 4.104) that if an anti-symplectic involution σ acts on a simply-connected, closed, symplectic 4-manifold X (that is, $\sigma^*\omega = -\omega$ for a symplectic structure ω) with a 2-dimensional smooth surface as a fixed point set, then $X/\sigma = r\mathbb{C}P^2 \sharp s\mathbb{C}\bar{P}^2$ or $nS^2 \times S^2$, for some $r, s, n, \in \mathbb{N}$.

Akbulut [**1**] showed that if σ is a complex conjugation over a complex algebraic surface X with a real algebraic surface as a fixed point set then $X/\sigma = r\mathbb{C}P^2 \sharp s\mathbb{C}\bar{P}^2$ or $nS^2 \times S^2$ for many cases.

In Section 3, we consider an anti-symplectic involution σ over a symplectic 4-manifold X with a 2-dimensional compact submanifold as a fixed point set. By using

Theorem 1.1, we show that if the fixed point set contains a Riemann surface with genus greater than 1 representing non trivial homology class in $H_2(X/\sigma;\mathbb{Z})$, then the quotient X/σ with $b_2^+(X/\sigma) > 1$ has a vanishing Seiberg-Witten invariant.

Let X' be a closed, smooth, oriented 4-manifold with a smoothly embedded 2-torus T^2 with self-intersection number 0 and let $\pi : X \to X'$ be a double cover branched along T^2. In Section 4, we prove a relation between the Seiberg-Witten invariants on X and those of X' when $b_2^+(X), b_2^+(X') > 2$. Ruan and Wang [**21**] proved the same results when the genus of the fixed point set is greater than 1 and the fixed point set has self-intersection number 0. In particular, if $\sigma : X \to X$ is an anti-symplectic involution on a closed symplectic 4-manifold X whose fixed point set is a torus with self-intersection number 0, we get a relation between the Seiberg-Witten invariants on X and those of X/σ.

In Section 5, we calculate Theorem 4.7 for some cases.

2. Seiberg-Witten invariant of the quotient manifold under a symplectic involution with a 2-dimensional fixed point set. Let X be a closed symplectic 4-manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is a symplectic involution if and only if $\sigma^*\omega = \omega$ and $\sigma^2 = \mathrm{Id}$ on X.

PROPOSITION 2.1. *Let X be a closed symplectic 4-manifold with a symplectic structure ω. Suppose that $\sigma : X \to X$ is a symplectic involution with a 2-dimensional, compact, oriented submanifold Σ. Then Σ is a symplectic submanifold.*

Proof. By definition, J is a ω-compatible almost complex structure if and only if $\omega(v, Jv) > 0$, for all $v \neq 0 \in TX$, and $\omega(Jv, Jw) = \omega(v, w)$, for all $v, w \in TX$. It is known that the set of all ω-compatible almost complex structures is not empty and contractible. Then we can find a ω-compatible metric g such that $g(v, w) = \omega(v, Jw)$ and ω is self-dual with respect to g.

Let $T\Sigma$ and N_Σ be, respectively, the tangent and normal complex line bundles of Σ in X. The induced map σ_* on $TX|_\Sigma = T\Sigma \oplus N_\Sigma$ satisfies $\sigma_*|_{T\Sigma} = \mathrm{Id}$ and $\sigma_*|_{N_\Sigma} = -\mathrm{Id}$.

Then σ acts as an isometry over $TX|_\Sigma$ for the ω-compatible metric g. Indeed, for all $v_1, v_2 \in T\Sigma$ and $w_1, w_2 \in N_\Sigma$, $g(v_i, w_j) = 0$, $i, j = 1, 2$, and

$$\begin{aligned}\sigma^*g(v_1, v_2) &= g(\sigma_*v_1, \sigma_*v_2) = g(v_1, v_2),\\ \sigma^*g(w_1, w_2) &= g(\sigma_*w_1, \sigma_*w_2) = g(-w_1, -w_2) = g(w_1, w_2).\end{aligned}$$

Then we have

$$\begin{aligned}g(J\sigma_*v, w) &= \omega(\sigma_*v, w) = \sigma^*\omega(\sigma_*v, w) = \omega(v, \sigma_*w) = g(Jv, \sigma_*w)\\ &= \sigma^*g(Jv, \sigma_*w) = g(\sigma_*Jv, w), \quad \text{for all} \quad v, w \in TX|_\Sigma.\end{aligned}$$

Thus $g(J\sigma_*v, w) = g(\sigma_*Jv, w)$, for all $v, w \in TX|_\Sigma$, and $J \circ \sigma_* = \sigma_* \circ J$ on $TX|_\Sigma$. Then for all $v \in T\Sigma$, $Jv \in T\Sigma$ and so $T\Sigma$ is a complex vector space.

For any non zero $v \in T\Sigma$, we have $Jv \in T\Sigma$ and $\omega(v, Jv) = g(v, v) > 0$. Thus the restriction of ω on Σ is a symplectic structure on Σ. □

PROPOSITION 2.2. *Let X be a closed symplectic 4-manifold with a symplectic structure ω. Suppose that $\sigma : X \to X$ is a symplectic involution with a 2-dimensional compact, oriented submanifold Σ as a fixed point set. Then the quotient X/σ with $b_2^+(X/\sigma) \geq 1$ is a closed symplectic 4-manifold.*

https://doi.org/10.1017/S0017089503001344 Published online by Cambridge University Press

Proof. Let $\pi : X \to X/\sigma$ be the projection and the image of the fixed point set $\pi(\Sigma) = \Sigma'$. Then by **[24]** and **[5]**, the quotient X/σ is a closed, smooth 4-manifold.

By **[4]**, there is a σ-invariant tubular neighborhood $N(\Sigma)$ of Σ in X such that the restriction $\pi|_{N(\Sigma)} : N(\Sigma) \to \pi(N(\Sigma)) = N(\Sigma')$ is a double covering with the branch set Σ' that is locally $\pi(z, v) = \pi(z, -v) = [z, v]$, for all $z \in \Sigma$ and v in the normal fiber.

Since $\sigma^*\omega = \omega$, the σ-invariant symplectic form ω on X defines naturally a symplectic form ω' on X/σ by $\omega'(v', w') = \omega(v, w)$ if $\pi_*(v) = v', \pi_*(w) = w'$ for all $v, w \in TX$.

Indeed, for all $x' \in \Sigma' \subset X/\sigma$, the tangent space $T_{x'}X' = T_{x'}\Sigma' \oplus N_{\Sigma'}|_{x'}$ and locally $\omega' = dx'_1 dx'_2 + dx'_3 dx'_4$, where $x' = (x'_1, x'_2)$ and (x'_3, x'_4) is a coordinate of the normal fiber. Then there is an element $x \in \Sigma \subset X$ such that $\pi(x) = x'$, $T_x X = T_x\Sigma \oplus N_\Sigma|_x$ and locally $\omega = dx_1 dx_2 + dx_3 dx_4$.

Let $v = (v_1, v_2)$, $w = (w_1, w_2) \in T_x X = T_x\Sigma \oplus N_\Sigma|_x$ and $\pi_* v = v'$, $\pi_* w = w'$. Then $\sigma_*(v_1, v_2) = (v_1, -v_2)$ and we have

$$\omega(v, w) = (dx_1dx_2 + dx_3dx_4)(v, w) = dx_1dx_2(v_1, w_1) + dx_3dx_4(v_2, w_2),$$
$$\sigma^*\omega(v, w) = \omega(\sigma_* v, \sigma_* w) = \omega((v_1, -v_2),$$
$$(w_1, -w_2)) = dx_1dx_2(v_1, w_1) + dx_3dx_4(-v_2, -w_2) = dx_1dx_2(v_1, w_1) + dx_3dx_4(v_2, w_2).$$

Thus ω' is well-defined on $\Sigma' \subset X/\sigma$. The other case $x \in X/\sigma - \Sigma'$ is clear since $\sigma^*\omega = \omega$. Thus we have completed the proof.

EXAMPLE 2.3. **[7]**. Let $X = S^2 \times S^2$ be the symplectic 4-manifold with the standard product symplectic form $\omega = \omega_1 + \omega_2$, where ω_1 and ω_2 are the standard symplectic forms on S^2.

The involution $\sigma : X \to X$ is given by $\sigma(x, y) = (y, x)$. Then σ is clearly a symplectic involution and its fixed point set is the diagram $\triangle(X) = S^2$. Then the quotient $X/\sigma = \mathbb{C}P^2$ is symplectic. □

3. Seiberg-Witten invariant of the quotient manifold under an anti-symplectic involution with a 2-dimensional fixed point set. Let X be a closed symplectic 4-manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is an anti-symplectic involution if and only if it satisfies $\sigma^*\omega = -\omega$ and $\sigma^2 = \mathrm{Id}$ on X. If X is a Kähler surface then σ is anti-symplectic if and only if σ is anti-holomorphic; that is, $\sigma_* \circ J = -J \circ \sigma_*$ for the complex structure J on X. For an example of an anti-holomorphic involution, we can consider a complex conjugation over a complex algebraic surface.

From now on suppose that there is an anti-symplectic involution $\sigma : X \to X$ with a 2-dimensional, compact submanifold X^σ as a fixed point set. Then we have the following result.

LEMMA 3.1. *Each connected, oriented* 2*-dimensional component* $\Sigma \subset X^\sigma$ *is a Lagrangian surface.*

Proof. Since σ is anti-symplectic, $\sigma^*\omega = -\omega$ and so $\sigma^*\omega|_\Sigma = -\omega|_\Sigma$. However, over the fixed point set Σ, we have

$$\sigma^*\omega|_\Sigma = \omega|_{\sigma(\Sigma)} = \omega|_\Sigma.$$

Thus $\omega|_\Sigma = 0$ and Σ is a Lagrangian surface in X. □

https://doi.org/10.1017/S0017089503001344 Published online by Cambridge University Press

PROPOSITION 3.2. *For an anti-symplectic involution σ, we have $\sigma_* \circ J = -J \circ \sigma_*$ for a ω-compatible almost complex structure J as long as σ is an isometry for the ω-compatible metric g.*

Proof. Let g be a ω-compatible metric such that $g(v, w) = \omega(v, Jw)$, for all $v, w \in TX$, and ω is self-dual with respect to g.

Since σ is anti-symplectic and acts as an isometry for the ω-compatible metric g, we have

$$\begin{aligned} g(J\sigma_* v, w) &= \omega(\sigma_* v, w) = -\sigma^*\omega(\sigma_* v, w) = \omega(-v, \sigma_* w) = g(-Jv, \sigma_* w) \\ &= \sigma^* g(-Jv, \sigma_* w) = g(-\sigma_* Jv, w), \quad \text{for all} \quad v, w \in TX. \end{aligned}$$

Thus we have $J \circ \sigma_* = -\sigma_* \circ J$ on TX. □

LEMMA 3.3. *Each connected, oriented 2-dimensional component $\Sigma \in X^\sigma$ satisfies $\chi(\Sigma) + \Sigma \cdot \Sigma = 0$.*

Proof. Let J be the ω-compatible almost complex structure and g be the compatible metric.

Over $TX|_\Sigma = T\Sigma \oplus N_\Sigma$, the induced map σ_* acts as $\sigma_*|_{T\Sigma} = \mathrm{Id}$ and $\sigma_*|_{N_\Sigma} = -\mathrm{Id}$ where $T\Sigma$ and N_Σ are the tangent and normal complex line bundle of Σ in X, respectively. Then σ acts as an isometry on $TX|_\Sigma$ for the ω-compatible metric g. Indeed, for all $v_1, v_2 \in T\Sigma$ and $w_1, w_2 \in N_\Sigma$, $g(v_i, w_j) = 0$, for $i, j = 1, 2$, and

$$\begin{aligned} \sigma^* g(v_1, v_2) &= g(\sigma_* v_1, \sigma_* v_2) = g(v_1, v_2), \\ \sigma^* g(w_1, w_2) &= g(\sigma_* w_1, \sigma_* w_2) = g(-w_1, -w_2) = g(w_1, w_2). \end{aligned}$$

By Proposition 3.2, we have $J \circ \sigma_* = -\sigma_* \circ J$ on $TX|_\Sigma$ and so J is an orientation reversing isomorphism $J : T_x\Sigma \to N_\Sigma|_x$, for each $x \in \Sigma$. Thus we have $\chi(\Sigma) = -\Sigma \cdot \Sigma$. □

THEOREM 3.4. *Let (X, ω) be a symplectic 4-manifold and $\sigma : X \to X$ be an anti-symplectic involution with a 2-dimensional compact submanifold as a fixed point set. If the fixed point set contains a Riemann surface Σ with genus $g(\Sigma) \geq 2$ and $0 \neq [\Sigma] \in H_2(X : \mathbb{Z})$, then the quotient manifold X/σ with $b_2^+(X/\sigma) > 1$ has a vanishing Seiberg-Witten invariant.*

Proof. Let $\pi : X \to X/\sigma$ be the projection map and $\pi(\Sigma) = \Sigma'$. By **[5]** and **[24]**, we have $\Sigma' \cdot \Sigma' = 2\Sigma \cdot \Sigma$.

If the quotient X/σ has a Seiberg-Witten basic class L then, by Theorem 1.1, we have

$$|c_1(L)[\Sigma']| + \Sigma' \cdot \Sigma' \leq -\chi(\Sigma'). \tag{1}$$

By Lemma 3.3, $\chi(\Sigma) + \Sigma \cdot \Sigma = 0$ and so the equation (1) implies that

$$|c_1(L)[\Sigma']| + 2\Sigma \cdot \Sigma + \chi(\Sigma) = |c_1(L)[\Sigma']| + \Sigma \cdot \Sigma \leq 0.$$

Then we have

$$|c_1(L)[\Sigma']| \leq -\Sigma \cdot \Sigma. \tag{2}$$

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Since $g(\Sigma) \geq 2$, we have $\Sigma \cdot \Sigma = -\chi(\Sigma) > 0$ and so equation (2) yields a contradiction. Thus there is no Seiberg-Witten basic class over X/σ. □

REMARK 3.5. Let X be a closed, smooth, almost complex 4-manifold with an almost complex structure J. Assume that $\sigma : X \to X$ is an anti-holomorphic involution with a 2-dimensional, compact submanifold as a fixed point set. Then, since $J \circ \sigma_* = -\sigma_* \circ J$ on TX, we have an orientation reversing isomorphism $J : T_x\Sigma \to N_\Sigma|_x$, for all $x \in \Sigma$.

With the same conditions on Σ as in Theorem 3.4, the quotient X/σ with $b_2^+(X/\sigma) > 1$ has a vanishing Seiberg-Witten invariant. □

4. Relationship between Seiberg-Witten invariants on X and X' when $g(\Sigma) = 1$ and $\Sigma \cdot \Sigma = 0$. Let X' be a closed smooth 4-manifold and $\pi : X \to X'$ be a double branched cover along a surface Σ'. Ruan and Wang [**21**] established a formula between the Seiberg-Witten invariants on X and X' with $b_2^+(X'), b_2^+(X) > 1$ when Σ' has genus greater than 1 and $\Sigma' \cdot \Sigma' = 0$. Suppose that $H_2(X;\mathbb{Z})$ has no 2-torsion. Let $\pi^{-1}(\Sigma') = \Sigma$ and Y_0 be the complement of a tubular neighborhood of Σ'.

THEOREM 4.1. [**21**]. *Let $\pi : X \to X'$ be a double cover branched along a surface Σ' with genus greater than 1, $[\Sigma']^2 = 0$, and such that $b_2^+(X'), b_2^+(X) > 1$. Suppose that ξ is a Spin^c-structure on X' satisfying $c_1(\det\xi)\cdot[\Sigma'] \leq 0$, and the virtual dimension of the Seiberg-Witten moduli space and the adjunction term, $|c_1(\det\xi)[\Sigma']| + \Sigma' \cdot \Sigma' + \chi(\Sigma')$ both vanish. Moreover let $\tilde{\xi}$ be a Spin^c-structure on X whose determinant bundle is $\det\tilde{\xi} = \pi^*(\det\xi) \otimes PD[\Sigma]^{-1}$ and whose restriction to $\tilde{Y}_0 = \pi^{-1}(Y_0)$ is the pull-back of $\xi|_{Y_0}$. Then the following equality holds*:

$$SW_X(\tilde{\xi}) = SW_{X'}(\xi) + k_\xi(X', \Sigma') \mod 2,$$

where $k_\xi(X', \Sigma') = \Sigma_{[\gamma]\in K_Y^*} SW_c(\xi|_Y \otimes \gamma)$ is an invariant of the triple $(X', \Sigma', \frac{[\Sigma']}{2})$ and $c = c_1(\det\xi)^2 - 2\chi(\Sigma')$.

Ruan and Wang proved Theorem 4.1 by using the relative Seiberg-Witten invariants formula [**19**]. Their idea is to rewrite the Seiberg-Witten invariants on X and X' in terms of relative Seiberg-Witten invariants and relate the relative Seiberg-Witten invariants using the Seiberg-Witten theory with a $\mathbb{Z}_2$-action. Under the conditions of Theorem 4.1, all finite energy solutions of the Seiberg-Witten moduli space defined over the cylindrical end space are irreducible. They exclude the case in which Σ' is a torus, because here we have reducible solutions of the Seiberg-Witten equations over the cylindrical extensions of the complement of T^2.

In Section 4, we prove a formula between the Seiberg-Witten invariants of X and X' when the genus of Σ' is 1 and $\Sigma' \cdot \Sigma' = 0$ by using [**18**] and [**21**].

Assume that X' is a closed, smooth, oriented 4-manifold with a smoothly embedded 2-torus T^2 with self-intersection number 0. Then X' is diffeomorphic to a 4-manifold $Y \cup_\phi (D^2 \times T^2)$, where Y is a smooth, compact, oriented 4-manifold with boundary $\partial Y \cong T^3$, $\phi : \partial(D^2 \times T^2) \to \partial Y$ is an orientation reversing diffeomorphism and D^2 is a disk in $\mathbb{R}^2$ with $\partial D^2 \cong S^1$. We identify $X' = Y \cup_\phi (D^2 \times T^2)$.

By Hirzebruch [**14**], if $[T^2] = 2a$, $a \in H_2(X';\mathbb{Z})$, then there is a branched double cover $\pi : X \to X'$ along $[T^2]$. Then X is a closed, oriented, smooth 4-manifold with a smoothly embedded torus $\tilde{T}^2 = \pi^{-1}(T^2)$ with self-intersection number 0 and X is diffeomorphic to $\tilde{Y}\cup_{\psi}(D^2\times\tilde{T}^2)$, where $\tilde{Y}$ is an unramified 2-fold cover of Y with $\partial\tilde{Y}=\pi^{-1}(\partial Y)$ and $\psi:\partial(D^2\times\tilde{T}^2)\to\partial(\tilde{Y})$ is an orientation reversing diffeomorphism.

Suppose that $b_2^+(Y), b_2^+(\tilde{Y})>1$. Since $b_2^+(X')=b_2^+(Y)+1$ and $b_2^+(X)=b_2^+(\tilde{Y})+1$, we have $b_2^+(X'), b_2^+(X)>2$.

Let $\tilde{\gamma}$ be a representative of the homology class $\psi_*[\partial D^2\times\{\mathrm{pt}\}]\in H_1(\partial\tilde{Y};\mathbb{Z})$. Suppose that $\tilde{\gamma}\in\ker(\tilde{i}_*)$, where $\tilde{i}_*:H_1(\partial\tilde{Y};\mathbb{R})\to H_1(\tilde{Y};\mathbb{R})$ is induced from the inclusion $\tilde{i}:\partial\tilde{Y}\to\tilde{Y}$.

Then we can fix a $\tilde{b}\in H_2(\tilde{Y},\partial\tilde{Y};\mathbb{R})$ such that the boundary $\partial\tilde{b}=\tilde{\gamma}$ and $\pi_*\tilde{\gamma}=\gamma=\phi_*(2[\partial D^2\times\{\mathrm{pt}\}])\in\ker(i_*)$. Thus there exists $\pi_*\tilde{b}=b\in H_2(Y,\partial Y;\mathbb{R})$ such that $\partial b=\gamma$, where $i_*:H_1(\partial Y;\mathbb{R})\to H_1(Y;\mathbb{R})$.

Let ξ be a Spin^c-structure over X' with determinant bundle $\det\xi=L$ and the restriction $\xi|_{D^2\times T^2}$ of ξ to $D^2\times T^2$ is trivial. Take $Y_0=X'\backslash(D^2\times T^2)$ and $\tilde{Y}_0=\pi^{-1}(Y_0)$.

LEMMA 4.2. *In the same situations as above, there exists a Spin^c-structure $\tilde{\xi}$ on X whose restriction to $D^2\times\tilde{T}^2$ is trivial,* $\det\tilde{\xi}=\tilde{L}\cong\pi^*L\otimes PD^{-1}[\tilde{T}^2]$ *and* $\tilde{\xi}|_{\tilde{Y}_0}\cong\pi^*(\xi|_{Y_0})$.

Proof. For the existence of the Spin^c-structure $\tilde{\xi}$ with determinant bundle $\pi^*L\otimes\mathrm{PD}^{-1}[\tilde{T}^2]$ and $\pi^*\xi|_{Y_0}\cong\tilde{\xi}|_{\tilde{Y}_0}$, see Proposition 5.11 [**21**]. We only check that $\tilde{\xi}|_{D^2\times\tilde{T}^2}$ is trivial.

The subspace $D^2\times\tilde{T}^2$ is a symplectic 4-manifold with a symplectic structure $\omega=dx_1dx_2+dy_1dy_2$, where $(x_1,x_2)\in\tilde{T}^2$ and $(y_1,y_2)\in D^2$ are coordinates. Then the positive spinor field $W^+(\tilde{\xi})$ and the determinant bundle $\tilde{L}$ over $D^2\times\tilde{T}^2$ can be decomposed by

$$W^+(\tilde{\xi})|_{D^2\times\tilde{T}^2}=E\otimes(\mathrm{II}\oplus K^*_{D^2\times\tilde{T}^2}),\quad \tilde{L}|_{D^2\times\tilde{T}^2}=E^2\otimes K^*_{D^2\times\tilde{T}^2},$$

for some complex line bundle $E\to D^2\times\tilde{T}^2$, where $K_{D^2\times\tilde{T}^2}$ is the canonical class and II is a trivial line bundle over $D^2\times\tilde{T}^2$.

Since $L|_{D^2\times T^2}$ is trivial and $\tilde{L}|_{D^2\times\tilde{T}^2}=\pi^*L|_{D^2\times\tilde{T}^2}\otimes\mathrm{PD}^{-1}[\tilde{T}^2]|_{D^2\times\tilde{T}^2}$, $\tilde{L}|_{D^2\times\tilde{T}^2}$ is trivial and $2c_1(E)=c_1(K_{D^2\times\tilde{T}^2})$.

Because

$$c_1(K_{D^2\times\tilde{T}^2})[D^2\times\tilde{T}^2]=-c_1(T(D^2\times\tilde{T}^2))[D^2\times\tilde{T}^2]=-(\chi(\tilde{T}^2)+\tilde{T}^2\cdot\tilde{T}^2)=0,$$

$W^+(\tilde{\xi})|_{D^2\times\tilde{T}^2}$ and $\tilde{L}|_{D^2\times\tilde{T}^2}$ are all trivial, where $T(D^2\times\tilde{T}^2)$ is the tangent bundle of $D^2\times\tilde{T}^2$. Since $W^+(\tilde{\xi})=\tilde{\xi}\times_{\mathrm{Spin}^c(4)}\mathbb{C}^2$, we conclude that $\tilde{\xi}|_{D^2\times\tilde{T}^2}$ is trivial. □

From now on let $\xi|_{Y_0}=\xi_0$, $\tilde{\xi}_0=\pi^*(\xi_0)$, $\det(\xi_0)=L_0$, and $\det(\tilde{\xi}_0)=\tilde{L}_0$. Denote $\tilde{Y}'=\tilde{Y}\cup_{T^3}T^3\times[0,\infty)=\mathrm{cl}(\tilde{Y}_0)\cup_{T^3}T^3\times[0,\infty)$. Fix a flat metric $\tilde{h}$ on T^3 and a corresponding cylindrical end metric $\tilde{g}$ on $\tilde{Y}'$ such that $\tilde{g}=\tilde{h}+dt^2$ near the end of $\tilde{Y}'$, $t\in[0,\infty)$.

Since $\tilde{\xi}_0|_{T^3}$ is trivial, $\tilde{\xi}_0$ is a Spin^c-structure on $\tilde{Y}'$. Over the space $\mathcal{A}(\tilde{L}_0)\times\Gamma(W^+_{\tilde{Y}'}(\tilde{\xi}_0))$, we define Seiberg-Witten equations. For any compactly supported, real-valued, smooth, self-dual two-form $\tilde{\zeta}\in\Omega_+{}^2(\tilde{Y}';\mathbb{R})$, let $\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})$ be the moduli space of all finite energy solutions of the perturbed Seiberg-Witten equations by the action of the gauge group, where the energy of a pair (A,ψ) is defined by $\int_{\tilde{Y}'}|F_A|^2\mathrm{dvol}$. Then, by [**18**], there is a continuous map $\tilde{\partial}_\infty$ and a covering map $\tilde{p}$

$$\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\xrightarrow{\tilde{\partial}_\infty}\chi_0(T^3)\xrightarrow{\tilde{p}}\chi(T^3),$$

where $\chi(T^3)$ is the moduli space of the 3-dimensional Seiberg-Witten equations for the trivial Spin^c-structure and a flat metric over T^3 and $\chi_0(T^3)$ is a covering space of $\chi(T^3)$ defined in Section 2 [**18**].

By Lemma 2.3 [**18**], there is a unique singular point $\tilde{\theta}=(\tilde{\theta}_0,0)\in\chi(T^3)$ such that $\ker D_{\tilde{\theta}_0}\neq 0$ and $\mathcal{M}_{\tilde{Y}'}(\tilde{\mathcal{L}}_0,\tilde{g},\tilde{\zeta})$ has singularities induced from $\tilde{\theta}$ and it is a compact manifold with boundary, the boundary mapping to the singular point $\tilde{\theta}$.

Let $\mathcal{C}_{\tilde{Y}}$ be the set of isomorphism classes of Spin^c-structures $\tilde{\xi}_0$ on the space $\tilde{Y}=\mathrm{cl}(X\backslash(D^2\times\tilde{T}^2))$ such that $\tilde{\xi}_0|_{\partial\tilde{Y}}$ is trivial. By [**18**], $\tilde{p}^{-1}(\tilde{\theta})$ is in one-to-one correspondence with the set $\tilde{r}^{-1}(\tilde{\xi}_0)$, where $\tilde{r}:\mathcal{C}_{\tilde{Y},\partial\tilde{Y}}\to\mathcal{C}_{\tilde{Y}}$ is the forgetful map. For the set $\mathcal{C}_{\tilde{Y},\partial\tilde{Y}}$, see [**18**].

By definition [**18**], the Spin^c-structure $\tilde{\xi}'\in\wedge^+(\tilde{\xi})$ if and only if $\tilde{\xi}'\in\tilde{r}^{-1}(\tilde{\xi}_0)$ and for all points in $\tilde{T}^2$,

$$\langle c_1(\tilde{L}),\tilde{b}+[D^2\times \mathrm{pt}]\rangle < \langle c_1(\det\tilde{\xi}'),\tilde{b}\rangle.$$

Then the relative Seiberg-Witten invariant $\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)$ over $\tilde{Y}'$ is defined by

$$\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)=\Sigma_{\tilde{\xi}'\in\wedge^+(\tilde{\xi})}\sharp(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\cap\tilde{D}\cap\tilde{\partial}_\infty^{-1}(\tilde{\theta}_{\tilde{\xi}'}));$$

that is the sum of the counting numbers of a smooth, compact, zero-dimensional manifold $\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\cap\tilde{D}\cap\tilde{\partial}_\infty^{-1}(\tilde{\theta}_{\tilde{\xi}'})$, where $\tilde{\theta}_{\tilde{\xi}'}\in p^{-1}(\tilde{\theta})$ is the element corresponding to $\tilde{\xi}'\in\tilde{r}^{-1}(\tilde{\xi}_0)$.

Remark 4.3. $\tilde{D}$ is a geometric representative of $\tilde{\mu}(\mathrm{pt})^{\frac{\tilde{d}}{2}}$ that is similar to the geometric representative defined in the Donaldson invariant and $\tilde{\mu}$ is a map

$$\tilde{\mu}:H_0(\tilde{Y}';\mathbb{Q})\to H^2(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta});\mathbb{Q})$$

defined by $\tilde{\mu}(\mathrm{pt})=c_1(\mathbb{L})$, where $\mathbb{L}=\pi^*(\tilde{L}_0)$ is the bundle over $\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\times\tilde{Y}'$, $\pi:\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\times\tilde{Y}'\to\tilde{Y}'$ is the projection map and $\tilde{d}$ is given by $\tilde{d}=\frac{1}{4}(c_1(\tilde{L}_0)^2-2\chi(\tilde{Y})-3\mathrm{Sign}(\tilde{Y}))$. □

Since X is a 2-fold branched cover of X', there is an involution $\sigma:X\to X$ with a fixed point set $\tilde{T}^2$. Then σ acts freely over $\tilde{Y}'$ and there are involutions $\tilde{\tau}:\tilde{\xi}_0=\pi^*\xi_0\to\tilde{\xi}_0$ and $\tau=\det\tilde{\tau}:\tilde{L}_0\to\tilde{L}_0$ induced from the involution σ_* on the orthonormal frame bundle of $\tilde{Y}'$. Then we have an involution

$$\tau^*:\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\to\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})$$

and a $\mathbb{Z}_2$-invariant moduli space $\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}$, which is the fixed point set of τ^* and is independent of the choice of τ.

Proposition 4.4. *The maps $\tilde{\partial}_\infty$ and $\tilde{p}$ are $\mathbb{Z}_2$-equivariant. Furthermore the singular point $\tilde{\theta}\in\chi(T^3)^{\mathbb{Z}_2}$.*

Proof. Every Spin^c-structure $\xi_{\mathbb{R}\times T^3}\to\mathbb{R}\times T^3$ is a pull back from a Spin^c-structure ξ_{T^3} on T^3. From the embedding $\mathrm{Spin}^c(3)\to\mathrm{Spin}^c(4)$ sending (q,x) to (q,q,x), we have an identification between the positive and negative spinor spaces $W^+(\xi_{\mathbb{R}\times T^3})\cong W^-(\xi_{\mathbb{R}\times T^3})\cong\pi^*\xi_{T^3}$.

Let $\det\xi_{\mathbb{R}\times T^3}=L_{\mathbb{R}\times T^3}$ and $\det\xi_{T^3}=L_{T^3}$. Then there is a gauge transformation $g:\mathbb{R}\times T^3\to S^1$ such that for all connections $A\in\mathcal{A}(L_{\mathbb{R}\times T^3})$, $g(A)$ has no dt-component, $t\in\mathbb{R}$, which is said to be in *temporal gauge*. Then the Seiberg-Witten equations over

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$\mathbb{R}\times T^3$ can be written as the gradient flow equations. The critical points of the gradient flow equation are solutions of 3-dimensional Seiberg-Witten equations over T^3.

Thus if we consider a $\mathbb{Z}_2$-invariant solution of the Seiberg-Witten equations over $\mathbb{R}\times T^3$, then it induces a $\mathbb{Z}_2$-invariant solution of the 3-dimensional Seiberg-Witten equations over T^3. Thus we have the restrictions of $\tilde{\partial}_\infty$ and $\tilde{\pi}$ such that

$$\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}\xrightarrow{\tilde{\partial}_\infty}\chi_0(T^3)^{\mathbb{Z}_2}\xrightarrow{\tilde{p}}\chi(T^3)^{\mathbb{Z}_2}.$$

If there is a $u\neq 0$ in $\ker D_{\tilde{\theta}_0}$, then $h(u)\neq 0$, for all $h\in\mathbb{Z}_2$, because h is an involution. Since the Dirac operator D_A is $\mathbb{Z}_2$-equivariant, $D_{h(\tilde{\theta}_0)}h(u)=h(D_{\tilde{\theta}_0}u)=0$, for all $h\in\mathbb{Z}_2$. Thus there is a $h(u)\neq 0$ such that $h(u)\in\ker D_{h(\tilde{\theta}_0)}$. Since $\tilde{\theta}=(\tilde{\theta}_0,0)$ is the unique point such that $\ker D_{\tilde{\theta}_0}\neq 0$, we conclude that, for all $h\in\mathbb{Z}_2$, $h(\tilde{\theta}_0)=\tilde{\theta}_0$ in $\chi(T^3)$. □

REMARK 4.5. By [**5**] and [**21**], we can choose a generic, $\mathbb{Z}_2$-invariant Riemannian metric $\tilde{g}$ and a $\mathbb{Z}_2$-invariant self-dual two-form $\tilde{\zeta}$ over $\tilde{Y}'$ such that $(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}\cap D\cap(\tilde{\partial}_\infty)^{-1}(\tilde{\theta}_{\tilde{\xi}'}))$ is a smooth, compact, zero-dimensional manifold, where D is the geometric representative of $\mu(\mathrm{pt})^{\frac{\tilde{d}}{4}}$, μ is given by $\mu: H_0(\tilde{Y}';\mathbb{Q})\to H^2(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2};\mathbb{Q})$ and $\dim\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}=\frac{\tilde{d}}{2}$.

However, in this case, the space $(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})\cap\tilde{D}\cap\tilde{\partial}_\infty^{-1}(\tilde{\theta}_{\tilde{\xi}'}))$ may not be smooth. □

As in Theorem 3.8 of [**21**], by comparing the $\mathbb{Z}_2$-invariant moduli space over $\tilde{Y}'$ with the moduli space over Y', for $\tilde{g}=p^*g$ and $\tilde{\zeta}=p^*\zeta$, there is a homeomorphism

$$\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}\cong\mathcal{M}_{Y'}(\xi_0,g,\zeta)\amalg\left(\amalg_{\eta\in\mathcal{K}_{Y'}}\mathcal{M}_{Y'}(\xi_0\otimes\eta,g,\zeta)\right),$$

where $\mathcal{K}_{Y'}$ is a subspace of $H^2(Y';\mathbb{Z})$ consisting of isomorphic line bundles η on Y' that pull back to the trivial line bundle on $\tilde{Y}'$ and $\eta|_{T^3}$ is trivial.

Then by [**18**] there are continuous maps $\partial_\infty:\mathcal{M}_{Y'}(\xi_0,g,\zeta)\to\chi_0(T^3)$ and $\partial'_\infty:\mathcal{M}_{Y'}(\xi_0\otimes\eta,g,\zeta)\to\chi_0(T^3)$.

Let D' and D'' be geometric representatives of $\mu(\mathrm{pt})\in H^2(\mathcal{M}_{Y'}(\xi_0,g,\zeta):\mathbb{Q})$ and $\mu'(\mathrm{pt})\in H^2(\mathcal{M}_{Y'}(\xi_0\otimes\eta,g,\zeta):\mathbb{Q})$, respectively.

Then we can define the relative Seiberg-Witten invariant on Y' by

$$\begin{aligned}\mathcal{S}W_{Y'}(\xi_0)&=\Sigma_{\xi'\in\wedge^+(\xi)}\sharp\left(\mathcal{M}_{Y'}(\xi_0,g,\zeta)\cap D'\cap\partial_\infty^{-1}(\theta_{\xi'})\right),\\ \mathcal{S}W_{Y'}(\xi_0\otimes\eta)&=\Sigma_{\xi_\eta'\in\wedge^+(\xi\otimes\tilde{\eta})}\sharp\left(\mathcal{M}_{Y'}(\xi_0\otimes\eta,g,\zeta)\cap D''\cap{\partial'_\infty}^{-1}(\theta_{\xi'_\eta})\right),\end{aligned}\tag{1}$$

where $\tilde{\eta}\to X$ is an extension of the bundle $\eta\to Y'$.

By Proposition 4.4, we can define the $\mathbb{Z}_2$-invariant, relative Seiberg-Witten invariant $\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)^{\mathbb{Z}_2}$ by

$$\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)^{\mathbb{Z}_2}=\Sigma_{\tilde{\xi}'\in\wedge^+(\tilde{\xi})}\sharp\left(\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0,\tilde{g},\tilde{\zeta})^{\mathbb{Z}_2}\cap D\cap(\tilde{\partial}_\infty)^{-1}(\tilde{\theta}_{\tilde{\xi}'})\right),$$

where $\pi^*\xi'=\pi^*\xi'_\eta=\tilde{\xi}'$.

By using the method of proof as in Theorem 2.2 of [**21**], for a generic, $\mathbb{Z}_2$-invariant, self-dual two-form $\tilde{\zeta}$, we have

$$\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)^{\mathbb{Z}_2}=\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)\mod 2.\tag{2}$$

Let $\det\tilde{\xi}'=\tilde{L}'$, $\det\xi'=L'$, and $\det\xi'_\eta=L'_\eta$.

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PROPOSITION 4.6. *$\tilde{\xi}' \in \wedge^+(\tilde{\xi})$ if and only if*

$$\xi' \in \wedge^+(\xi) \quad \textit{and} \quad \langle c_1(L'_\eta), \pi_* \tilde{b}\rangle > \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2 \times pt]\rangle.$$

Proof. For all $A \in \mathcal{A}(\tilde{L})$ and all points pt $\in \tilde{T}^2$, we have

$$\langle c_1(\tilde{L}), \tilde{b} + [D^2 \times \mathrm{pt}]\rangle = \int_{\tilde{b}+[D^2\times \mathrm{pt}]} \frac{i}{2\pi} F_A.$$

Since $\tilde{L}|_{D^2\times\tilde{T}^2}$ and $L|_{D^2\times T^2}$ are trivial and $\pi^*(\xi_0) \cong \tilde{\xi}_0$, we have

$$\langle c_1(\tilde{L}), \tilde{b} + [D^2 \times \mathrm{pt}]\rangle = \langle c_1(\tilde{L}_0), \tilde{b}\rangle,$$
$$\langle c_1(\tilde{L}_0), \tilde{b}\rangle = \langle \pi^* c_1(L_0), \tilde{b}\rangle = \langle c_1(L_0), \pi_*\tilde{b}\rangle = \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2 \times \mathrm{pt}]\rangle.$$

Thus we have

$$\langle c_1(\tilde{L}), \tilde{b} + [D^2 \times \mathrm{pt}]\rangle = \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2 \times \mathrm{pt}]\rangle. \tag{3}$$

Furthermore,

$$\begin{aligned} \langle c_1(\tilde{L}'), \tilde{b}\rangle &= \langle \pi^* c_1(L'), \tilde{b}\rangle = \langle c_1(L'), \pi_*\tilde{b}\rangle = \langle c_1(L'), b\rangle, \\ \langle c_1(\tilde{L}'), \tilde{b}\rangle &= \langle \pi^* c_1(L'_\eta), \tilde{b}\rangle = \langle c_1(L'_\eta), \pi_*\tilde{b}\rangle = \langle c_1(L'_\eta), b\rangle, \end{aligned} \tag{4}$$

where $\partial b = \gamma \in \ker i_*$.

By equations (3) and (4) we conclude that $\tilde{\xi}' \in \wedge^+(\xi)$ if and only if

$$\xi' \in \wedge^+(\xi) \quad \text{and} \quad \langle c_1(L'_\eta), \pi_*\tilde{b}\rangle > \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2 \times \mathrm{pt}]\rangle,$$

where $\pi^*\xi' = \pi^*\xi'_\eta = \tilde{\xi}'$. □

Let $\mathbb{O}_\eta = \{\xi'_\eta \in r^{-1}(\xi_0 \otimes \eta) | \langle c_1(L'_\eta, \pi_*\tilde{b}\rangle > \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2 \times \mathrm{pt}]\rangle\}$.

We now come to our main theorem.

THEOREM 4.7. *Let $\pi : X \to X'$ be a double cover branched along a torus T^2 with self-intersection 0 and $b_2^+(X'), b_2^+(X) > 2$. Suppose that $H_2(X;\mathbb{Z})$ has no 2-torsion and ξ is a Spin^c-structure on X' such that $\xi|_{D^2\times T^2}$ is trivial. Let $\tilde{\xi}$ be a Spin^c-structure on X whose restriction to $D^2 \times \tilde{T}^2$ is trivial, the determinant bundle $\tilde{L} \cong \pi^*L \otimes PD^{-1}[\tilde{T}^2]$ and $\tilde{\xi}|_{\tilde{Y}_0} \cong \pi^*(\xi|_{Y_0})$. Then we have a relation between the Seiberg-Witten invariants of X and those of X' such that*

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{X'}(\xi) + k(X', T^2, a) \mod 2,$$

where

$$k(X', T^2, a) = \Sigma_{\eta\in\mathcal{K}_{Y'}} \Sigma_{\xi'_\eta\in\mathbb{O}_\eta} \sharp\big(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta) \cap D'' \cap (\partial'_\infty)^{-1}\big(\theta_{\xi'_\eta}\big)\big).$$

Proof. By Proposition 4.6 we have a homeomorphism between smooth, compact, zero-dimensional spaces

$$\begin{aligned} &\amalg_{\tilde{\xi}'\in\wedge^+(\tilde{\xi})} (\mathcal{M}_{\tilde{Y}'}(\tilde{\xi}_0, \tilde{g}, \tilde{\zeta})^{\mathbb{Z}_2} \cap D \cap (\tilde{\partial}_\infty)^{-1}(\tilde{\theta}_{\tilde{\xi}'})) \\ &\quad \cong \amalg_{\xi'\in\wedge^+(\xi)} (\mathcal{M}_{Y'}(\xi_0, g, \zeta) \cap D' \cap (\partial_\infty)^{-1}(\theta_{\xi'}))\amalg, \\ &\amalg_{\eta\in\mathcal{K}_{Y'}} \amalg_{\xi'_\eta\in\mathbb{O}_\eta} \big(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta) \cap D'' \cap (\partial'_\infty)^{-1}\big(\theta_{\xi'_\eta}\big)\big), \end{aligned}$$

where $\pi^*\xi' = \pi^*\xi'_\eta = \tilde{\xi}'$.

Thus, under mod 2 we have

$$\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)^{\mathbb{Z}_2} = \mathcal{S}W_{Y'}(\xi_0) + \Sigma_{\eta\in\mathcal{K}_{Y'}} \Sigma_{\xi'_\eta\in\mathbb{O}_\eta} \sharp\big(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta)\cap D''\cap(\partial'_\infty)^{-1}\big(\theta_{\xi'_\eta}\big)\big). \quad (5)$$

The equation (2) implies that

$$\mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0)^{\mathbb{Z}_2} = \mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0) \mod 2. \quad (6)$$

By using Theorem 4.1 of [**18**] we have

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{\tilde{Y}'}(\tilde{\xi}_0), \quad \mathcal{S}W_{X'}(\xi) = \mathcal{S}W_{Y'}(\xi_0) \mod 2. \quad (7)$$

From equations (5), (6), and (7) we conclude that under mod 2,

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{X'}(\xi) + \Sigma_{\eta\in\mathcal{K}_{Y'}} \Sigma_{\xi'_\eta\in\mathbb{O}_\eta} \sharp\big(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta)\cap D''\cap(\partial'_\infty)^{-1}\big(\theta_{\xi'_\eta}\big)\big),$$

completing the proof. □

Remark 4.8. By definition, $\xi''\in\wedge^+(\xi\otimes\tilde{\eta})$ if and only if $\xi''\in r^{-1}((\xi\otimes\tilde{\eta})|_Y) = r^{-1}(\xi_0\otimes\eta)$ and $\langle c_1(\det\xi''), \pi_*\tilde{b}\rangle > \langle c_1(\det(\xi\otimes\tilde{\eta})), \pi_*\tilde{b} + \pi_*[D^2\times \mathrm{pt}]\rangle$.

Because we do not know the action of the line bundle $\tilde{\eta}$ over $D^2\times T^2$ and ξ'_η in Theorem 4.7 only satisfies

$$\langle c_1(L'_\eta), \pi_*\tilde{b}\rangle > \langle c_1(L), \pi_*\tilde{b} + \pi_*[D^2\times \mathrm{pt}]\rangle,$$

we conclude that in general, $\wedge^+(\xi\otimes\tilde{\eta}) \neq \mathbb{O}_\eta$ and

$$\Sigma_{\xi'_\eta\in\mathbb{O}_\eta}\sharp(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta)\cap D''\cap(\partial'_\infty)^{-1}(\theta(\xi'_\eta))) \neq \mathcal{S}W_{Y'}(\xi_0\otimes\eta).$$

Thus $\Sigma_{\eta\in\mathcal{K}_{Y'}}\Sigma_{\xi'_\eta\in\mathbb{O}_\eta}\sharp(\mathcal{M}_{Y'}(\xi_0\otimes\eta, g, \zeta)\cap D'\cap(\partial'_\infty)^{-1}(\theta_{\xi'_\eta}))$ cannot be extended to a Seiberg-Witten invariant $\mathcal{S}W_{X'}(\xi\otimes\tilde{\eta})$ and, as in [**21**], it is an invariant of (X', T^2, a), where $2a = [T^2]\in 2H_2(X';\mathbb{Z})$. □

Corollary 4.9. *Let X be a closed symplectic 4-manifold with $b_2^+(X) > 2$. Suppose that $\sigma : X\to X$ is an anti-symplectic involution with a torus T^2 as a fixed point set. Under the conditions for the Spin^c-structures ξ and $\tilde{\xi}$ of Theorem 4.7, we have a relation between the Seiberg-Witten invariants of X and the quotient $X/\sigma = X'$ with $b_2^+(X') > 2$ such that*

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{X'}(\xi) + k(X', T^2, a) \mod 2.$$

Proof. There is a branched double cover $\pi : X\to X'$ along $\pi(T^2) = T^2$, and so the required result follows. □

5. Applications. To find the relationship between the Seiberg-Witten invariants on X and X' of Theorem 4.7, we have to calculate the invariant $k(X', T^2, a)$. As in [**21**] we can show that $k(X', T^2, a) = 0$ for many cases.

Proposition 5.1. *Let X be a Kähler surface with $b_2^+(X) > 3$ and with the canonical class K_X satisfying $K_X^2 > 0$. Let $\sigma : X\to X$ be an anti-holomorphic involution with a smoothly embedded torus as a fixed point set. Then the Seiberg-Witten invariant on the*

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quotient X' is

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{X'}(\xi) \mod 2$$

for the Spin^c-structures $\tilde{\xi}$ and ξ of Theorem 4.7.

Proof. Consider a projection map $\pi : X \to X' = X/\sigma$. Then, by [**24**], we have $b_2^+(X) = 2b_2^+(X') + 1$ and $b_2^+(X') > 1$.

Since σ acts freely on Y' and

$$2\chi(Y') + 3\mathrm{Sign}(Y') = 2\chi(X') + 3\mathrm{Sign}(X') = K_{X'}^2 > 0,$$

by [**25**] there is no reducible or irreducible solution of the Seiberg-Witten equations over the cylindrical end space Y'. Thus the moduli space $\mathcal{M}_{Y'}(\xi_0 \otimes \eta, g, \mu)$ is empty and hence the invariant

$$k(X', T^2, a) = \Sigma_{\xi'_\eta \in \mathbb{O}} \Sigma_{\eta \in \mathcal{K}_{Y'}} \sharp\big(\mathcal{M}_{Y'}(\xi_0 \otimes \eta, g, \mu) \cap D' \cap \partial_\infty'^{-1}(\theta_{\xi'_\eta})\big) = 0,$$

for all cases. Thus the Seiberg-Witten invariant on the quotient X' is

$$\mathcal{S}W_X(\tilde{\xi}) = \mathcal{S}W_{X'}(\xi) \mod 2$$

for the Spin^c-structures $\tilde{\xi}$ and ξ in Theorem 4.7. □

In the case considered in [**21**], $g(\Sigma') > 1$ and $\Sigma' \cdot \Sigma' = 0$. They did not find an example such that $k(X', \Sigma', a) \neq 0$ mod 2 although they believe such an example should exist. When $g(\Sigma') = 1$ and $\Sigma' \cdot \Sigma' = 0$, there is an example such that $k(X', T^2, a) \neq 0$ mod 2.

EXAMPLE 5.2. Let $\sigma : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ be an involution defined by the diagonal complex conjugation. Then the fixed point set of σ is a torus and $\mathbb{C}P^1 \times \mathbb{C}P^1/\sigma = S^4$. See Section 6 of [**21**] for this construction.

Let X' be a closed symplectic 4-manifold with $b_2^+(X') > 1$. Now we take a connected sum $X = \mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2X'$ which is taken away from the branch set T^2. Then there is a double cover $X \to X' \sharp S^4 = X'$ branched along T^2.

By the Seiberg-Witten vanishing theorem [**22**], there is no Seiberg-Witten basic class on X. Thus we have $\mathcal{S}W_X(\tilde{\xi}) = 0$ and $\mathcal{S}W_{X'}(\xi) = k(X', T^2, a)$ mod 2.

Since X' is a closed symplectic 4-manifold with $b_2^+(X') > 1$, $\mathcal{S}W_{X'}(\xi) \neq 0$ mod 2 for a Seiberg-Witten basic class ξ. Thus we have

$$\mathcal{S}W_{X'}(\xi) = k(X', T^2, a) \neq 0 \mod 2.$$ □

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