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Extension of Maps to Nilpotent Spaces

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Abstract. We show that every compactum has cohomological dimension 1 with respect to a finitely generated nilpotent group G whenever it has cohomological dimension 1 with respect to the abelianization of G. This is applied to the extension theory to obtain a cohomological dimension theory condition for a finite-dimensional compactum X for extendability of every map from a closed subset of X into a nilpotent CW-complex M with finitely generated homotopy groups over all of X.

1 Dimension Over Nilpotent Groups

Recall that for a group G and a space X the cohomological dimension $\dim_G X$ is defined by $\dim_G X \leq n$ whenever $X\tau K(G,n)$ (this is Kuratowski's notation for the case every map from a closed subspace of X to $K(G,n)$ can be extended over all of X). Since all homotopy groups in dimension ≥ 2 are abelian, for a non-abelian group G the cohomological dimension $\dim_G X$ can only take values 0, 1 or ∞.

Theorem 1 *For a finitely generated nilpotent group G the following equality holds for every compactum X:*

$$\dim_G X = \dim_{\mathrm{Ab}\,G} X,$$

where Ab G *means the abelianization of G; for a non-abelian G this equality holds if on the right-hand side we identify all dimensions greater than* 1 *with infinity.*

In order to prove this theorem we generalize Bockstein's basis theory to finitely generated nilpotent groups. For such a group G define a family $\tilde{\sigma}(G) \subset \{\mathbb{Z}\} \cup \{\mathbb{Z}_p; p \text{ prime}\}$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, by: $\mathbb{Z} \in \tilde{\sigma}(G)$ whenever G_0 (the group G localized at 0) is non-trivial and $\mathbb{Z}_p \in \tilde{\sigma}(G)$ whenever G_p (the group G localized at the prime p) is non-trivial.

Proposition 2 *Let G be a nilpotent group of class k and denote by $\Gamma^i G$ the i-th group of the lower central series of G. Then there is an inclusion*

$$\tilde{\sigma}(\Gamma^k G) \subset \tilde{\sigma}(G/\Gamma^k G).$$

Proof Localizing the exact sequence

$$1 \to \Gamma^k(G) \to G \to G/\Gamma^k(G) \to 1$$

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with respect to a prime p we obtain the exact sequence

$$1 \to \Gamma^k(G)_p \to G_p \to \big(G/\Gamma^k(G)\big)_p \to 1$$

which can be written also as

$$1 \to \Gamma^k(G_p) \to G_p \to G_p/\Gamma^k(G_p) \to 1. \tag{1}$$

Note that $\Gamma^i(G)_p = \Gamma^i(G_p)$, for every positive integer i. If $\mathbb{Z}_p \in \tilde{\sigma}\big(\Gamma^k(G)\big)$ then $\Gamma^k(G)_p \neq 1$ and $G_p/\Gamma^k(G_p) = 1$, which is equivalent to $\mathbb{Z}_p \notin \tilde{\sigma}\big(G/\Gamma^k(G)\big)$, would lead to $1 \neq \Gamma^k(G_p) = G_p$ which contradicts the fact that G (and thus G_p) is nilpotent.

The above holds also for localization at $p = 0$ with the group $\mathbb{Z}_p$ replaced by $\mathbb{Z}$ in this case. ∎

Proposition 3 *Let G be a nilpotent group of class k. Then*

$$\tilde{\sigma}(G) = \tilde{\sigma}\big(G/\Gamma^k(G)\big).$$

Proof If $\mathbb{Z}_p \in \tilde{\sigma}\big(G/\Gamma^k(G)\big)$ then $\big(G/\Gamma^k(G)\big)_p \neq 1$ and thus $G_p \neq 1$ implying $\mathbb{Z}_p \in \tilde{\sigma}(G)$. If $\mathbb{Z}_p \in \tilde{\sigma}(G)$ then $G_p \neq 1$ implying that $\big(G/\Gamma^k(G)\big)_p \neq 1$ or $\Gamma^k(G)_p \neq 1$. In the former case obtain $\mathbb{Z}_p \in \tilde{\sigma}\big(G/\Gamma^k(G)\big)$ immediately, in the latter case apply Proposition 2 to G_p and obtain the same result. ∎

From the structure of finitely generated abelian groups we directly obtain the following proposition.

Proposition 4 *For a finitely generated abelian group G and any compactum X the cohomological dimension* $\dim_G X$ *equals the maximum of* $\dim_H X$ *over all* $H \in \tilde{\sigma}(G)$. ∎

We generalize this proposition to nilpotent groups (with dimension taking only values 0, 1 or ∞ if the group is non-abelian).

Proposition 5 *If G is a finitely generated nilpotent group then*

$$\dim_G X = \max\{\dim_H X; H \in \tilde{\sigma}(G)\}.$$

Proof Let G be nilpotent of class k and we prove the proposition by induction on k.

The exact sequence

$$1 \to \Gamma^k(G) \to G \to G/\Gamma^k(G) \to 1$$

gives rise to the fibration

$$K\big(\Gamma^k(G), 1\big) \to K(G, 1) \to K\big(G/\Gamma^k(G), 1\big).$$

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It is a well-known fact that for a fibration $F \to E \to B$ the properties $X\tau B$ and $X\tau F$ imply $X\tau E$. Thus

$$\dim_G X \leq \max\{\dim_{\Gamma^k(G)} X, \dim_{G/\Gamma^k(G)} X\}.$$

Since $\Gamma^k(G)$ is abelian and $G/\Gamma^k(G)$ is nilpotent of class $k-1$, the inductive assumption implies that the maximum above equals to

$$\max\Big\{\max\{\dim_H X; H \in \tilde{\sigma}\big(\Gamma^k(G)\big)\}, \max\{\dim_H X; H \in \tilde{\sigma}\big(G/\Gamma^k(G)\big)\}\Big\}.$$

By Proposition 2 this equals to

$$\max\{\dim_H X; H \in \tilde{\sigma}\big(G/\Gamma^k(G)\big)\}$$

and by Proposition 3 this equals to

$$\max\{\dim_H X; H \in \tilde{\sigma}(G)\}. \qquad \blacksquare$$

Proof of Theorem 1 First show that $X\tau K(G,1)$ implies $X\tau K(\mathrm{Ab}\, G,1)$ for an arbitrary group G. From [1] (Theorem 6) it follows that $X\tau\, \mathrm{SP}^\infty K(G,1)$, by [2], [3], we obtain $X\tau K\Big(H_1\big(K(G,1)\big), 1\Big)$ which is equivalent to $X\tau K(\mathrm{Ab}\, G,1)$.

Now we show the opposite inequality. Proposition 5 implies that $\dim_G X$ is less or equal to

$$\max\{\dim_H X; H \in \tilde{\sigma}(G)\} = \max\{\dim_H X; H \in \tilde{\sigma}(\mathrm{Ab}\, G)\}$$

(the latter equality follows from Proposition 3) which equals to $\dim_{\mathrm{Ab}\, G} X$. $\blacksquare$

2 Extension of Maps

First we prove a version of the Hurewicz theorem in cohomological dimension theory.

Theorem 6 *Let M be a nilpotent* CW*-complex with finitely generated homotopy groups and X a compactum. Then the following are equivalent:*

1. $\dim_{H_i(M)} X \leq k$ *for every* $k > 0$;
2. $\dim_{\pi_i(M)} X \leq k$ *for every* $k > 0$.

Proof Let $\pi_k = \pi_k(M)$ and $H_k = H_k(M;\mathbb{Z})$. Define $h(G) = \{\min k : G \in \tilde{\sigma}(H_k)\}$ and $\pi(G) = \{\min k : G \in \tilde{\sigma}(\pi_k)\}$. Note that given any generalized Serre class $\mathcal{C}$ of groups and any positive integer k one obtains the Hurewicz theorem modulo $\mathcal{C}$ from Theorem II.2.16 of [4] applied to the k-th stage of the Postnikov system.

Thus, in particular, for the generalized Serre class of groups whose elements have finite order we obtain $h(\mathbb{Z}) = \pi(\mathbb{Z})$. From the Hurewicz theorem modulo the generalized Serre class of groups whose elements have orders q^k, for q prime different from p, we obtain $h(\mathbb{Z}_p) = \pi(\mathbb{Z}_p)$.

Let us show that 2 implies 1. Proposition 5 implies that $\dim_{H_k} X = \dim_G X$ for some group $G \in \tilde{\sigma}(H_k)$. Therefore $l = h(G) \leq k$. From $h(G) = \pi(G)$ we obtain

$$\dim_G X \leq \dim_{\pi_l} X \leq l \leq k.$$

The other implication can be proved similarly. ■

As a corollary we obtain the following variation of the main theorem of [1].

Theorem 7 *For any nilpotent* CW*-complex M with finitely generated homotopy groups and finite-dimensional compactum X, the following are equivalent:*

1. *$X\tau M$;*
2. *$X\tau$ $\mathrm{SP}^\infty M$;*
3. *$\dim_{H_i(M)} X \leq i$ for every $i > 0$;*
4. *$\dim_{\pi_i(M)} X \leq i$ for every $i > 0$.*

Proof The proof follows the original proof of [1] except in the step $3 \Rightarrow 4$ which follows from the Theorem 6 above. ■

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