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THE ATIYAH–PATODI–SINGER RHO INVARIANT AND SIGNATURES OF LINKS

ENRICO TOFFOLI

Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany
(enricotoffoli@gmail.com)

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Abstract Relations between the Atiyah–Patodi–Singer rho invariant and signatures of links have been known for a long time, but they were only partially investigated. In order to explore them further, we develop a versatile cut-and-paste formula for the rho invariant, which allows us to manipulate manifolds in a convenient way. With the help of this tool, we give a description of the multivariable signature of a link L as the rho invariant of some closed three-manifold Y_L intrinsically associated with L. We study then the rho invariant of the manifolds obtained by the Dehn surgery on L along integer and rational framings. Inspired by the results of Casson and Gordon and Cimasoni and Florens, we give formulas expressing this value as a sum of the multivariable signature of L and some easy-to-compute extra terms.

Keywords: rho invariant; aps; Atiyah–Patodi–Singer; Levine–Tristram; Cimasoni–Florens; multivariable signature; cut-and-paste; Dehn surgery

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1. Introduction

Given a closed, oriented manifold N of odd dimension, together with a representation $\alpha\colon \pi_1(N) \to U(n)$ for some $n \in \mathbb{N}$, we can consider the Atiyah–Patodi–Singer rho invariant $\rho_\alpha(N)$. This is a real number defined as the difference between the eta invariant of the twisted and the untwisted odd signature operator associated with any Riemannian metric on N, and it turns out to be independent of the choice of the metric [1, 2]. The rho invariant is thus defined as a spectral invariant, but it has the following fundamental property that relates it to well-studied topological invariants: if N is the boundary of a compact, oriented manifold M such that the representation α extends to $\pi_1(M)$, then it can be computed as

$$\rho_\alpha(N) = n\sigma(M) - \sigma_\alpha(M), \tag{1.1}$$

where $\sigma(M)$ denotes the ordinary signature of M, and $\sigma_\alpha(M)$ is the twisted signature associated with the chosen extension $\alpha\colon \pi_1(M) \to U(n)$.

In knot theory, rho invariants are used to give descriptions and generalizations of knot and link signatures. One of the basic observations is that, if S_K is the closed manifold obtained by 0-framed surgery on K and $\alpha\colon \pi_1(S_K) \to U(1)$ is the representation sending the meridian of K to $\omega \in U(1)$, then

$$\rho_\alpha(S_K) = -\sigma_K(\omega), \tag{1.2}$$

where σ_K is the Levine–Tristram signature function of K. Atiyah–Patodi–Singer rho invariants associated with higher-dimensional non-abelian representations were used in the knot theory by Levine [22, 23] and Friedl [16, 17] as obstructions to knot and link concordance. Multivariable signatures of links, defined by Cimasoni and Florens [9] using generalized Seifert surfaces, were given a description as twisted signatures of four manifolds and also employed as concordance invariants [9, 12, 13]. However, a description of them as Atiyah–Patodi–Singer rho invariants was not investigated thoroughly. One of the goals of this paper is to fill this gap.

Given the difficulty in computing rho invariants directly, it can be very useful to study their behaviour under modifications of the original manifold. One possible way to simplify manifolds is what we will call *cut-and-paste* through the rest of the paper. If a closed $(2k+1)$-dimensional manifold is split by a codimension-one closed manifold Σ into a union $X_1 \cup_\Sigma X_2$, we can often find a manifold X_0 with $\partial X_0 = -\Sigma$ such that $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$ are "simpler" than $X_1 \cup_\Sigma X_2$. Schematically, we write this modification as

$$X_1 \cup_\Sigma X_2 \quad \rightsquigarrow \quad X_1 \cup_\Sigma X_0 \;\sqcup\; -X_0 \cup_\Sigma X_2.$$

It is then useful to compare the rho invariant of $X_1 \cup_\Sigma X_2$ with the sum of the rho invariants of the two other manifolds. Recall that, when X_i is any of the three manifolds X_0, X_1, or X_2, the subspace

$$V^\alpha_{X_i} = \ker(H_k(\Sigma;\mathbb{C}^n_\alpha) \to H_k(X_i;\mathbb{C}^n_\alpha)).$$

is Lagrangian in $H_k(\Sigma;\mathbb{C}^n_\alpha)$ with respect to the symplectic form given essentially by the twisted intersection form (we will omit α from the notation when we are considering the trivial one-dimensional representation). In particular, we can compute their *Maslov triple index* $\tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2})$, which is an integer-valued function defined on triples of Lagrangian subspaces (see Definition 3.1). We prove the following cut-and-paste formula for the Atiyah–Patodi–Singer rho invariant.

Theorem 3.9. *Let X_1, X_2 and X_0 be compact, oriented manifolds of dimension $2k+1$ with $\partial X_1 = -\partial X_0 = -\partial X_2 =: \Sigma$, and let $\alpha\colon \pi_1(X_1 \cup_\partial X_2) \to U(n)$ be a representation that extends to $\pi_1(X_0)$. Then, for every choice of such an extension, we have*

$$\rho_\alpha(X_1 \cup_\Sigma X_2) = \rho_\alpha(X_1 \cup_\Sigma X_0) + \rho_\alpha(-X_0 \cup_\Sigma X_2) + C,$$

where

$$C = \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) - n\,\tau(V_{X_0}, V_{X_1}, V_{X_2}),$$

with the first Maslov triple index performed on $H_k(\partial X_1;\mathbb{C}^n_\alpha)$, and the second on $H_k(\partial X_1;\mathbb{C})$.

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An analogous cut-and-paste formula for the untwisted eta invariant was proved by Bunke [5, 2.5]. Formulas related to Theorem 3.9 were also discussed by Kirk and Lesch in connection with their gluing formulas for eta invariants for manifolds with boundary [21, Section 8.3]. In fact, our cut-and-paste formula can also be proved using their results (see [28, Section 2.3.4]). Here, however, we give a simple proof which does not involve rho invariants with manifolds with boundary, and it is based on Wall's non-additivity for the signature instead.

The second part of the paper is made up of applications of Theorem 3.9 in the context of link theory, which help relate multivariable signatures to rho invariants. We consider from now on only rho invariants of three-manifolds with one-dimensional representations of the fundamental group. As such representations factor through the first homology in a unique way, we can simply see them as representations of the first homology group. We recall that a k-component link $L = K_1 \cup \cdots \cup K_k$ is said to be *n-coloured* if it is considered together with a surjective map $c\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$, which partitions it naturally into n sublinks $L_1, \ldots, L_n$. A component K_i is then said to have colour $c(i)$. Given an n-coloured link L in S^3, the Cimasoni–Florens signature is a function

$$\sigma_L\colon \mathbb{T}^n_* \to \mathbb{Z},$$

where $\mathbb{T}^n_* := (S^1 \setminus \{1\})^n$. If $n = 1$, σ_L coincides with the Levine–Tristram signature function. Now, let X_L be the exterior of L, i.e. the complement of an open tubular neighbourhood of L in S^3. Then, X_L is a compact, oriented three manifolds with boundary, whose first homology group is a free abelian group generated by the meridians of L. Observe that $\mathbb{T}^n_*$ has a natural bijection with the set of representation $H_1(X_L; \mathbb{Z}^n) \to U(1)$ that send the meridians of components of the same colour s to a same value $\omega_s \in S^1 \setminus \{1\}$ (we call these *coloured representations*): given an element $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_*$, we have an *associated* coloured representation α defined by sending the meridian of a component K_i to to $\omega_{c(i)}$. The following result, which can be seen as a generalization of (1.2), expresses the multivariable signature of L as the rho invariant of a suitable closed three-manifold Y_L which only depends on L. This manifold Y_L is built by gluing the link exterior X_L together with a three-manifold with boundary obtained by plumbing punctured disks in a way that is prescribed by the linking numbers of L. For a precise definition of Y_L, see Construction 4.17.

Proposition 4.18. *Let L be an n-coloured link. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L; \mathbb{Z}) \to U(1)$ be the associated coloured representation. Then, α can be extended to a representation of $H_1(Y_L; \mathbb{Z})$ and, for any choice of an extension, we have*

$$\rho_\alpha(Y_L) = -\sigma_L(\omega).$$

The manifold Y_L has a very simple description in the case when L is *colour-to-colour algebraically split*, i.e. when the total linking number $\mathrm{lk}(L_s, L_t)$ is 0 for every pair of distinct colours s, t. Under this assumption, we prove two formulas relating the rho invariants of the closed manifolds obtained by the Dehn surgery on L with the multivariable signature of L. Given a link $L = K_1 \cup \cdots \cup K_k$ with a rational framing $r = (r_1, \ldots, r_k) \in \mathbb{Q}^k$,

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let Λ_r be the rational matrix with coefficients

$$\Lambda_{ij} = \begin{cases} \mathrm{lk}(K_i, K_j), & \text{if } i \neq j, \\ r_i, & \text{if } i = j. \end{cases}$$

As Λ_r is a symmetric matrix, we can compute its signature $\operatorname{sign}\Lambda_r$. Let $S_L(r)$ be the closed three-manifold obtained by the Dehn surgery on L along the rational framing r. We say that a representation $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ is compatible with r, if it extends to $H_1(S_L(r);\mathbb{Z})$. First, we focus on surgery along integral framings. In this context, a fairly general formula was given by Cimasoni and Florens [9, Theorem 6.7], extending to the multivariable setting a result of Casson and Gordon [7, Lemma 3.1] (see also [19, Theorem 3.6]). These formulas only consider representations with a finite image and are written in terms of the invariant $\sigma(N, \alpha)$ of Casson and Gordon (see § 2.3). Rewritten in terms of the rho invariant, the result of Cimasoni and Florens reads as follows.

Theorem 4.21 (Cimasoni–Florens). *Let L be a k-coloured k-component link. Let $q \in \mathbb{N}$ a positive integer and let $n_1, \ldots, n_k \in \{1, \ldots, p-1\}$ be integers, each of which is coprime with q. Let $\omega = (e^{2\pi i n_1/q}, \ldots, e^{2\pi i n_k/q}) \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Let g be a compatible integral framing on L. Then, we have*

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) + \sum_{i<j} \Lambda_{ij} + \operatorname{sign}\Lambda_g - \frac{2}{q^2}\sum_{i=1}^{k}(q-n_i)n_j\Lambda_{ij}.$$

Theorem 4.21 applies to the signature associated with the *maximal colouring*, where all distinct link components have different colours, but a formula for any colouring can be easily deduced from it. However, there are some restrictions on the values of ω that limit its use. Under the assumption of L being colour-to-colour algebraically split, we prove the following formula with no restrictions on the values of ω.

Theorem 4.24. *Let L be an n-coloured link which is colour-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Let g be a compatible integral framing on L. Then, we have*

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_g - 2\sum_{s=1}^{n} h_s\theta_s(1-\theta_s),$$

where, for each colour s, the values h_s and θ_s are determined by

$$h_s := \sum_{c(i)=c(j)=s} \Lambda_{ij}, \qquad \theta_s \in (0,1) \text{ such that } \omega_s = e^{2\pi i\theta_s}.$$

Theorem 4.24 is proved by first describing the multivariable signature as a rho invariant using Proposition 4.18, and then applying the cut-and-paste formula (Theorem 3.9) to modify Y_L into a disjoint union of $S_L(g)$ and lens spaces $L(h_s, 1)$, for which we can write the rho invariant very explicitly. The proof is completed by a careful computation of the Maslov triple index involved in the cut-and-paste formula.

Suppose now that a stronger assumption on the linking numbers is satisfied, namely that every link component K_i has total linking number 0 with all sublinks L_s such that $s \neq c(i)$. We say in this case that L is *component-to-colour algebraically split*. Then, as it is easy to verify, there is a particular integer framing f_L (the *Seifert framing*) which is compatible with all coloured representations $H_1(X_L;\mathbb{Z}) \to U(1)$. An immediate consequence of Theorem 4.24 is the following.

Corollary 4.28. *Let L be an n-coloured link which is component-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Then, α extends to $H_1(S_L(f_L);\mathbb{Z})$ and we have*

$$\rho_\alpha(S_L(f_L)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_{f_L}.$$

The analog to Corollary 4.28 for the onecoloured setting was used by Nagel and Powell in studying concordance properties of the Levine–Tristram signature [25]. Besides being a useful formula on its own, moreover, Corollary 4.28 is our starting point to prove a result that takes into account (non-integral) rational framings. This is expressed by the next and final result of this paper.

Theorem 4.31. *Let L be an n-coloured, k-component link that is component-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon \pi_1(X_L) \to U(1)$ be the associated coloured representation. Let r be a compatible rational framing on L. Then, we have*

$$\rho_\alpha(S_L(r)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_r - \sum_{i=1}^{k}(\rho(L(p_i,q_i),\omega_{c(i)}) + \operatorname{sgn}(p_i/q_i)),$$

where p_i, q_i are coprime integers such that $r_i - f_i = p_i/q_i$ (here, f_i is the i-th coefficient of the Seifert framing).

As in the proof Theorem 4.24, lens spaces arise from the cut-and-paste construction. In this case, we do not spell out the values of their rho invariant in the statement of the theorem, as this would make the formula more cumbersome. Note, however, that these values can always be easily computed (see Proposition 2.10).

Observe that onecoloured links are a special case of component-to-colour algebraically split links. In particular, Theorem 4.31 gives a new formula relating rho invariants with the Levine–Tristram signature.

Remark. Whenever not stated otherwise, all manifolds are assumed to be smooth.

Outline of the paper

In §2, we review the basics about twisted signatures and Atiyah–Patodi–Singer rho invariants. We illustrate then a formula for the rho invariant of a three-dimensional lens space. In §3, we review the Maslov triple index of Lagrangian subspaces and Wall's non-additivity theorem for the signature. We prove then our cut-and-paste formula (Theorem 3.9). In §4, we develop the applications in the knot theory. We first prove the basic formula relating rho invariants and multivariable signatures (Proposition 4.18), and then use it to show results about integer (Theorem 4.24) and rational (Theorem 4.31) Dehn surgery.

2. Twisted signatures and rho invariants

In § 2.1, we review the definition of twisted homology and twisted signatures and set notation for these. In § 2.2, we recall the basics about Atiyah–Patodi–Singer eta and rho invariants. In § 2.3, we underline the relation between these invariants and an invariant of Casson and Gordon. In § 2.4, we give a reinterpretation of a well-known computation for the rho invariant of three-dimensional lens spaces.

2.1. Twisted intersection forms and signatures

Let M be a connected, compact, oriented manifold of dimension $2k$ with a representation $\alpha\colon \pi_1(M) \to U(n)$ for some $n \in \mathbb{N}$. Let $\pi := \pi_1(M)$. Let $\widetilde{M}$ be the universal cover of M, so that π acts on the left on $C_*(\widetilde{M})$ by deck transformations, and on $\mathbb{C}^n$ through the representation α (we write $\mathbb{C}^n_\alpha$ for the left $\mathbb{Z}[\pi]$-module coming from this action). We consider the twisted homology groups

$$H_i(M; \mathbb{C}^n_\alpha) := H_i((\mathbb{C}^n_\alpha)^t \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})),$$
$$H^i(M; \mathbb{C}^n_{\xi^\alpha}) := H^i(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{M}), \mathbb{C}^n_\alpha)),$$

where the t denotes the fact that we are turning the left $\mathbb{Z}[\pi]$-module structure of $\mathbb{C}^n$ into a right one by means of the involution $g \to g^{-1}$ in π. Relative homology and cohomology groups are defined in a similar manner, as well as twisted homology groups with different $\mathbb{Z}[\pi]$-modules (see e.g [18, Definition A.2]).

Remark 2.1. If $M = M_1 \sqcup \cdots \sqcup M_N$ is a disjoint union of connected manifolds M_j, we will make the abuse of notation of writing $\alpha\colon \pi_1(M) \to U(n)$ to denote a collection of representations $\alpha_j\colon \pi_1(M_j) \to U(n)$. In this case, we define then

$$H_i(M; \mathbb{C}^n_\alpha) := \bigoplus_{j=1}^{N} H_i(M_j; \mathbb{C}^n_\alpha).$$

We recall the following facts.

Fact 1. Given a subset $N \subseteq M$, there is a well-defined cup product [18, Lemma A.11]

$$\cup\colon H^p(M, N; \mathbb{C}^n_\alpha) \times H^q(M, N; \mathbb{C}^n_{\overline{\alpha}}) \to H^{p+q}(M, N; \mathbb{C}^n_\alpha \otimes_{\mathbb{Z}} \mathbb{C}^n_{\overline{\alpha}}),$$

where $\overline{\alpha}$ denotes the complex-conjugate representation of α.

Fact 2. As the representation α is Hermitian, the standard Hermitian product of $\mathbb{C}^n$ gives rise to a well-defined map of $\mathbb{Z}[\pi]$-modules $\mathbb{C}^n_\alpha \otimes_{\mathbb{Z}} \mathbb{C}^n_{\overline{\alpha}} \to \mathbb{C}$ (where $\mathbb{C}$ is the $\mathbb{Z}[\pi]$-module associated with the trivial one-dimensional representation). This induces a group homomorphism

$$H^{p+q}(M, N; \mathbb{C}^n_\alpha \otimes_{\mathbb{Z}} \mathbb{C}^n_{\overline{\alpha}}) \to H^{p+q}(M, N; \mathbb{C}).$$

Composing the above map with the cup product of Fact 1, and observing that $H^q(M, N; \mathbb{C}^n_{\overline{\alpha}})$ coincides with the complex-conjugate vector space $\overline{H^q(M, N; \mathbb{C}^n_\alpha)}$, we get a bilinear map

$$\cup_{\mathbb{C}}\colon H^p(M, N; \mathbb{C}^n_\alpha) \times \overline{H^q(M, N; \mathbb{C}^n_\alpha)} \to H^{p+q}(M, N; \mathbb{C}).$$

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Fact 3. There are twisted Poincaré duality isomorphisms [18, Theorem A.15]

$$\mathrm{PD}\colon H^p(M,\partial M;\mathbb{C}^n_\alpha)\xrightarrow{\sim} H_{2k-p}(M;\mathbb{C}^n_\alpha).$$

Fact 4. Since the representation α is unitary, the evaluation map in twisted homology gives rise to an isomorphism [11, Lemma 2.7]

$$\mathrm{ev}_{\mathbb{C}}\colon H^q(M;\mathbb{C}^n_\alpha)\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{C}}(\overline{H_q(M;\mathbb{C}^n_\alpha)},\mathbb{C}).$$

Using the first three facts, we define the *twisted intersection form* as the sesquilinear form

$$I^\alpha_M\colon H_k(M;\mathbb{C}^n_\alpha)\times H_k(M;\mathbb{C}^n_\alpha)\to\mathbb{C},$$

given by

$$I^\alpha_M(a,b)=\langle\mathrm{PD}^{-1}(a)\cup_{\mathbb{C}}\overline{\mathrm{PD}^{-1}(b)},[M]\rangle$$

(compare e.g. with [18, Section 13]). As for the ordinary intersection pairing, the properties of the cup product make the form I^α_M Hermitian if k is even and skew-Hermitian if k is odd. This leads to the following definition.

Definition 2.2. Let M be a compact oriented manifold of dimension $2k$ with a representation $\alpha\colon\pi_1(M)\to U(n)$. The *signature of M twisted by α* is the integer

$$\sigma_\alpha(M):=\begin{cases}\mathrm{sign}(I^\alpha_M), & \text{if } k \text{ is even},\\ \mathrm{sign}(i\,I^\alpha_M), & \text{if } k \text{ is odd},\end{cases}$$

where sign denotes the signature of a Hermitian form.

Observe that the form I^α_M can be defined alternatively using the sequence of maps

$$H_k(M;\mathbb{C}^n_\alpha)\xrightarrow{j_*}H_k(M,\partial M;\mathbb{C}^n_\alpha)\xrightarrow{\mathrm{PD}^{-1}}H^k(M;\mathbb{C}^n_\alpha)\xrightarrow{\mathrm{ev}_{\mathbb{C}}}\mathrm{Hom}(\overline{H_k(M;\mathbb{C}^n_\alpha)},\mathbb{C})$$

(see e.g. [12, Definition 2.7]), where $\mathrm{ev}_{\mathbb{C}}$ is the map discussed in Fact 4. From this point of view, we see that the radical of I^α_M coincides with the kernel of j_*, as the two other maps in the composition are isomorphisms. As a consequence, I^α_M is non-degenerate whenever M is closed.

2.2. Basics on the rho invariant

Let N be a closed, oriented, Riemannian manifold of dimension $2k+1$, with a representation $\alpha\colon\pi_1(N)\to U(n)$. Let $E_\alpha\to N$ be the associated flat vector bundle, and consider the subspace

$$\Omega^{\mathrm{ev}}(N,E_\alpha):=\bigoplus_{q=0}^{k}\Omega^{2q}(N,E_\alpha)$$

of twisted differential forms of even degree (see [2, Section 2] for details). Let D^α_N be the *twisted odd signature operator*, i.e. the first-order differential operator on $\Omega^{\mathrm{ev}}(N,E_\alpha)$

defined by

$$D_N^\alpha \phi := (-1)^{q+1} i^k (\star d - d\star)\phi, \qquad \text{for } \phi \in \Omega^{2q}(N, E_\alpha).$$

The operator D_N^α can be extended to a self-adjoint elliptic operator with discrete spectrum and, by the results of Atiyah, Patodi, and Singer [3, Theorem 4.5], its eta function

$$\eta(s) = \sum_{\substack{\lambda \in \mathrm{Spec}(D_N^\alpha) \\ \lambda \neq 0}} \mathrm{sgn}\lambda \, |\lambda|^{-s}$$

has a meromorphic extension which is holomorphic at $s = 0$, leading to the *eta invariant*

$$\eta_\alpha(N) := \eta(0) \in \mathbb{R}.$$

We say that a compact Riemannian manifold M has metric of *product form* near the boundary, if there exists a neighbourhood of ∂M that is isometric to $(-\varepsilon, 0] \times \partial M$ with the product metric. The main result about the eta invariant of the twisted signature operator is the following [2, Theorem 2.2].

Theorem 2.3 (Atiyah–Patodi–Singer). *Let M be an even-dimensional compact, oriented manifold with $\partial M = N$, equipped with Riemannian metric of product form near N, and let $\alpha\colon \pi_1(M) \to U(n)$ be a representation. Then*

$$\sigma_\alpha(M) = n \int_M L(p) - \eta_\alpha(N),$$

where $L(p)$ is the Hirzebruch L-polynomial in the Pontryagin forms of M.

Note that both summands on the right-hand term depend on the Riemannian metric on N. We shall not dwell upon the geometrical significance of the integral of the L-polynomial, as it is going to get simplified soon.

Remark 2.4. The restriction of $\alpha\colon \pi_1(M) \to U(n)$ to a representation of $\pi_1(N)$ is made by composing α with the natural map $\pi_1(N) \to \pi_1(M)$. If N is not connected, a map $\pi_1(N_1) \to \pi_1(M)$ for each connected component N_i of N can be obtained by choosing appropriate paths between base points. The restriction $\alpha\colon \pi_1(N) \to U(n)$ must be interpreted as the collection of the representations $\pi_1(N_i) \to U(n)$ for all connected components N_i. The invariant $\eta_\alpha(N)$ is defined in this case as the sum of the eta invariants of the N_i's.

For the eta invariant associated with the untwisted odd signature operator D_N on $\Omega^{\mathrm{ev}}(N, \mathbb{C})$, we use the notation $\eta(N) := \eta(D_N)$. Following Atiyah, Patodi, and Singer [2], we are now going to define the rho invariant.

Definition 2.5. Let N be a closed, oriented manifold of odd dimension, and let $\alpha\colon \pi_1(N) \to U(n)$ be a representation. The *Atiyah–Patodi–Singer rho invariant* of N associated with α is the real number

$$\rho_\alpha(N) := \eta_\alpha(N) - n\,\eta(N),$$

where the eta invariants are computed for an arbitrary Riemannian metric on N.

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We shall see in a moment that the difference $\eta_\alpha(N) - n\,\eta(N)$ is independent of the Riemannian metric, so that $\rho_\alpha(N)$ is well defined. Moreover, $\rho_\tau(N)$ is 0 for trivial representations τ, and it satisfies

$$\rho_\alpha(-N) = -\rho_\alpha(N). \tag{2.1}$$

The main theorem about the rho invariant is the following.

Theorem 2.6 (Atiyah–Patodi–Singer). *(i) $\rho_\alpha(N)$ is independent of the Riemannian metric on N.*

(ii) If M is a compact, oriented manifold with $\partial M = N$ and α extends to M, then

$$\rho_\alpha(N) = n\,\sigma(M) - \sigma_\alpha(M).$$

Proof. Both statements are easy consequences of Theorem 2.3. See [2, Theorem 2.4]. □

We state one more well-known result that will turn useful later on.

Proposition 2.7. *Let Σ be a closed, oriented surface, and let $\psi\colon \pi_1(\Sigma \times S^1) \to U(1)$ be a representation. Then $\rho_\psi(\Sigma \times S^1) = 0$.*

Proof. See e.g. [12, Lemma 4.2]. □

2.3. Rho invariants and Casson–Gordon invariants

We will now review the definition of an invariant of Casson and Gordon and relate it to the Atiyah–Patodi–Singer rho invariant. Let N be a closed, oriented three manifolds, and let $\alpha\colon H_1(N;\mathbb{Z}) \to U(1)$ be a representation. Assume that the image of α is finite. Using a bordism argument, Casson and Gordon observe that there exists a compact, oriented four-manifold W such that the boundary of W is the disjoint union of r copies of N for some $r \in \mathbb{N}$ (we will write $\partial W = rN$) with a representation $\alpha'\colon H_1(W;\mathbb{Z}) \to U(1)$ that restrict to α on each boundary component [8, p. 183]. They define then an invariant as

$$\sigma(N,\alpha) := \frac{1}{r}(\sigma_{\alpha'}(W) - \sigma(W)). \tag{2.2}$$

By additivity of the signature and again some bordism theory, they show that the invariant $\sigma(N,\,\alpha)$ is independent of the choice of W and of the extension α' [8, p. 183–184] (see also [12, Corollary 2.11] for a more detailed version of their proof). Using the Atiyah–Patodi–Singer index theorem, their invariant can be immediately reinterpreted as an Atiyah–Patodi–Singer rho invariant. We state this explicitly for further reference, albeit it is surely known to the experts.

Proposition 2.8. *Let N be a closed, oriented three-manifold, and let $\alpha\colon H_1(N;\mathbb{Z}) \to U(1)$ be a representation with finite image. Then, we have*

$$\sigma(N,\alpha) = -\rho_\alpha(N).$$

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Proof. Let W be a compact, oriented four-manifold with $\partial W = rN$, with a representation $\alpha'\colon H_1(W;\mathbb{Z}) \to U(1)$ that restricts to α as discussed above, so that $\sigma(N, \alpha)$ is described by (2.2). Using Theorem 2.6, on the other hand, we have

$$r\rho_\alpha(N) = \rho_\alpha(rN) = \sigma(W) - \sigma_{\alpha'}(W).$$

Comparing this with (2.2), we obtain the desired statement. □

2.4. The rho invariant of lens spaces

Given coprime integers p and q, the three-dimensional lens space $L(p, q)$ can be built as the union of two solid tori Y_1, Y_2 along any orientation-reversing diffeomorphism $f\colon \partial Y_2 \to \partial Y_1$ such that

$$f_*(\mu_2) = -q\mu_1 + p\lambda_1,$$

where μ_1, μ_2 are the meridians, respectively, of Y_1, Y_2, and λ_1 is a longitude of Y_1. This construction is well defined for both positive and negative values of p and q. For positive p it coincides, up to some explicit orientation-preserving diffeomorphism, with the classical definition of $L(p, q)$ as a quotient of S^3 (compare with [15, Lemma 91.3]). In general, there are orientation-preserving diffeomorphisms

$$L(-p, q) \cong L(p, -q) \cong -L(p, q).$$

In particular, for both positive and negative p, we have an identification of $\pi_1(L(p, q))$ with $\mathbb{Z}/p$, and under this identification, the element $[1] \in \mathbb{Z}/p$ corresponds to the generator of $\pi_1(Y_1)$.

Rho invariants of three-dimensional lens spaces can be computed explicitly. As every representation $\alpha\colon \mathbb{Z}/p \to U(n)$ can be written as a direct sum of one-dimensional representations, we shall focus on one-dimensional representations. Moreover, we shall exclude the case $p = 0$, as $L(0, 1)$ is diffeomorphic to $S^2 \times S^1$ and its rho invariant is 0 for any choice of α. We observe in this case that the representations $\alpha\colon \mathbb{Z}/p \to U(1)$ are in a natural bijection with the set of $|p|^{\text{th}}$ roots of unity: to each such root ω, we associate the representation α_ω sending 1 to ω.

Notation 2.9. Given a $|p|^{\text{th}}$ root of unity ω, we write

$$\rho(L(p, q), \omega) := \rho_{\alpha_\omega}(L(p, q)).$$

Formulas for the rho invariants of lens spaces were given since the original paper of Atiyah, Patodi and Singer [2, Proposition 2.12]. Introducing the periodic sawtooth function $(\!(\cdot)\!) : \mathbb{R} \to (-\frac{1}{2}, \frac{1}{2})$ defined by

$$(\!(x)\!) := \begin{cases} x - \lfloor x \rfloor - \dfrac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases},$$

we give the following description.

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Proposition 2.10. *Let p, q be two coprime integers with $p \neq 0$, and let $\zeta = e^{2\pi i/p}$. Then, for $k \in \{0, 1, \ldots, |p|-1\}$, we have*

$$\rho(L(p,q),\zeta^{kq}) = -4\sum_{j=1}^{k-1}\left(\!\left(\frac{qj}{p}\right)\!\right) - 2\left(\!\left(\frac{qk}{p}\right)\!\right).$$

Proof. We first suppose p, $q > 0$. Set $z := e^{2\pi i n/p}$, with $n = \gcd(p, k)$, and set $r := k/n$. Then, we have $\zeta^{kq} = z^{rq}$. The representation sending 1 to ω^q has as its image the set of m^{th} roots of unity, with m such that $p = mn$. In this setting, Casson and Gordon [8, pp.187–188] proved that

$$\sigma(L(p,q),\alpha_\omega) = 4\left(\text{area}\Delta\left(nr, \frac{rq}{m}\right) - \text{int}\Delta\left(nr, \frac{rq}{m}\right)\right), \tag{2.3}$$

where $\Delta(x, y)$ is the triangle with vertices $(0, 0)$, $(x, 0)$ and (x, y) and the number $\text{int}(\Delta(x, y))$ is given by counting:

- $+1$ for every point of $\mathbb{Z}^2$ that lies in the interior of $\Delta(x, y)$;
- $+1/2$ for every point of $\mathbb{Z}^2$ that lies in the interior of its edges;
- $+1/4$ for every point of $\mathbb{Z}^2 \setminus \{(0, 0)\}$ that coincides with one of the vertices.

Using Proposition 2.8 and the definition of r, we rewrite (2.3) as

$$\rho(L(p,q),\zeta^{kq}) = 4(\text{int}\Delta(k, \frac{kq}{p}) - \text{area}\Delta(k, \frac{kq}{p})). \tag{2.4}$$

We will now express the right-hand term of (2.4) in a more explicit way. First of all, it is clear that $4\,\text{area}\,\Delta(k, \frac{kq}{p}) = \frac{2q}{p}k^2$. Moreover, we can count the lattice points inside the triangle by following vertical lines $\{(x, y) \,|\, x = j\}$, for $j = 1, \ldots, k$, and then summing over j. We obtain

$$\begin{aligned} 4\text{int}\Delta\left(k, \frac{kq}{p}\right) &= 4\sum_{j=1}^{k-1}\left(\frac{1}{2} + \left\lfloor\frac{jq}{p}\right\rfloor\right) + 4\left(\frac{1}{4} + \frac{1}{2}\left\lfloor\frac{kq}{p}\right\rfloor\right) \\ &= 2k - 1 + 4\sum_{j=1}^{k-1}\left\lfloor\frac{jq}{p}\right\rfloor + 2\left\lfloor\frac{kq}{p}\right\rfloor. \end{aligned}$$

Taking the difference, we obtain thus

$$\rho(L(p,q),\zeta^{kq}) = -\frac{2q}{p}k^2 + 2k - 1 + 4\sum_{j=1}^{k-1}\left\lfloor\frac{jq}{p}\right\rfloor + 2\left\lfloor\frac{kq}{p}\right\rfloor. \tag{2.5}$$

Expanding the expression in the statement, it is now immediate to see that it coincides with (2.5). As the sawtooth function $(\!(\cdot)\!)$ is odd, we see that both sides of the identity

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change sign when either p or q is changed of sign. As a consequence, the result keeps holding for non-positive choices of p and q. $\square$

Corollary 2.11. *Let n be any integer, and let $\omega \in U(1)$ be an $|n|^{\text{th}}$ root of unity. Then, we have*

$$\rho(L(n,1),\omega) = 2n\theta(1-\theta) - \operatorname{sgn}(n),$$

where $\theta \in [0,\,1)$ is such that $\omega = e^{2\pi i\theta}$.

Proof. For $n = 0$, we have $\rho(L(n,\,1),\,\omega) = 0$ for all $\omega \in U(1)$ because $L(0,\,1) = S^2 \times S^1$, so that the result is trivially satisfied in this case. For $n > 0$, there has to be a $k \in \{0,\,1,\,\cdots n-1\}$ such that $\theta = k/n$, with $k \in \{0,\,1,\,\cdots n-1\}$. From (2.5), we easily see that

$$\rho(L(n,1),\omega) = -\frac{2k^2}{n} + 2k - 1 = 2n\theta(1-\theta) - 1,$$

and the desired formula is satisfied in this case. For negative n, we obtain now from the last equation that

$$\rho(L(n,1),\omega) = -\rho(L(-n,1),\omega) = 2n\theta(1-\theta) + 1,$$

which leads to the general formula of the statement. $\square$

Remark 2.12. Proposition 2.10 can be used to write rho invariants of lens spaces as a difference of Dedekind–Rademacher sums. See [28, Theorem 3.3.27].

3. The cut-and-paste formula

In § 3.1, we recall the definition of the Maslov triple index of three Lagrangian subspaces in a complex symplectic space. In § 3.3, we review a (non-)additivity theorem of Wall for signatures of manifolds under some fairly general notion of gluing. In Section 3.4, we prove the cut-and-paste formula for the Atiyah–Patodi–Singer rho invariants (Theorem 3.9), which is the main result of this section.

3.1. Complex symplectic spaces and the Maslov triple index

A *complex symplectic space* is a pair $(H,\,\omega)$ such that H is a finite-dimensional complex vector space, and $\omega\colon H \times H \to \mathbb{C}$ is a non-degenerate skew-Hermitian form, called the *symplectic form*. We shall often omit ω from the notation and simply call H a complex symplectic space. We recall that a subspace $L \subseteq H$ is *Lagrangian* if it coincides with its orthogonal complement with respect to the symplectic form ω. Let $\mathcal{L}ag(H)$ denote the set of all Lagrangian subspaces of H. We are now going to define a function

$$\tau\colon \mathcal{L}ag(H) \times \mathcal{L}ag(H) \times \mathcal{L}ag(H) \to \mathbb{Z}.$$

Given three Lagrangian subspaces $L_1,\,L_2,\,L_3 \in \mathcal{L}ag(H)$, it is immediate to verify that the sesquilinear form

$$\begin{aligned}\psi_{L_1L_2L_3}\colon (L_1+L_2)\cap L_3 \times (L_1+L_2)\cap L_3 &\to \mathbb{C}\\ (a_1+a_2, b_1+b_2) &\mapsto \omega(a_1,b_2)\end{aligned}$$

(with a_1, $b_1 \in L_1$ and a_2, $b_2 \in L_2$) is well defined and Hermitian. In particular, we can give the following definition.

Definition 3.1. The *Maslov triple index* of (L_1, L_2, L_3) is the integer

$$\tau(L_1, L_2, L_3) := \operatorname{sign}\psi_{L_1 L_2 L_3}.$$

The Maslov triple index satisfies several elementary properties, among which are the following two.

Proposition 3.2.

(i) Let L_1, L_2, $L_3 \in \mathcal{L}ag(H)$, and let α be a permutation of the set $\{1, 2, 3\}$. Then

$$\tau(L_{\alpha(1)}, L_{\alpha(2)}, L_{\alpha(3)}) = \operatorname{sgn}(\alpha)\, \tau(L_1, L_2, L_3).$$

In particular, the Maslov triple index vanishes whenever two of the Lagrangians coincide.

(ii) Let L_1, L_2, L_3, $L_4 \in \mathcal{L}ag(H)$. Then, τ satisfies the cocycle equation

$$\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0.$$

Proof. See e.g. [6, Section 8] for proofs in the real symplectic setting. These can be transferred verbatim to the complex symplectic one. □

Example 3.3. Suppose that (H, ω) is a complex symplectic space of dimension 2. Let (μ, λ) be an ordered basis in which ω is represented by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. such that

$$\omega(\mu, \mu) = \omega(\lambda, \lambda) = 0, \quad \omega(\mu, \lambda) = -1$$

we shall call such pair a *symplectic basis*. Then, it is easy to verify that a one-dimensional subspace is Lagrangian if and only if it is the span of some vector $a\mu + b\lambda$ with a, $b \in \mathbb{R}$. We set in this case the notation

$$\tau(v_1, v_2, v_3) := \tau(\operatorname{Span}_{\mathbb{C}}\{v_1\}, \operatorname{Span}_{\mathbb{C}}\{v_2\}, \operatorname{Span}_{\mathbb{C}}\{v_3\}).$$

Using the definition of the Maslov triple index, we easily compute that

$$\tau(\mu, \lambda, a\mu + b\lambda) = -\operatorname{sgn}(ab).$$

3.2. Complex symplectic spaces and twisted homology

In the applications, complex symplectic spaces will arise from the following setting. Let Σ be a $2k$-dimensional closed, oriented manifold, and let $\alpha\colon \pi_1(\Sigma) \to U(n)$ be a representation. As we have seen, the twisted intersection form on $H := H_k(\Sigma; \mathbb{C}^n_\alpha)$ is Hermitian if k is even, and it is skew-Hermitian if k is odd. Moreover, it is non-degenerate

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because Σ is closed. We can thus always consider the non-degenerate, skew-Hermitian form

$$\omega := \begin{cases} I^{\alpha}_{\Sigma}, & \text{if } k \text{ is odd,} \\ i\, I^{\alpha}_{\Sigma}, & \text{if } k \text{ is even,} \end{cases}$$

which makes the pair (H, ω) a complex symplectic space. We introduce the following notation.

Notation 3.4. Given a compact, connected $(2k+1)$-dimensional manifold X with a representation $\alpha\colon \pi_1(X) \to U(n)$, we set

$$V^{\alpha}_{X} = \ker(H_k(\partial X; \mathbb{C}^n_{\alpha}) \to H_k(X; \mathbb{C}^n_{\alpha})).$$

A well-known argument based on Poincaré duality shows that V^{α}_X is a Lagrangian subspace of $H_k(\partial X; \mathbb{C}^n_{\alpha})$ with respect to ω (compare with [14, Lemma 2.14] and [28, Proposition 1.4.6]). We can thus give the following definition.

Definition 3.5. Let X be compact, connected manifold of dimension $2k+1$, and let $\alpha\colon \pi_1(X) \to U(n)$ be a representation. We refer to V^{α}_X as the *canonical Lagrangian* associated with X and α.

3.3. Wall's non-additivity of the signature

We shall now review a result of Wall. We start with some notation that will be useful throughout the paper.

Notation 3.6. Given two topological spaces X, Y with a common subspace A, we set

$$X \cup_A Y := (X \sqcup Y)\,/\!\sim,$$

where $\sim$ is the relation that identifies every element in $A \subseteq X$ with its copy in $A \subseteq Y$. We say that $X \cup_A Y$ is obtained by *gluing* X and Y along A. If X and Y are manifolds with $\partial X = \partial Y$, we also write $X \cup_{\partial} Y$ to denote the gluing along their common boundary.

Suppose that M is a compact, oriented manifold of even dimension that is split as

$$M = M_1 \cup_{X_0} M_2$$

along a properly embedded submanifold X_0 of codimension 1, which is allowed to have boundary Σ. Let $X_1 := \partial M_1 \setminus \operatorname{int}(X_0)$ and $X_2 := \partial M_2 \setminus \operatorname{int}(X_0)$. As unoriented manifolds, we have then $\partial X_0 = \partial X_1 = \partial X_2 = \Sigma$ and

$$\partial M_1 = X_1 \cup_{\Sigma} X_0, \quad \partial M_2 = X_0 \cup_{\Sigma} X_2, \quad \partial M = X_1 \cup_{\Sigma} X_2. \tag{3.1}$$

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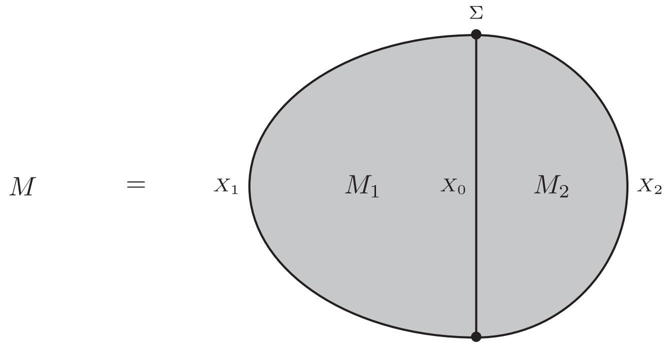


We pick on X_1 the orientation coming from being a codimension 0 submanifold of ∂M_1, and we give Σ the orientation coming from being the boundary of X_1. Suppose now that $\alpha\colon \pi_1(M) \to U(n)$ is a representation. In our setting, Σ is (up to orientation) the common boundary of X_0, X_1 and X_2, so that the canonical Lagrangians $V^\alpha_{X_0}$, $V^\alpha_{X_1}$ and $V^\alpha_{X_2}$ all live in the same space $H_k(\Sigma; \mathbb{C}^n_\alpha)$. In particular, it makes sense to compute their Maslov triple index. In fact, the following result holds.

Theorem 3.7 (Wall's non-additivity). *Let M be a closed, oriented, even-dimensional manifold, and let $\alpha\colon \pi_1(M) \to U(n)$ be a representation. Then, if M decomposes as $M = M_1 \cup_{X_0} M_2$ as above, we have*

$$\sigma_\alpha(M) = \sigma_\alpha(M_1) + \sigma_\alpha(M_2) - \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}).$$

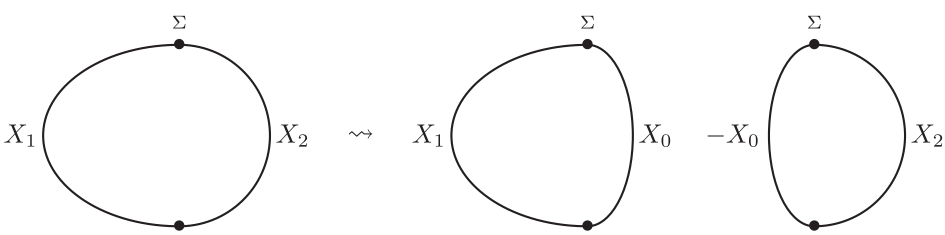
Remark 3.8. Theorem 3.7 was originally proved by Wall for the untwisted signature [30] (see also the paper of Py [27, (3.2)] for a more detailed proof), and it can be easily checked that the result extends to twisted signatures. See [12–14] for further references and uses of the twisted version of the theorem.

3.4. The cut-and-paste formula for the rho invariant

Suppose to have a closed, oriented $(2k+1)$-dimensional manifold that is split by a codimension-one closed manifold Σ, yielding a decomposition $X_1 \cup_\Sigma X_2$. Let X_0 be a compact, oriented manifold with $\partial X_0 = -\Sigma$. Then, we can replace $X_1 \cup_\Sigma X_2$ with the disjoint union of $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$. We will call this manipulation *cut-and-paste*. Schematically, we have

$$X_1 \cup_\Sigma X_2 \quad \rightsquigarrow \quad X_1 \cup_\Sigma X_0 \ \sqcup \ -X_0 \cup_\Sigma X_2,$$

and pictorially we can represent the operation as in the next figure.



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Suppose now that $\alpha\colon \pi_1(X_1 \cup_\Sigma X_2) \to U(n)$ is a representation. In particular, α is defined on $\pi_1(X_1)$ and on $\pi_1(\Sigma)$. In order to have rho invariants of $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$ to be compared to $\rho_\alpha(X_1 \cup_\Sigma X_2)$, we need to extend α to the fundamental groups of these manifolds. This is possible if and only if we can construct an extension of α from $\pi_1(\Sigma)$ to $\pi_1(X_0)$: then, using Seifert–Van Kampen's theorem, this will be patched with $\alpha\colon \pi_1(X_1) \to U(n)$ to produce representations $\pi_1(X_1 \cup_\Sigma X_0) \to U(n)$ and $\pi_1(-X_0 \cup_\Sigma X_2) \to U(n)$. For simplicity, we will use the same notation α for all of these representations. Then, we want to compute the correction term C in the formula

$$\rho_\alpha(X_1 \cup_\Sigma X_2) = \rho_\alpha(X_1 \cup_\Sigma X_0) + \rho_\alpha(-X_0 \cup_\Sigma X_2) + C. \tag{3.2}$$

Now, if $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$ bound manifolds W_1 and W_2 such that the representation extends, then Wall's theorem, together with the Atiyah–Patodi–Singer signature theorem, tells us how to compute the correction term. Namely, in that case, we have

$$C = \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) - n\,\tau(V_{X_0}, V_{X_1}, V_{X_2}). \tag{3.3}$$

The content of the main result of this section is that the correction term C of (3.2) is always given by (3.3), no matter whether the manifolds W_1 and W_2 exist. This should be compared with an analogous result for the untwisted eta invariant [5, 2.5].

Theorem 3.9. *Let X_1, X_2 and X_0 be compact, oriented manifolds of dimension $2k+1$ with $\partial X_1 = -\partial X_0 = -\partial X_2 =: \Sigma$, and let $\alpha\colon \pi_1(X_1 \cup_\partial X_2) \to U(n)$ be a representation that extends to $\pi_1(X_0)$. Then, for every choice of such an extension, we have*

$$\rho_\alpha(X_1 \cup_\Sigma X_2) = \rho_\alpha(X_1 \cup_\Sigma X_0) + \rho_\alpha(-X_0 \cup_\Sigma X_2) + C,$$

where

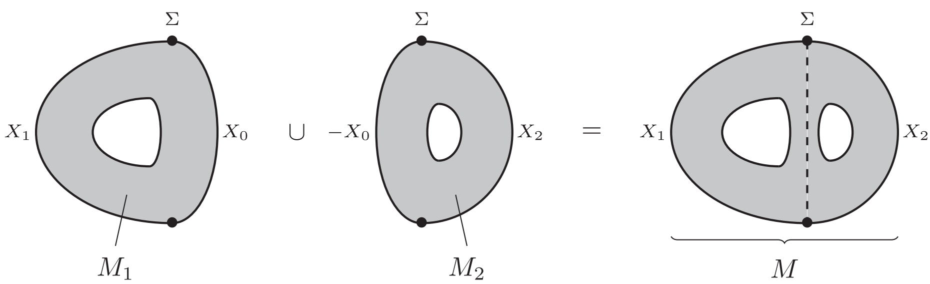
$$C = \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) - n\,\tau(V_{X_0}, V_{X_1}, V_{X_2}),$$

with the first Maslov triple index performed on $H_k(\partial X_1; \mathbb{C}^n_\alpha)$, and the second on $H_k(\partial X_1; \mathbb{C})$.

Proof. Consider the oriented manifolds

$$M_1 := [0,1] \times (X_1 \cup_\Sigma X_0), \quad M_2 := [0,1] \times (-X_0 \cup_\Sigma X_2).$$

We glue then M_1 with M_2 along $\{1\} \times X_0$, obtaining a topological oriented manifold M to which α extends.



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The boundary of M can be described topologically as

$$\partial M = (-(X_1 \cup_\Sigma X_0) \sqcup -(-X_0 \cup_\Sigma X_2)) \sqcup (X_1 \cup_\Sigma X_2), \tag{3.4}$$

and we can equip M with a smooth structure such that (3.4) is satisfied in the smooth sense [29, 15.10.3]. Thanks to Theorem 2.6 (ii), we have then

$$\rho_\alpha(\partial M) = n\,\sigma(M) - \sigma_\alpha(M). \tag{3.5}$$

By (3.4), the left-hand term is given by

$$\rho_\alpha(\partial M) = -\rho_\alpha(X_1 \cup_\Sigma X_0) - \rho_\alpha(-X_0 \cup_\Sigma X_2) + \rho_\alpha(X_1 \cup_\Sigma X_2). \tag{3.6}$$

By Wall's non-additivity (Theorem 3.7), we can compute the twisted and untwisted signature of M as

$$\begin{aligned}\sigma(M) &= \sigma(M_1) + \sigma(M_2) - \tau(V_{X_0}, V_{X_1}, V_{X_2}),\\ \sigma_\alpha(M) &= \sigma_\alpha(M_1) + \sigma_\alpha(M_2) - \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}).\end{aligned}$$

The manifolds M_1 and M_2 are of the form $[0, 1] \times X$, with X a closed manifold. In particular, they admit an orientation-reversing self-diffeomorphism defined by

$$\begin{aligned}[0,1]\times X &\to [0,1]\times X\\ (t,x) &\mapsto (1-t,x).\end{aligned}$$

As orientation-reversing diffeomorphisms have the effect of changing sign to the signature, this shows that the ordinary signatures of M_1 and M_2 vanish. Since the above diffeomorphism is trivial on the fundamental group, the same argument can be applied to the twisted signatures, which therefore also vanish. The computation of $\sigma(M)$ and $\sigma_\alpha(M)$ is hence reduced to

$$\sigma(M) = -\tau(V_{X_0}, V_{X_1}, V_{X_2}), \quad \sigma_\alpha(M) = -\tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we get the desired formula. □

Remark 3.10. An alternative approach to proving Theorem 3.9 would be by using gluing formulas for rho invariants for manifolds with boundary, as defined by Kirk and Lesch [20, 21] (the discussion in [21, Section 8.3] might be hinting in this direction). This approach allows to prove the cut-and-paste formula at the level of eta invariants, leading to a slightly stronger result, albeit at the cost of more sophisticated tools to be introduced. See [28, Section 2.3.4] for details about this point of view.

4. Signatures of links and rho invariants

In § 4.1, we introduce rationally framed links and set up some notation and easy results. In § 4.2, we introduce the multivariable signatures of Cimasoni and Florens and recall a four-dimensional description for them. In § 4.3, we prove Proposition 4.18, which reinterprets the multivariable signature of a coloured link as the rho invariants of some closed three-manifold associated with the link. In § 4.4, we prove Theorem 4.24, which is a formula

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relating the multivariable signature of a link with the rho invariant of the three manifolds obtained by integer surgery on the link. In § 4.5, we prove Theorem 4.31, which under some additional hypotheses does the same for rational surgery.

Remark 4.1. In this section, given a manifold X, we will only deal with one-dimensional unitary representations of $\pi_1(X)$. As $U(1)$ is an abelian group, every representation $\alpha'\colon \pi_1(X) \to U(1)$ factors through the abelianization $\mathrm{ab}\colon \pi_1(X) \to H_1(X;\mathbb{Z})$, and we can thus focus on representations $\alpha\colon H_1(X;\mathbb{Z}) \to U(1)$. We will normally write ρ_α and H_α to denote rho invariants and twisted homology associated with the representation $\alpha' = \alpha \circ \mathrm{ab}$.

Remark 4.2. A straightforward computation in twisted homology leads to the following well-known fact: for the two-dimensional torus T^2, we have $H_*(T^2;\mathbb{C}^n_\alpha) = 0$ for all non-trivial representation $\alpha\colon H_1(T;\mathbb{Z}) \to U(1)$. In this section, we will always use Theorem 3.9 in the situation where Σ is a disjoint union of two-dimensional tori and where the restriction of α to the first homology of these is non-trivial. As a consequence, the Maslov triple index in twisted homology will always be 0.

4.1. Links and framings

Let $L = K_1 \cup \cdots \cup K_k$ be an oriented link in S^3 (from now on, just a *link*). By removing from S^3 the interior of a closed tubular neighbourhood $N(L)$, we get its *link exterior*

$$X_L := S^3 \setminus \mathrm{int}(N(L)).$$

The link exterior X_L is a compact, oriented three-manifold, whose boundary is a union of tori: to each link component $K_i \subseteq L$, there corresponds a boundary component $T_i = -\partial(N(K_i))$ (this is the orientation coming from being part of the boundary of X_L, and it is the one we shall always consider). The link component K_i determines the following two elements:

- the *meridian* of K_i is the only element $\mu_i \in H_1(T_i;\mathbb{Z})$ whose image in $H_1(N(K_i);\mathbb{Z})$ is 0 and such that $\mathrm{lk}(\mu_i,\, K_i) = 1$;
- the *standard longitude* of K_i is the only element $\lambda_i^s \in H_1(T_i;\mathbb{Z})$ whose image in $H_1(N(K_i);\mathbb{Z})$ is homologous to K_i and such that $\mathrm{lk}(\lambda_i^s,\, K_i) = 0$.

For a knot K, we shall often just use the notation T, μ and λ for the boundary torus, the meridian and the standard longitude. Observe that the algebraic intersection of these two elements is given by

$$\mu_i \cdot \lambda_i^s = -1. \tag{4.1}$$

We shall often consider the images of the above elements into homology with rational or complex coefficients without changing their names.

Definition 4.3. A *rational framing* on a link $L = K_1 \cup \cdots \cup K_k$ is a k-tuple of rational numbers $r = (r_1,\, \ldots,\, r_k)$. The pair $(L,\, r)$ is called a *rationally framed link*. If all r_i's are

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integers, we say that r is an *integer framing*. The *framed longitudes* of a rationally framed link (L, r) are the elements

$$\lambda_i := \lambda_i^s + r_i\mu_i \in H_1(T_i; \mathbb{Q}).$$

Without changing notation for them, we shall now consider the images of the meridians and framed longitudes in the homology of the link complement X_L. It is an elementary well-known fact that $H_1(X_L; \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by the meridians, and that each standard longitude satisfies

$$\lambda_i^s = \sum_{j\neq i} \mathrm{lk}(K_i, K_j)\, \mu_j \in H_1(X_L; \mathbb{Z}). \tag{4.2}$$

Next, we define the following matrix associated with a rationally framed link.

Definition 4.4. The *framed linking matrix* of a framed k-component link (L, r) is the symmetric matrix $\Lambda_r = (\Lambda_{ij})_{i,j} \in \mathbb{Q}^{k\times k}$ defined by

$$\Lambda_{ij} = \begin{cases} \mathrm{lk}(K_i, K_j), & \text{if } i \neq j, \\ r_i, & \text{if } i = j. \end{cases}$$

Example 4.5. The *Seifert framing* on a link $L = K_1 \cup \cdots \cup K_k$ is the integer framing $(f_1, \cdots, f_k)$ defined by

$$f_i := -\sum_{j\neq i} \mathrm{lk}(K_i, K_j).$$

In particular, the coefficients of its framed linking matrix satisfy

$$\Lambda_{ii} = -\sum_{j\neq i} \Lambda_{ij}.$$

The framed longitudes $\lambda_i = \lambda_i^s + f_i\mu_i$ associated with the Seifert framing correspond to the intersections of a Seifert surface with the boundary tori T_i.

From (4.2), together with the definition of the framed longitudes and of the framed linking matrix, it follows now immediately that the framed longitudes in the rational homology of X_L are equal to

$$\lambda_i = \sum_{j=1}^{k} \Lambda_{ij}\, \mu_j \in H_1(X_L; \mathbb{Q}). \tag{4.3}$$

For the computations involving the Maslov triple index, a good understanding of the complex symplectic space $H_1(\partial X_L; \mathbb{C})$ is required. The next result summarizes some easy facts that we will need later.

Lemma 4.6. *Let (L, r) be a k-component rationally framed link. Then:*

(i) the collection $\{\mu_1, \dots \mu_k,\ \lambda_1, \dots \lambda_k\}$ forms a basis for $H_1(\partial X_L; \mathbb{C})$ which satisfies

$$\begin{cases} \mu_i \cdot \mu_j = \lambda_i \cdot \lambda_j = 0, \\ \mu_i \cdot \lambda_j = -\delta_{ij} \end{cases} \qquad \text{for all } i, j;$$

(ii) the canonical Lagrangian V_{X_L} can be described explicitly as

$$V_{X_L} = \mathrm{Span}_{\mathbb{C}}\{v_1, \dots, v_k\}, \quad \text{where } v_i = \lambda_i - \sum_{s=1}^{k} \Lambda_{is}\mu_s;$$

(iii) the subspaces $\mathcal{M} := \mathrm{Span}_{\mathbb{C}}\{\mu_1, \dots, \mu_k\}$ and $\mathcal{L}_r = \mathrm{Span}_{\mathbb{C}}\{\lambda_1, \dots, \lambda_k\}$ are Lagrangians, and their triple Maslov index with the canonical Lagrangian is given by

$$\tau(\mathcal{M}, \mathcal{L}_r, V_{X_L}) = \mathrm{sign}\Lambda_r.$$

Proof. (i) is an immediate consequence of the definition of the framed meridians together with (4.1). (ii) is an immediate consequence of (4.3). The fact that $\mathcal{M}$ and $\mathcal{L}_r$ are Lagrangians is obvious from (i). In order to prove (iii), we compute the Maslov triple index using the definition. As $\mathcal{M}$ and $\mathcal{L}_r$ are transverse, we have $(\mathcal{M} + \mathcal{L}_r) \cap V_{X_L} = V_{X_L}$, and every generator v_i can be written in a unique, obvious way as the sum of an element in $\mathcal{M}$ and one in $\mathcal{L}_r$. By Definition 3.1, then, $\tau(\mathcal{M}, \mathcal{L}_r, V_{X_L})$ is the signature of the Hermitian form $\psi\colon V_{X_L} \times V_{X_L} \to \mathbb{C}$ defined on the basis elements of V_{X_L} by

$$\psi(v_i, v_j) = \left(-\sum_{s=1}^{k} \Lambda_{is}\mu_s\right) \cdot v_j = -\sum_{s=1}^{k} \Lambda_{is}(\mu_s \cdot \lambda_j) = \Lambda_{ij}.$$

It follows that $\tau(\mathcal{M}, \mathcal{L}_r, V_{X_L}) = \mathrm{sign}\,\Lambda_r$, as it was desired. □

Given a rationally framed link (L, r), we can consider the closed manifold $S_L(r)$ obtained by the *Dehn surgery* along the framing. This is done in the following way: for each link component K_i, we choose coprime integers (p_i, q_i) such that $p_i/q_i = r_i$, and glue a solid torus Y_i to X_L along the boundary component T_i in such a way that the meridian of the solid torus is identified with the element $p_i\mu_i + q_i\lambda_i^s \in H_1(T_i; \mathbb{Z})$. In particular, $H_1(S_L(r); \mathbb{Z})$ can be described as a quotient of $H_1(X_L; \mathbb{Z})$.

Definition 4.7. Given a link L with a representation $\alpha\colon H_1(X_L; \mathbb{Z}) \to U(1)$, we say that a rational framing r on L is *compatible* with α if α factors through $H_1(S_L(r); \mathbb{Z})$.

Remark 4.8. From the definition of surgery, it is clear that a rational framing $r = (r_1, \dots, r_k)$ with $r_i = p_i/q_i$ as above is compatible with α if and only if $\alpha(p_i\mu_i + q_i\lambda_i^s) = 1$ for all i. Using (4.2), we see that in terms of the coefficients of the framed linking matrix we have

$$r \text{ is compatible with } \alpha \iff \prod_{j=1}^{k} \alpha(\mu_j)^{q_i\Lambda_{ij}} = 1\ \forall i.$$

Note that, in general, given a representation α, there might be no rational framing that is compatible with it.

4.2. Coloured links and signatures

A *n-colouring* on a link L, for $n \in \mathbb{N}$, is a partition of its components into n non-empty sublinks. Given a k-component link $L = K_1 \cup \cdots \cup K_k$, we identify the colouring with a surjective function $c\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$. The latter is the set of colours, and for every $1 \leq s \leq n$, we define

$$L_s := \bigcup_{c(j)=s} K_j$$

to be the sublink *of colour* s. The pair (L, c) is called an *n-coloured* link. We shall systematically omit c (which has to be considered as fixed) and simply call L an n-coloured link.

Notation 4.9. For $n \in \mathbb{N}$, set $\mathbb{T}^n := (U(1))^n$ and $\mathbb{T}^n_* := (U(1) \setminus \{1\})^n$.

Using a generalization of the concept of Seifert surfaces (called *C-complexes*), Cimasoni and Florens defined a multivariable version of the Levine–Tristram signature of a link [9]. Given an n-coloured link L, their multivariable signature is a function

$$\sigma_L\colon \mathbb{T}^n_* \to \mathbb{Z},$$

which coincides for $n = 1$ with the Levine–Tristram signature function. We recall now the following four-dimensional description of the multivariable signature. We first need a definition.

Definition 4.10. A *bounding surface* for an n-coloured link L is a union $F = F_1 \cup \cdots \cup F_n$ of properly embedded, locally flat, compact, oriented surfaces $F_i \subseteq D^4$ with $\partial F_i = L_i \in \partial D^4 = S^3$ and that only intersect each other transversally in double points.

A bounding surface for L can be obtained for example by pushing the interior of a C-complex into the interior of D^4 (see e.g. [10, Section 3] for details). Given a bounding surface $F = F_1 \cup \cdots \cup F_n \subseteq D^4$ for a link L, we can take a small tubular neighbourhood $N(F_i)$ of each surface F_i and define the *exterior* of F in D^4 as the four-manifold with boundary

$$W_F := D^4 \setminus (N(F_1) \cup \cdots \cup N(F_n)).$$

It is easy to show that $H_1(W_F; \mathbb{Z})$ is freely generated by the meridians of the surfaces $F_1, \ldots, F_n$ (the meridian of F_i being the image in $H_1(W_F; \mathbb{Z})$ of any of the meridians of L_i). The following description of the Cimasoni–Florens signature is known [12, Proposition 3.5].

Proposition 4.11. *Let L be a an n-coloured link in S^3 and let $F = F_1 \cup \cdots \cup F_n$ be a bounding surface for L. Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(W_F; \mathbb{Z}) \to U(1)$ be the representation that sends the meridian of F_i to ω_i. Then, we have*

$$\sigma_L(\omega) = \sigma_\alpha(W_F).$$

We conclude this part with a couple of definitions.

Definition 4.12. Let $L = K_1 \cup \cdots \cup K_k$ be an n-coloured link. The *coloured Seifert framing* on L is the integer framing $f_L = (f_1, \ldots, f_k)$ given by

$$f_i := - \sum_{\substack{j \neq i \text{ s.t.} \\ c(j)=c(i)}} \operatorname{lk}(K_i, K_j).$$

In other words, it is the framing obtained by providing each coloured sublink L_i with its Seifert framing (see e.g. 4.5).

Observe that the coloured Seifert framing on L needs not coincide with the Seifert framing of the underlying link. In fact, its framed longitudes correspond to the intersection of ∂X_L with a C-complex. From the definition of the coloured Seifert framing, it is immediate to see that the coefficients of the associated framed linking matrix satisfy

$$\forall\, 1 \leq i \leq k: \qquad \sum_{\substack{j \text{ s.t.} \\ c(j)=c(i)}} \Lambda_{ij} = 0. \tag{4.4}$$

Definition 4.13. Given an n-coloured link L, a representation $\alpha\colon H_1(X_L; \mathbb{Z}) \to U(1)$ is said to be *coloured* if it sends meridians of the same colour to the same value, i.e. if

$$c(i) = c(j) \implies \alpha(\mu_i) = \alpha(\mu_j).$$

Given an element $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n$, the coloured representation α defined by $\alpha(\mu_i) := \omega_{c(i)}$ is said to be the representation *associated* to ω.

It is immediate to see that the above association gives a natural bijection between elements $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n$ and coloured representations $\alpha\colon H_1(X_L; \mathbb{Z}) \to U(1)$.

4.3. Multivariable signatures as rho invariants

As the untwisted signature of W_F is 0 (see for example [12, Proposition 3.3]), Proposition 4.11 is equivalent to the formula

$$\sigma_L(\omega) = \sigma_\alpha(W_F) - \sigma(W_F).$$

In particular, by the Atiyah–Patodi–Singer theorem, we have

$$\sigma_L(\omega) = -\rho_\alpha(\partial W_F). \tag{4.5}$$

We will give now a more explicit description of ∂W_F, and see how to replace it with a manifold which is independent of the choice of F.

We start by recalling the following construction, which is a special case of plumbing. Let Γ be a graph whose set of vertices is $\{1, \ldots, n\}$ and such that:

- each vertex i is decorated by a compact, oriented surface Σ_i (or, equivalently, by pair of natural numbers $[g_i, r_i]$ corresponding to the genus and number of boundary components of Σ_i);
- each edge is decorated by a number $\varepsilon = \pm 1$.

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In the following, we shall refer to a graph with the decorations described above as a *plumbing graph*. We construct then an oriented three-manifold P_Γ by the following process.

(1) For each edge with endpoints i and j, remove a small open disk from Σ_i and one from Σ_j. Let $\Sigma'_1, \ldots, \Sigma'_j$ be the resulting surfaces.

(2) For each edge with endpoints i and j and decoration $\varepsilon = \pm 1$, glue the three-manifolds $\Sigma'_i \times S^1$ and $\Sigma'_j \times S^1$ along the boundary components coming from the two corresponding removed disks, according to the diffeomorphism $\varphi\colon S^1 \times S^1 \to S^1 \times S^1$ given by $\varphi(x, y) = (y^\varepsilon, x^\varepsilon)$.

(3) Set

$$P_\Gamma := \Big(\bigsqcup_{i=1}^{n} \Sigma'_i \times S^1 \Big) / \sim,$$

where $\sim$ is the equivalence relation given by the above gluings.

This construction coincides with that of [12, Section 4]. With respect to the general plumbing construction [26], it corresponds to the special case of all Euler numbers equal to 0, and our definition of plumbing graph also reflects this specialization. The boundary of P_Γ is a disjoint union of $r = \sum_i r_i$ tori. These tori maintain a preferred product structure, and we can describe the boundary of P_Γ as

$$\partial P_\Gamma = \bigsqcup_{i=1}^{n} \partial \Sigma_i \times S^1. \tag{4.6}$$

As in [12], we give the following definition.

Definition 4.14. Given a plumbing graph Γ, the *total weight* of a pair of vertices $\{s, t\}$, denoted by $p_\Gamma(s, t)$, is the integer obtained as the sum of the ± 1-decorations of all the edges with endpoints s and t. If all total weights of Γ are 0, the plumbing graph Γ is called *balanced*.

Example 4.15. Let $F = F_1 \cup \cdots \cup F_n \subseteq D^4$ be a bounding surface for L. The boundary of W_F then is given up to orientation-preserving diffeomorphism as

$$\partial W_F = X_L \cup_\partial (-P_{\Gamma_F}), \tag{4.7}$$

where Γ_F is a graph whose vertices $\{1, \ldots, n\}$ are decorated by the surfaces F_i's, and whose edges correspond to the intersection points between them, with the sign of the intersection as decoration (see [12, Example 4.12]). Observe that the total weights of Γ_F are given by

$$p_{\Gamma_F}(s,t) = F_s \cdot F_t = \mathrm{lk}(L_s, L_t).$$

The boundary of $-P_{\Gamma_F}$, which can be described as in (4.6), is identified in the gluing (4.7) with ∂X_L as follows. The boundary piece $-\partial F_s \times S^1$ is identified with the boundary tori

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of colour s in ∂X_L, in such a way that:

(1) the classes of the S^1-factors are glued to the meridians;

(2) the classes of the boundary components of F_s are glued to the framed longitudes associated with the coloured Seifert framing.

As we have seen in (4.5), the multivariable signature of L can be expressed up to sign as the rho invariant of ∂W_F. In turn, as it appears from Example 4.15, ∂W_F can be described as the union of X_L with some plumbed three-manifold which depends on the choice of a bounding surface for L. In the next construction, we will build a plumbing graph Γ_L associated with L. The associated closed three-manifold Y_L, obtained by gluing P_{Γ_L} with X_L, will play the role of ∂W_F with the advantage of being unequivocally determined by the link L.

Notation 4.16. Given a link L in S^3, let $|L| \in \mathbb{N}$ be its number of components.

Construction 4.17. Given an n-coloured link L, we construct the associated plumbing graph Γ_L in the following way:

- the set of vertices is $\{1, \ldots n\}$, and the vertex i is decorated by a genus-0 surface with $|L_i|$ boundary components;
- between each pair of distinct vertices i, j there are exactly $|\mathrm{lk}(L_i, L_j)|$ edges, and they are all decorated by $\varepsilon = \mathrm{sign}(\mathrm{lk}(L_i, L_j))$.

In other words, we are plumbing spheres with the appropriate number of punctures along the smallest graph Γ whose total weights satisfy the condition $p_\Gamma(i, j) = \mathrm{lk}(L_i, L_j)$. Then, we form a closed three-manifold Y_L as

$$Y_L := X_L \cup_\partial (-P_{\Gamma_L}).$$

The identification along the boundary is defined in the exact same way as the one arising in Example 4.15, i.e. identifying the classes corresponding to the S^1-factors of $-\partial P_{\Gamma_L}$ with the meridians of L of the appropriate colour, and the classes corresponding to the boundary components of the punctured sphere with the longitudes associated with the coloured Seifert framing.

Using some of the ideas of [12] together with the cut-and-paste formula of § 2, we will now show that the multivariable signature can be written as the rho invariant of Y_L.

Proposition 4.18. *Let L be an n-coloured link. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Then, α can be extended to a representation of $H_1(Y_L;\mathbb{Z})$ and, for any choice of an extension, we have*

$$\rho_\alpha(Y_L) = -\sigma_L(\omega).$$

Proof. Let $F = F_1 \cup \cdots \cup F_n \subseteq D^4$ be a bounding surface for L. Then, as α is a coloured representation, it extends to a representation $H_1(W_F;\mathbb{Z}) \to U(1)$ that, for all colour s, sends the meridian of F_s to $\omega_s \in U(1)$. By (4.5), we have hence

$$\rho_\alpha(\partial W_F) = -\sigma_L(\omega). \tag{4.8}$$

As we have seen in Example 4.15, we have $\partial W_F = X_L \cup_\partial (-P_{\Gamma_F})$, where Γ_F is a plumbing graph determined by F. We perform now cut-and-paste by replacing P_{Γ_F} with P_{Γ_L}. Schematically, this is

$$X_L \cup_\partial (-P_{\Gamma_F}) \quad \rightsquigarrow \quad X_L \cup_\partial (-P_{\Gamma_L}) \ \sqcup \ P_{\Gamma_L} \cup_\partial (-P_{\Gamma_F}). \tag{4.9}$$

The manifold $P_{\Gamma_L} \cup_\partial (-P_{\Gamma_F})$ can be seen as the plumbing along the graph Γ whose vertices $\{1, \ldots, n\}$ are decorated by the closed surfaces $\Sigma_s \cup_\partial (-F_s)$ and whose set of edges is the union of all edges of Γ_L and Γ_F, with the decorations of the edges of Γ_F changed of sign. In particular, (4.9) can be rewritten as

$$\partial W_F \quad \rightsquigarrow \quad Y_L \ \sqcup \ P_\Gamma. \tag{4.10}$$

Observe that, by construction, for each pair of vertices $\{s, t\}$ we have

$$p_{\Gamma_L}(s,t) = p_{\Gamma_F}(s,t), \tag{4.11}$$

so that

$$p_\Gamma(s,t) = p_{\Gamma_L}(s,t) - p_{\Gamma_F}(s,t) = 0$$

i.e. Γ is balanced.

We will now prove that the representation α can be extended to $H_1(Y_L;\mathbb{Z})$. In the gluings, the boundary of X_L is identified with the boundaries of P_{Γ_F} and P_{Γ_L}, leading to natural maps

$$\varphi\colon H_1(\partial X_L;\mathbb{Z}) \to H_1(P_{\Gamma_F};\mathbb{Z}), \quad \psi\colon H_1(\partial X_L;\mathbb{Z}) \to H_1(P_{\Gamma_L};\mathbb{Z}) \tag{4.12}$$

induced by the inclusions. Standard Mayer–Vietoris computations, together with the equality (4.11), show that $\ker\varphi$ and $\ker\psi$ coincide, as both are generated by the following elements (compare with [12, Lemma 4.7]):

(i) the differences $\mu_i - \mu_j$ with $c(i) = c(j)$;

(ii) for each colour s, the element

$$\sum_{c(i)=s} \lambda_i - \sum_{t\neq s} \mathrm{lk}(L_s, L_t)\mu_{j_t},$$

where μ_{j_t} is any meridian of colour t.

As α extends to $H_1(W_F;\mathbb{Z})$, it is also defined on $H_1(\partial W_F;\mathbb{Z}) = H_1(P_{\Gamma_F};\mathbb{Z})$. In particular, α has to be trivial on $\ker\varphi$ (this can be verified using the explicit description of the generators). From the fact that $\ker\varphi = \ker\psi$, we see then that α also admits an extension to $H_1(P_{\Gamma_L};\mathbb{Z})$, because $U(1)$ is divisible.

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Pick any extension $\alpha\colon H_1(P_{\Gamma_L};\mathbb{Z}) \to U(1)$ of α. We can now apply the cut-and-paste formula of Theorem 3.9 to (4.10), obtaining

$$\rho_\alpha(\partial W_F) = \rho_\alpha(Y_L) + \rho_\alpha(P_\Gamma) - \tau(V_{P_{\Gamma_L}}, V_{X_L}, V_{P_{\Gamma_F}}) \tag{4.13}$$

(the Maslov triple index in twisted homology is 0 because all ω_i's are non-trivial by assumption; see Remark 4.2). As we have seen, the plumbing graph Γ is balanced. As a consequence, we have $\rho_\alpha(P_\Gamma) = 0$ by a computation of Conway, Nagel and the author [12, Proposition 4.10]. The Lagrangians $V_{P_{\Gamma_F}}$ and $V_{P_{\Gamma_L}}$ are identified under the gluing to

$$V_{P_{\Gamma_F}} = \ker\varphi \otimes \mathbb{C}, \quad V_{P_\Gamma} = \ker\psi \otimes \mathbb{C},$$

where φ and ψ are the maps defined in (4.12). From the fact that $\ker\varphi = \ker\psi$ it follows then that $V_{P_{\Gamma_F}} = V_{P_\Gamma}$, and hence we have $\tau(V_{X_L}, V_{P_{\Gamma_F}}, V_{P_{\Gamma_L}}) = 0$. The equality (4.13) gets thus rewritten as

$$\rho_\alpha(\partial W_F) = \rho_\alpha(Y_L).$$

Substituting this into (4.8), the proof is complete. □

We introduce now the following definitions.

Definition 4.19. Let $L = K_1 \cup \cdots \cup K_k$ be an n-coloured link. Then

(i) if $\mathrm{lk}(L_s, L_t) = 0$ for all pairs (s, t) of distinct colours, we say that L is *colour-to-colour algebraically split*;

(ii) if every link component K_i satisfies $\mathrm{lk}(K_i, L_s) = 0$ for all $s \neq c(i)$, we say that L is *component-to-colour algebraically split*.

Of course, being component-to-colour algebraically split is a stronger condition than being colour-to-colour algebraically split. Any onecoloured link is component-to-colour algebraically split, and so is any link with vanishing linking numbers, no matter what colouring it is assigned.

Remark 4.20. If L is colour-to-colour algebraically split, then Y_L has a very simple description. In fact, in this case, Γ_L is a graph with no edges, and thus the associated plumbed three-manifold is

$$P_{\Gamma_L} = \bigsqcup_{s=1}^{n} \Sigma_s \times S^1,$$

where Σ_s is a sphere with $|L_s|$ punctures. As a consequence of Proposition 4.18, the multivariable signature is hence (up to sign) just the rho invariant of the closed three-manifold obtained by gluing these products $\Sigma_s \times S^1$ to the link exterior X_L. This holds, in particular, if L is onecoloured, so that we can always express the Levine–Tristram signature of L as

$$\sigma_L(\omega) = -\rho_\alpha(X_L \cup_\partial (-\Sigma \times S^1)),$$

where Σ is a punctured sphere. In the case of a knot, this gives the well-known description of the Levine–Tristram signature as the rho invariant of the manifold obtained by 0-framed surgery. In the next two sections, we will study the relationship between rho invariants and the Dehn surgery in higher generality.

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4.4. Integral surgery

We will now study the value of the rho invariant of manifolds obtained by integral surgery on a link L. We start by recalling the following result of Cimasoni and Florens [9, Theorem 6.7].

Theorem 4.21 (Cimasoni–Florens). *Let L be a k-coloured k-component link. Let $q \in \mathbb{N}$ a positive integer and let $n_1, \ldots, n_k \in \{1, \ldots, p-1\}$ be integers, each of which is coprime with q. Let $\omega = (e^{2\pi i n_1/q}, \ldots, e^{2\pi i n_k/q}) \in (S^1 \setminus \{1\})^n$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Let g be a compatible integral framing on L. Then, we have*

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) + \sum_{i<j} \Lambda_{ij} + \mathrm{sign}\Lambda_g - \frac{2}{q^2}\sum_{i=1}^{k}(q-n_i)n_j\Lambda_{ij}.$$

Remark 4.22. The result of Cimasoni and Florens was originally written in terms of the Casson–Gordon invariant of § 2.3. We have translated it into a result about the rho invariant by using Proposition 2.8.

Remark 4.23. Observe that formulas about any colouring of L can be extracted from Theorem 4.21, as the signature function associated with any colouring can be easily deduced from the one associated with the maximal colouring [9, Proposition 2.5]. For the onecolouring, this gives back a result of Casson and Gordon [7, Lemma 3.1] about the Levine–Tristram signature.

We would like to remove the restrictions on the values of ω in the statement of Theorem 4.21. In order to be able to do so, we need to assume some conditions about the linking numbers between components of different colours.

The main result of this section is the following, which holds in the case of colour-to-colour algebraically split links.

Theorem 4.24. *Let L be an n-coloured link which is colour-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Let g be a compatible integral framing on L. Then, we have*

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) + \mathrm{sign}\Lambda_g - 2\sum_{s=1}^{n} h_s\theta_s(1-\theta_s),$$

where, for each colour s, $\theta_s \in (0,1)$ is such that $\omega_s = e^{2\pi i\theta_s}$, and h_s is the sum of all the coefficients of the framed linking matrix of the sublink L_s.

Proof. Set $P := P_{\Gamma_L}$. Let k be the number of components of L. We perform the cut-and-paste illustrated schematically by

$$X_L \cup_\partial Y \quad \rightsquigarrow \quad X_L \cup_\partial (-P) \;\sqcup\; P \cup_\partial Y, \tag{4.14}$$

where $Y = Y_1 \sqcup \cdots \sqcup Y_k$ is a disjoint union of k solid tori, glued along the framing g, and P is glued as prescribed by Construction 4.17. As L is colour-to-colour algebraically split,

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by Remark 4.20, we have

$$P = \bigsqcup_{s=1}^{n} \Sigma_s \times S^1,$$

where, for each colour s, Σ_s is a sphere with $|L_s|$ punctures. The manifold $P \cup_\partial Y$ is thus the disjoint union of the n closed manifolds obtained by capping all of the $\Sigma_s \times S^1$'s appropriately with solid tori.

Claim 1. Up to an orientation-preserving diffeomorphism, we have

$$P \cup_\partial Y = -\bigsqcup_{s=1}^{n} L(h_s, 1),$$

in such a way that the element $1 \in \mathbb{Z}/h_s = \pi_1(L(h_s, 1))$ is given by the class $[S^1] \in H_1(\Sigma_s \times S^1; \mathbb{Z})$.

We postpone for the moment the proof of this claim. As a consequence of Claim 1, (4.14) can be rewritten up to orientation-preserving diffeomorphism as

$$S_L(g) \quad \rightsquigarrow \; Y_L \;\sqcup\; -\bigsqcup_{s=1}^{n} L(h_s, 1).$$

Observe that the restriction of α to $H_1(\Sigma_s \times S^1)$ sends the class $[S^1]$ to ω_s, as the boundary circles of the form $\{p_i\} \times S^1$ are identified with the meridians of the link. The representation α extends thus to $L(h_s, 1)$ in such a way that, for each colour s, the element $1 \in \mathbb{Z}/h_s = H_1(L(h_s, 1); \mathbb{Z}))$ is sent to ω_s. We can now apply Theorem 3.9 and get

$$\rho_\alpha(S_L(g)) = \rho_\alpha(Y_L) - \sum_{s=1}^{n} \rho(L(h_s, 1), \omega_s) - \tau(V_P, V_{X_L}, V_Y) \tag{4.15}$$

(as usual, there is no Maslov triple index in twisted cohomology because of Remark 4.2). The first summand in the right-hand term of (4.15) is minus the multivariable signature of L thanks to Proposition 4.18. Using Corollary 2.11 to describe the rho invariant of the lens spaces at hand, and swapping the second and third variables of the Maslov triple index (see Proposition 3.2 (i)), we can rewrite (4.15) as

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) - \sum_{s=1}^{n} (2h_s\theta_s(1-\theta_s) - \operatorname{sgn}(h_s)) + \tau(V_P, V_Y, V_{X_L}). \tag{4.16}$$

Hence, in order to conclude, we need to identify the Maslov triple index term in (4.16). The rest of the proof is devoted to this.

Instead of trying to calculate the Maslov triple index directly, we use the cocycle property of our three Lagrangians together with $\mathcal{M} := \operatorname{Span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_k\}$ to simplify this task. Namely, using Proposition 3.2 (ii), we find

$$\tau(V_P, V_Y, V_{X_L}) = \tau(\mathcal{M}, V_P, V_Y) - \tau(\mathcal{M}, V_P, V_{X_L}) + \tau(\mathcal{M}, V_Y, V_{X_L}). \tag{4.17}$$

We set the following notation. Let $\lambda_1, \ldots, \lambda_k$ be the framed longitudes associated with the surgery framing $g = (g_1, \ldots, g_k)$, and let $\lambda_1, \ldots, \lambda_k$ bet the framed longitudes associated

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with the coloured Seifert framing $f_L = (f_1, \ldots, f_k)$. By definition, then, we have

$$\lambda_i' = \lambda_i + (f_i - g_i)\mu_i \quad \text{for all } i = 1, \ldots, k. \tag{4.18}$$

We also use the notation Λ_{ij} for the coefficients of the framed linking matrix Λ_g, and Λ_{ij}' for those of the frame linking matrix Λ_{f_L} (these two matrices only differ on the diagonal). We have the following description of the four Lagrangian subspaces appearing in the right-hand term of (4.17):

$$\begin{aligned}
\mathcal{M} &= \mathrm{Span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_k\},\\
V_P &= \mathrm{Span}_{\mathbb{C}}\{\mu_i - \mu_j \mid c(i) = c(j)\} \oplus \mathrm{Span}_{\mathbb{C}}\{v_1, \ldots, v_n\}, \quad \text{where } v_s = \sum_{c(i)=s} \lambda_i',\\
V_Y &= \mathrm{Span}_{\mathbb{C}}\{\lambda_1, \ldots, \lambda_k\},\\
V_{X_L} &= \mathrm{Span}_{\mathbb{C}}\{w_1, \ldots, w_k\}, \quad \text{where } w_i = \lambda_i - \sum_{j=1}^{k} \Lambda_{ij}\mu_j = \lambda_i' - \sum_{j=1}^{k} \Lambda_{ij}'\mu_j.
\end{aligned}$$

We compute now the three summands separately. We will prove the following.

Claim 2. $\tau(\mathcal{M}, V_P, V_Y) = -\sum_{s=1}^{n} \mathrm{sgn}(h_s)$.

Claim 3. $\tau(\mathcal{M}, V_P, V_{X_L}) = 0$.

Claim 4. $\tau(\mathcal{M}, V_Y, V_{X_L}) = \mathrm{sign}(\Lambda_g)$.

These three claims, together with (4.16) and (4.17), lead to the desired formula. In order to conclude, we are only left with proving Claim 1 to 4.

Proof of Claim 2. Write $\tau(\mathcal{M}, V_P, V_Y) = \tau(V_Y, \mathcal{M}, V_P)$. Clearly, we have

$$(V_Y + \mathcal{M}) \cap V_P = V_P.$$

In fact, the generators of V_P of the form $\mu_i - \mu_j$ are in $\mathcal{M}$, and the generators v_s can be written as

$$v_s = \sum_{c(i)=s} \lambda_i + \sum_{c(i)=s} (f_i - g_i)\mu_i, \tag{4.19}$$

where the first summand is in V_Y, and the second summand is in $\mathcal{M}$. Let

$$\psi \colon V_P \times V_P \to \mathbb{C}$$

be the Hermitian form associated with the triple $(V_Y, \mathcal{M}, V_P)$, whose signature is $\tau(V_Y, \mathcal{M}, V_P)$. The generators of the form $\mu_i - \mu_j$ are clearly in the radical of ψ, as they belong to the Lagrangian $\mathcal{M}$. As a consequence, it is enough to study ψ on the span

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of $v_1, \ldots, v_n$. Using the definition of ψ and the decomposition (4.19), we compute

$$\psi(v_s, v_t) = \left(\sum_{c(i)=s} \lambda_i\right) \cdot \left(\sum_{c(j)=t} (f_j - g_j)\mu_j\right) = \begin{cases} \sum_{c(i)=s} (f_i - g_i), & \text{if } s = t, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, we have

$$\operatorname{sign}\psi = \sum_{s=1}^{n} \operatorname{sgn}\left(\sum_{c(i)=s} (f_i - g_i)\right).$$

By definition of the coloured Seifert framing, together with the fact that $g_i = \Lambda_{ii}$, we can now compute that, for each colour s, we have

$$\sum_{c(i)=s} (f_i - g_i) = \sum_{c(i)=s} \left(-\sum_{\substack{c(j)=s \\ j\neq i}} \operatorname{lk}(K_i, K_j)\right) - \sum_{c(i)=s} g_i = -\sum_{c(i)=c(j)=s} \Lambda_{ij} = -h_s.$$

Putting these computations together, we find the equation in the statement of the claim.

Proof of Claim 3. The space $(\mathcal{M} + V_P) \cap V_{X_L}$ is the n-dimensional subspace generated by the terms

$$z_s := \sum_{c(i)=s} w_i = -\sum_{c(i)=s} \sum_{j=1}^{k} \Lambda'_{ij}\mu_j + \sum_{c(i)=s} \lambda'_i \tag{4.20}$$

where the first summand is in $\mathcal{M}$ and the second summand is in V_P. Let

$$\varphi\colon \operatorname{Span}_{\mathbb{C}}\{z_1, \ldots, z_n\} \times \operatorname{Span}_{\mathbb{C}}\{z_1, \ldots, z_n\} \to \mathbb{C}$$

be the Hermitian form associated with the triple $(\mathcal{M}, V_P, V_{X_L})$. Then, from the decomposition (4.20), we can compute

$$\varphi(z_s, z_t) = \left(-\sum_{c(i)=s} \sum_{j=1}^{k} \Lambda'_{ij}\mu_j\right) \cdot \left(\sum_{c(i)=t} \lambda'_i\right) = \sum_{\substack{c(i)=s \\ c(j)=t}} \Lambda'_{ij}.$$

For $s = t$, this is 0 by definition of the coloured Seifert framing. For $s \neq t$, instead, it is equal to $\operatorname{lk}(L_s, L_t)$, which is 0 because the link is colour-to-colour algebraically split. In particular, the form φ is trivial and the Maslov triple index is 0 as claimed.

Proof of Claim 4. This follows immediately from Lemma 4.6 (iii), as $V_Y = \mathcal{L}_g$.

Proof of Claim 1. As we have observed, P_{Γ_L} is the disjoint union of manifolds of the form $\Sigma_s \times S^1$, where Σ_s is a punctured sphere. By construction, in the gluing $X_L \cup_\partial Y$, the meridian m_i of the solid torus Y_i is identified with the framed longitude λ_i of the surgery framing g. On the other hand, when $-P_{\Gamma_L}$ is glued to X_L, a boundary component $C_i \times S^1 \subseteq \partial\Sigma_s \times S^1$ is identified with the boundary torus K_i in such a way that, homologically,

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the class $[C_i]$ coincides with the framed longitude λ_i' of the coloured Seifert framing, and $[S^1]$ coincide with the meridian μ_i. The "by-product" gluing $P_{\Gamma_L} \cup_\partial Y$ is the result of capping the boundary $\Sigma_s \times S^1$ with solid tori. As a consequence of this and (4.18), these cappings are given by the identifications

$$m_i = \lambda_i = \lambda_i' + (g_i - f_i)\mu_i = [C_i] + (g_i - f_i)[S^1].$$

In particular, for each s, this construction leads to the lens space $L(-h_s', 1)$ or equivalently to $-L(h_s', 1)$, with

$$h_s' := \sum_{c(i)=s} (g_i - f_i).$$

It is also easy to see that the element $1 \in \mathbb{Z}/h_s'$ corresponds to $[S^1]$ as desired. The proof is concluded by proving that $h_s' = h_s$. This follows from the definition of the coloured Seifert framing and the sequence of equalities

$$\sum_{c(i)=s} (g_i - f_i) = \sum_{c(i)=s} g_i + \sum_{c(i)=s} \sum_{\substack{j \neq i \text{ s.t.} \\ c(j)=s}} \operatorname{lk}(K_i, K_j) = \sum_{c(i)=c(j)=s} \Lambda_{ij} = h_s.$$

□

Remark 4.25. Applying Theorem 4.24 in the onecoloured setting, where the first hypothesis is always satisfied, we get the following formula relating the rho invariant of the manifold obtained by surgery and the Levine–Tristram signature:

$$\rho_\alpha(S_L(g)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_g - 2(\sum_{i,j} \Lambda_{ij})\theta(1-\theta), \tag{4.21}$$

where $\theta \in (0, 1)$ is such that $\omega = e^{2\pi i\theta}$ and the sum is over the coefficients of Λ_g. For $\theta \in \mathbb{Q}$, this coincides with a formula of Casson and Gordon [7, Lemma 3.1].

Example 4.26. Let r, s be positive coprime integers, and let $T(r, s)$ denote the (r, s)-torus knot. The $(rs - 1)$-Dehn surgery on $T(r, s)$ gives a manifold which is orientation-preserving diffeomorphic to the lens space $L(rs - 1, s^2)$ [24, Proposition 3.2]. Let $\zeta := e^{2\pi i/(rs-1)}$, and let $0 \leq k \leq rs - 2$. Keeping track of the induced map on the fundamental group under this diffeomorphism, by (4.21) (applied with $\omega = \zeta^k$) we obtain

$$\rho(L(rs-1, s^2), \zeta^{krs^2}) = -\sigma_{T(r,s)}(\zeta^k) + 1 - \frac{2k(rs-1-k)}{rs-1}.$$

It might be interesting to compare this formula to other known computations for the Levine–Tristram signature of torus knots (see e.g. the paper of Borodzik and Oleszkiewicz [4]).

Suppose now that the n-coloured link L is component-to-colour algebraically split. Observe that, under this assumption, the coloured Seifert framing coincides with the usual Seifert framing, i.e. with the coloured Seifert framing associated with the onecolouring of the same underlying link. As explained by the next result, this framing has the important property of being compatible with all $U(1)$-representations of $H_1(X_L; \mathbb{Z}) \to U(1)$.

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Lemma 4.27. *Let L be an n-coloured link which is component-to-colour algebraically split. Then, the coloured Seifert framing f_L is compatible with all coloured representations $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$.*

Proof. By Remark 4.8, we need to prove that

$$\prod_{j=1}^{k} \alpha(\mu_j)^{\Lambda_{ij}} = 1 \qquad \text{for all } i. \tag{4.22}$$

Let $\omega \in \mathbb{T}^n$ be the element determined by the relations $\alpha(\mu_i) = \omega_{c(i)}$ for all i. We can write then

$$\prod_{j=1}^{k} \alpha(\mu_j)^{\Lambda_{ij}} = \prod_{s=1}^{n} \omega_s^{\sum_{c(j)=s} \Lambda_{ij}}. \tag{4.23}$$

Since L is component-to-colour algebraically split, for every s different from $s_i := c(i)$ we have

$$\sum_{c(j)=s} \Lambda_{ij} = \operatorname{lk}(K_i, L_s) = 0.$$

As a consequence, (4.23) can be rewritten as

$$\prod_{j=1}^{k} \alpha(\mu_j)^{\Lambda_{ij}} = \omega_{s_i}^{\sum_{c(j)=s_i} \Lambda_{ij}}. \tag{4.24}$$

By (4.4), moreover, the exponent in the right-hand term of (4.24) is 0, and thus (4.22) is satisfied. □

Corollary 4.28. *Let L be an n-coloured link which is component-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the associated coloured representation. Then, α extends to $H_1(S_L(f_L);\mathbb{Z})$ and we have*

$$\rho_\alpha(S_L(f_L)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_{f_L}.$$

Proof. The representation α extends to $H_1(S_L(f_L);\mathbb{Z})$ thanks to Lemma 4.27, so that we can apply Theorem 4.24. The desired formula follows then from the observation that, for the coloured Seifert framing, thanks to (4.4) for any colour s we have

$$h_s \overset{\text{def}}{=} \sum_{c(i)=c(j)=s} \Lambda_{ij} = \sum_{c(i)=s} \sum_{c(j)=s} \Lambda_{ij} = \sum_{c(i)=s} 0 = 0.$$

□

Remark 4.29. Both Lemma 4.27 and Corollary 4.28 hold in particular in the one-coloured setting, where they were proved by Nagel and Powell [25, Section 5]. Our work is a generalization of this to the multivariable setting.

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4.5. Rational surgery

We now want to study the rho invariant of the closed manifold obtained by surgery along a rational framing on a link. We start from the case of a knot, as the statement and the proof are a bit simpler in this setting. Observe that, for a knot K, a representation $\psi\colon H_1(X_K) \to U(1)$, extends to $M_K(p/q)$ if and only if $\psi(\mu)^p = 1$, i.e. if and only if $\omega := \psi(\mu)$ is a p^{th} root of unity.

Proposition 4.30. *Let K be a knot, and let $\alpha\colon H_1(X_K;\mathbb{Z}) \to U(1)$ be the representation defined by $\alpha(\mu) = \omega$, with ω being a p^{th} root of unity. Then, we have*

$$\rho_\alpha(S_K(p/q)) = -\sigma_K(\omega) - \rho(L(p,q),\omega).$$

Proof. We perform cut-and-paste on $S_K(p/q)$ in the following way: we cut out the solid torus $Y = D^2 \times S^1$ of the filling, and we replace it with another copy $Y' = D^2 \times S^1$, this time glued along the 0 framing. It is convenient to actually glue $-Y'$ instead of Y': in such a way, we can define an orientation-reversing diffeomorphism between $-\partial Y'$ and ∂X_K which gives the identifications

$$m' = \lambda, \quad l' = \mu$$

between the standard basis $(m',\, l')$ of $H_1(Y';\mathbb{Z})$ and the basis $(\mu,\, \lambda)$ of $H_1(\partial X_L;\mathbb{Z})$. On the other hand, the meridian m of Y is identified with $p\mu + q\lambda$. Schematically, we write

$$X_K \cup_\partial Y \quad \rightsquigarrow \quad X_K \cup_\partial (-Y') \,\sqcup\, Y' \cup_\partial Y. \tag{4.25}$$

The union of solid tori $Y' \cup_\delta Y$ is now given along a diffeomorphism which gives the identification of m with $qm' + pl'$, so that the resulting manifold is the lens space $L(p,\, -q) \cong -L(p,\, q)$ (see § 2.4). In particular, (4.25) can be rewritten as

$$S_K(p/q) \quad \rightsquigarrow \quad S_K(0) \,\sqcup\, -L(p,q).$$

Observe that the generator $1 \in \mathbb{Z}/p = \pi_1(L(p,\, q))$ corresponds by construction to the longitude l'. In turn, l' is glued in the surgery with the meridian μ of K. As a consequence, the extension of the representation α to $H_1(L(p,\, q);\mathbb{Z})$ is the representation $\mathbb{Z}/p \to U(1)$ given by $1 \mapsto \omega$. Before applying the cut-and-paste formula, we observe that the result is trivially true for $\omega = 1$. We shall hence suppose $\omega \neq 1$. Thanks to Theorem 3.9, we can now compute

$$\rho_\alpha(S_K(p/q)) = \rho_\alpha(S_K(0)) - \rho(L(p,q),\omega) - \tau(V_{Y'}, V_{X_K}, V_Y) \tag{4.26}$$

(as usual, the Maslov triple index in twisted homology is 0 thanks to Remark 4.2 and the assumption $\omega \neq 1$). Now, we know that the first summand in the right-hand term of (4.26) is minus the Levine–Tristram signature (see Remark 4.20), while the second summand coincides with the one of the statement. As a consequence, to complete the proof, it is enough to show that $\tau(V_{Y'},\, V_{X_K},\, V_Y) = 0$. We are going to describe the Lagrangians explicitly. Considering the identifications given by the gluings, in terms of the basis $(\mu,\, \lambda)$

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of $H_1(\partial X_K;\mathbb{C})$ we have

$$V_{Y'} = \mathrm{Span}_{\mathbb{C}}(\lambda), \quad V_{X_K} = \mathrm{Span}_{\mathbb{C}}(\lambda), \quad V_Y = \mathrm{Span}_{\mathbb{C}}(p\mu + q\lambda).$$

As the first two subspaces coincide, the Maslov triple index is 0, and the proof is complete. $\square$

We shall now prove the general version of Proposition 4.30, which holds for all component-to-colour algebraically split links. As we have seen, a special case of these are onecoloured links. In particular, this gives a general formula relating the Levine–Tristram signature of a link L with the Atiyah–Patodi–Singer rho invariant of the closed three-manifold obtained by rational surgery on L.

Theorem 4.31. *Let L be an n-coloured, k-component link that is component-to-colour algebraically split. Let $\omega \in \mathbb{T}^n_*$, and let $\alpha\colon \pi_1(X_L) \to U(1)$ be the associated coloured representation. Let r be a compatible rational framing on L. Then, we have*

$$\rho_\alpha(S_L(r)) = -\sigma_L(\omega) + \mathrm{sign}\Lambda_r - \sum_{i=1}^{k} (\rho(L(p_i, q_i), \omega_{c(i)}) + \mathrm{sgn}(p_i/q_i)),$$

where p_i, q_i are coprime integers such that $r_i - f_i = p_i/q_i$ (here, f_i is the i-th coefficient of the Seifert framing).

Proof. We generalize the cut-and-paste construction of Proposition 4.30, removing the union of solid tori Y coming from the r-framed filling of X_L, and replacing them with a union of solid tori Y' glued along the coloured Seifert framing. As the difference between these framings is now given by the k-tuple $(p_1/q_1, \ldots, p_k/q_k)$, the same argument used in the proof of Proposition 4.30 (repeated now for each link component) implies that this cut-and-paste can be written as

$$S_L(r) \quad \rightsquigarrow S_L(f_L) \;\sqcup\; \left(\bigsqcup_{i=1}^{k} -L(p_i, q_i) \right).$$

In particular, Theorem 3.9 gives in this case

$$\rho_\alpha(S_L(r)) = \rho_\alpha(S_L(f_L)) - \sum_{i=1}^{k} \rho_\alpha(L(p_i, q_i)) - \tau(V_{Y'}, V_{X_L}, V_Y).$$

Applying Corollary 4.28 and rearranging the variables of the Maslov triple index with Proposition 3.2 (i), we can rewrite the last equation as

$$\rho_\alpha(S_L(r)) = -\sigma_L(\omega) + \mathrm{sign}\Lambda_{f_L} - \sum_{i=1}^{k} \rho(L(p, q), \omega_{c(i)}) + \tau(V_{Y'}, V_Y, V_{X_L}). \tag{4.27}$$

The computation of the Maslov triple index is more involved than in the case of knots. As we did in the proof of Theorem 4.24, instead of trying to calculate it directly, we use the

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cocycle property of our three Lagrangians together with $\mathcal{M} := \{\mu_1, \ldots, \mu_k\}$ to simplify this task. Namely, using Proposition 3.2 (ii), we find

$$\tau(V_{Y'}, V_Y, V_{X_L}) = \tau(\mathcal{M}, V_{Y'}, V_Y) - \tau(\mathcal{M}, V_{Y'}, V_{X_L}) + \tau(\mathcal{M}, V_Y, V_{X_L}). \tag{4.28}$$

Claim 1. $\tau(\mathcal{M}, V_{Y'}, V_Y) = -\sum_{i=1}^k \operatorname{sgn}(p_i/q_i)$.

Claim 2. $\tau(\mathcal{M}, V_{Y'}, V_{X_L}) = \operatorname{sign}\Lambda_{f_L}$ and $\tau(\mathcal{M}, V_Y, V_{X_L}) = \operatorname{sign}\Lambda_r$.

Thanks to (4.28), the two claims (whose proof we postpone for the moment) combine to give

$$\tau(V_{Y'}, V_Y, V_{X_L}) = -\sum_{i=1}^{k} \operatorname{sgn}(p_i/q_i) - \operatorname{sign}\Lambda_{f_L} + \operatorname{sign}\Lambda_r.$$

Substituting this value into (4.27), we obtain the formula in the statement of the theorem.

Proof of Claim 1. Let $\lambda_1, \ldots, \lambda_k$ be the framed longitudes corresponding to the framing r, and let $\lambda'_1, \ldots, \lambda'_k$ be the framed longitudes of the Seifert framing. Then, it is clear that

$$V_Y = \operatorname{Span}_{\mathbb{C}}\{\lambda_1, \cdots, \lambda_k\}, \qquad V_{Y'} = \operatorname{Span}_{\mathbb{C}}\{\lambda'_1, \cdots, \lambda'_k\}.$$

Moreover, by definition, we have $\lambda_i = \lambda_i^s + r_i\mu_i$ and $\lambda'_i = \lambda_i^s + f_i\mu_i$, so that the two sets of longitudes are related by

$$\lambda_i = \lambda'_i + (r_i - f_i)\mu_i = \lambda'_i + \frac{p_i}{q_i}\mu_i.$$

In particular, the three Lagrangians in the first summand of (4.28) all split according to the symplectic decomposition

$$H_1(\partial X_L; \mathbb{C}) = \bigoplus_{i=1}^{k} \operatorname{Span}_{\mathbb{C}}\{\mu_i, \lambda_i\}.$$

As the pair (μ_i, λ'_i) is a symplectic basis for $(H_1(T_i; \mathbb{C}), \cdot)$ (see Example 3.3), it is immediate to compute

$$\tau(\mathcal{M}, V_{Y'}, V_Y) = \sum_{i=1}^{k} \tau\Big(\mu_i, \lambda'_i, \lambda'_i + \frac{p_i}{q_i}\mu_i\Big) = -\sum_{i=1}^{k} \operatorname{sgn}(p_i/q_i).$$

Proof of Claim 2. The claim follows immediately from Lemma 4.6 (iii), as we have $V_Y = \mathcal{L}_r$ and $V_{Y'} = \mathcal{L}_{f_L}$. □

Remark 4.32. The hypothesis of L being component-to-colour algebraically split allows us to obtain a single nice explicit formula, because, in this case, we can always apply the cut-and-paste formula after performing the Dehn surgery along the Seifert framing, which is compatible with all representations thanks to Lemma 4.27. More general cases of rational surgery can be faced by finding an appropriate compatible integral framing (together with Theorem 4.21 or Theorem 4.24) and then modifying it into the desired rational framing in the same way as in the proof of Theorem 4.31. However, this is better dealt with on a case-by-case basis, as a general formula would be quite cumbersome.

Remark 4.33. If L is component-to-colour algebraically split and g is an integer framing on L which is compatible with the given $\omega \in \mathbb{T}_*^n$, we can apply either Theorem 4.24 or Theorem 4.31. The fact that the resulting formulas are compatible can be verified by a quick computation using Corollary 2.11.

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