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A note on the values of Mahler measure in quadratic fields

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Abstract. In this note, we prove that quadratic algebraic integers, except for trivial cases, are not Mahler measures of algebraic integers and we also answer in negative the question of A. Schinzel [9] whether $1+\sqrt{17}$ is a Mahler measure of an algebraic number.

1 Introduction and the statement of the theorems

Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{Z}[x]$ be a polynomial. The Mahler measure of P is defined by

$$M(P) = |a_0| \prod_{i=1}^{n} \max\{|\alpha_i|, 1\},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the zeros of P.

If α is an algebraic number we define its Mahler measure by $M(\alpha) = M(P)$, where P is the minimal polynomial of α in $\mathbb{Z}[x]$. Two problems were considered:

(1) Which algebraic numbers are Mahler measures of integer polynomials?
(2) Which algebraic numbers are Mahler measures of algebraic numbers?

Let $\mathcal{M} = \{M(\alpha) : \alpha \in \bar{\mathbb{Q}}\}$, where $\bar{\mathbb{Q}}$ is the set of algebraic numbers. It is well known and easy to check (see [1]) that every $\beta \in \mathcal{M}$ is an algebraic integer and a Perron number. However, in [3], D. Boyd gives an example of Perron units that are not Mahler measures of an algebraic integers. Partial results are abundant, see, for example, [2–6]. In [5], the authors show that every real algebraic integer is a difference of two Mahler measures. The results presented in this article relate to the following two theorems of A. Schinzel proved in [9].

Theorem 1.1. [9] *A primitive real quadratic integer β is in $\mathcal{M}$ if and only if there exists a rational integer a such that $\beta > a > |\beta'|$ and $a \mid \beta\beta'$, where β' is the conjugate of β. If the condition is satisfied then $\beta = M(\beta/a)$ and $a = N(a,\beta)$, where N denotes the absolute norm.*

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For quadratic integers that are not primitive, he considers the numbers $p\beta$, where p is a rational prime and β a primitive algebraic integer, and proves the following theorem.

Theorem 1.2. [9] *Let K be a quadratic field with discriminant $\Delta > 0$, β, β' be primitive conjugate integers of K and p a prime. If:*

(1)

$$p\beta \in \mathcal{M},$$

then either there exists an integer r such that

(2)

$$p\beta > r > p|\beta'| \quad \text{and } r \mid \beta\beta', \quad p \nmid r,$$

or

(3)

$$\beta \in \mathcal{M} \text{ and } p \text{ splits in } K.$$

Conversely, (2) implies (1), while (3) implies (1) provided either

(4)

$$\beta > \max\left\{-4\beta', \left(\frac{1+\sqrt{\Delta}}{4}\right)^2\right\}$$

or

(5)

$$p > \sqrt{\Delta}.$$

Here, we focus only on quadratic irrationalities. Let $\beta > 1$ be an algebraic integer of degree two. Denote by β' its algebraic conjugate. If $|\beta'| < 1$ then, obviously β is a Mahler measure, that is, $\beta \in \mathcal{M}$. This covers the case when β is a unit. However, the case when $|\beta'| > 1$ is more interesting. We prove the following:

Theorem 1.3. *Let $\beta > 1$ be an algebraic integer and suppose that $|\beta'| > 1$. Then β is not a Mahler measure of an algebraic integer, that is, $\beta \neq M(\alpha)$ for any algebraic integer α.*

Our next theorem relates to Schinzel's result cited here as Theorem 1.2. He noted that there are algebraic integers of degree two that do not satisfy condition (2), and satisfy condition (3) without conditions (4) or (5) of this theorem. As an example, he cites the number $1+\sqrt{17}$ and asks if it is a Mahler measure of an algebraic number. This particular question was open since 2004 and was quoted by A. Dubickas, J. McKee and C. Smyth, P.A. Filli, L.Potmeyer, and M. Zhang [6–8] among others. In the following remark, we show that the list of numbers with the properties listed above and thus falling in the gap in Theorem 1.2 is in fact infinite.

Remark 1.4. There are infinitely many real algebraic integers β of degree two that together with suitable prime p do not satisfy condition (2), but satisfy condition (3) without conditions (4) or (5).

Proof Let $k \geq 4$ be a rational integer. Let $2^k + 1 = b^2 m$, where m is square-free and b a positive integer. Obviously, $m \neq 1$ and $m \equiv 1 \bmod 8$. Hence, 2 splits in $\mathbb{Q}(\sqrt{m})$. Let $\beta = (1 + b\sqrt{m})/2$ and $p = 2$. Then (2) fails because $\beta\beta' = (1 - b^2 m)/4 = -2^{k-2}$, so $p \mid r$. Further (3) holds, (4) fails because $\beta < -4\beta'$, and (5) fails. ■

It is easy to find other types examples than those listed in Remark 1.4 that also fail the assumptions of Theorems 1.1 and 1.2. For example, many numbers of the form $\beta = (1 + \sqrt{m})/2$ with square-free $m \equiv 1 \bmod 8$ and $p = 2$ fall to this category. This happens for $m = 17, 33, 41, 57, 65, 73$, and the list is most likely infinite. The number $1 + \sqrt{17} = 2\beta$ is of the type of numbers considered in the proof of Remark 1.4. Our next theorem shows that it is not a Mahler measure of an algebraic number. We focused on this number as a tribute to A. Schinzel who specifically asked about it.

It is easy to check that

$$M(4x^2 \pm 2x - 4) = 1 + \sqrt{17}$$

and also

$$M(4x^{2d} \pm 2x^d - 4) = 1 + \sqrt{17}$$

for every positive integer d as $M(f(x^d)) = M(f(x))$. Here, we prove the following theorem.

Theorem 1.5. *Let $f \in \mathbb{Z}[x]$. If $M(f) = 1 + \sqrt{17}$ and f is irreducible over $\mathbb{Q}$, then*

$$f(x) = 4x^{2d} \pm 2x^d - 4,$$

where d is a positive integer.

Since $4x^{2d} \pm 2x^d - 4 = 2(x^{2d} \pm x^d - 2)$, polynomials $f(x)$ are reducible in $\mathbb{Z}[x]$. Consequently, the answer to Schinzel's question is negative, $1 + \sqrt{17} \notin \mathcal{M}$.

The ideas used in the proof potentially can be used to investigate the numbers described in the remark, especially when $\mathbb{Q}(\sqrt{m})$ has class number 1.

2 Lemmas

We start with with the following lemma.

Lemma 2.1. *Let $\mathcal{O}_K$ be the ring of algebraic integers of a number field K. If*

$$f(x) = a\prod_{i=1}^{n}(x - \alpha_i) \in \mathcal{O}_K[x],$$

then $a\alpha_1 \ldots \alpha_s$ is an algebraic integer for $1 \leq s \leq n$.

This lemma is well known and widely used in case of $\mathcal{O}_K = \mathbb{Z}$. The version stated here is a slight generalization. The following lemma may be deduced from Dixon and Dubickas [4, Lemma 2].

Lemma 2.2. *Suppose that λ is a quadratic algebraic integer that is the Mahler measure of an algebraic number α, that is $\lambda = M(\alpha)$. Let $f(x) = a_0 \prod_{i=1}^{n}(x - \alpha_i)$ be the minimal polynomial of α in $\mathbb{Z}[x]$ and $a_0 > 0$, $a_n = f(0)$, λ' the algebraic conjugate of λ, and $N(\lambda) = \lambda\lambda'$. Then*

$$a_0^{2r}|a_n|^{2s} = |N(\lambda)^n|,$$

where s is the number of conjugates of α lying strictly outside the unit circle, and $r = n - s$.

The following is Schur's [10] lemma, employed in Schur–Cohn algorithm to determine the distribution of roots of a complex polynomial relative to the unit circle.

Lemma 2.3. *Let p be a complex polynomial of degree $n \geq 1$. Define its reciprocal adjoint polynomial $p*$ by $p*(z) = z^n \overline{p(\bar{z}^{-1})}$ and its Schur transform by $Tp = \overline{p(0)}p - \overline{p*(0)}p*$. Let $\delta = Tp(0)$. Then:*

(1) *If $\delta \neq 0$ then p, Tp, and $p*$ share zeros on the unit circle.*
(2) *If $\delta > 0$ then p and Tp have the same number of zeros inside the unit circle.*
(3) *If $\delta < 0$ then $p*$ and Tp have the same number of zeros inside the unit circle.*

3 Proof of Theorem 1.3

For a contradiction, suppose that $f \in \mathbb{Z}[x]$ is a monic irreducible polynomial and $M(f) = \beta$. Let

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i).$$

Suppose that $|\alpha_i| > 1$ for $i = 1 \ldots s$, and $|\alpha_i| \leq 1$ for $i = s + 1 \ldots n$. For convenience, we use notation $\gamma_i = \alpha_{s+i}$ for $i = 1 \ldots r$, where $r = n - s$. We define two sets

$$S = \{\alpha_1, \ldots, \alpha_s\} \text{ and } R = \{\gamma_1, \ldots, \gamma_r\}.$$

Further, let $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ be the splitting field of f, $K = \mathbb{Q}(\beta)$, $G = \mathrm{Gal}(L/\mathbb{Q})$, and $H = \mathrm{Gal}(L/K)$. We claim that

$$\text{every } \sigma \in H \text{ permutes } S \text{ and permutes } R. \tag{3.1}$$

Indeed, since H fixes β, for any $\sigma \in H$, we have

$$|\sigma(\alpha_i)||\sigma(\alpha_2)| \ldots |\sigma_s(\alpha_s)| = \beta = |\alpha_1||\alpha_2| \ldots |\alpha_s|.$$

Then if $\sigma(\alpha_i) \notin S$ we would have $\sigma(\alpha_i) \in R$ so the left-hand side would be strictly smaller than the right. Further σ is a one-to-one map, hence $\sigma(R) \cap S = \varnothing$, so $\sigma(R) \subseteq R$, and thus $\sigma(R) = R$. We have

$$\beta = \prod_{i=1}^{s}|\alpha_i|.$$

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We must have $s < n$ since otherwise $M(f)$ would be equal to the absolute value of the constant term of f which is a rational integer. We apply now Lemma 2.2 with $a_0 = 1$ and $\lambda = \beta$. We get $|a_n|^{2s} = |N(\beta)^n|$. Hence

$$|a_n|^{\frac{2s}{n}} = |N(\beta)| = |\beta\beta'| > \beta \text{ because } |\beta'| > 1.$$

However, $|a_n| = |\alpha_1 \dots \alpha_s||\alpha_{s+1} \dots \alpha_n| \leq \beta$. Thus

$$\beta \geq |a_n| > \beta^{\frac{n}{2s}}$$

and we conclude that $2s > n$, so $2s > s + r$, and $s > r$. We shall show that the last inequality contradicts the irreducibility of f. For this, let

$$f_1(x) = \prod_{i=1}^{s}(x - \alpha_i) \text{ and } f_2(x) = \prod_{i=1}^{r}(x - \gamma_i).$$

The coefficients of these polynomials are symmetric functions of $\alpha_1, \dots, \alpha_s$ and $\gamma_1, \dots, \gamma_r$, respectively. Since every σ from H permutes S and permutes R, these coefficients are in K, the fixed field of H. Now, let σ be any automorphism in $G\backslash H$, then we conclude that $f_1(x)\sigma(f_1(x))$ and $f_2(x)\sigma(f_2(x))$ both are in $\mathbb{Z}[x]$ as $\sigma(K) = K$ and σ is a non-identity automorphism of K. Further $f(x) = f_1(x)f_2(x)$. We get

$$f^2(x) = f(x)\sigma(f(x)) = (f_1(x)\sigma(f_1(x))(f_2(x)\sigma(f_2(x))).$$

The degree of integer polynomial $f_2(x)\sigma(f_2(x))$ is $2r < n$. However, $f^2(x)$ as a product of two irreducible polynomials of degree n cannot have a factor of degree $2r < n$, a contradiction.

4 Proof of Theorem 1.5

For a contradiction, suppose that $\lambda = 1 + \sqrt{17} = M(\alpha)$ and the minimal polynomial of α in $\mathbb{Z}[x]$ is

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n.$$

We also define a polynomial

$$g(x) = \eta x^n f(x^{-1}), \text{ where } \eta \in \{-1, +1\} \text{ is the sign of } a_n.$$

Hence

$$g(x) = \eta(a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0).$$

Clearly, both polynomials are irreducible, have positive leading coefficients, and $M(g) = M(f) = 1 + \sqrt{17}$. The interplay between f and g plays an important role in the proof. We use notation from the previous section: $S = \{\alpha_1, \dots, \alpha_s\}$ and $R = \{\gamma_1, \dots, \gamma_r\}$, where $\gamma_i = \alpha_{s+i}$ for $1 \leq i \leq r$. Also $L = \mathbb{Q}(\alpha_1, \dots, \alpha_n), K = \mathbb{Q}(\lambda)$, $G = \mathrm{Gal}(L/\mathbb{Q})$, and $H = \mathrm{Gal}(L/K)$. Again, the elements of S lie strictly outside the unit circle, while the elements of R lie inside or on the unit circle. The property (3.1) still holds.

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We first prove some basic properties of polynomials f and g.

Lemma 4.1. *We have:*

(1) $r = s$, *so* $\deg f = 2s = 2r$ *is even,*
(2) $a_0 = 4 = |a_n|$,
(3) *the polynomials f and g have no zeros on the unit circle.*

Proof (1) We let

$$f_1(x) = \prod_{i=1}^{s}(x - \alpha_i) \text{ and } f_2(x) = \prod_{i=1}^{r}(x - \gamma_i)$$

as in the end of the proof of Theorem 1.3. Then (3.1) still holds, so f_1 and f_2 are in $K[x]$, and $f = a_0 f_1 f_2$. Again as in the proof of Theorem 1.3, if $r \neq s$ we get a contradiction with the irreducibility of f.

(2) Lemma 2.2 with $\lambda = 1 + \sqrt{17}$ gives $a_0^{2r}|a_n|^{2s} = |N(\lambda)^n| = 16^n$, so

$$a_0^r |a_n|^s = 4^n.$$

Further

$$|a_n| = |a_0\alpha_1 \dots \alpha_n| \leq |a_0\alpha_1 \dots \alpha_s| = 1 + \sqrt{17}.$$

The first equality shows that $|a_n|$ is a power of 2, the second implies that $|a_n| \leq 4$. We shall show that also $a_0 \leq 4$. To see this, we apply Lemma 2.2 to g and get $|g(0)| = |a_0| \leq 4$ in the same way as we obtained $|f(0)| = |a_n| \leq 4$. Both inequalities $|a_n| \leq 4$ and $|a_0| \leq 4$ together with $a_0^r|a_n|^s = 4^n$ now give $|a_n| = a_0 = 4$.

(3) Clearly $f(-1) \neq 0$ and $f(1) \neq 0$ because f is irreducible. Suppose that $\zeta \in \mathbb{C}\backslash\mathbb{R}$ is a zero of f, and ζ lies on the unit circle. Then also $\zeta^{-1} = \bar{\zeta}$ lies on the unit circle and is a zero of f because the coefficients of f are real numbers. This shows that irreducible polynomials f and g share a zero, hence $f = g$. Thus $S^{-1} = \{\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_s^{-1}\}$ consists of zeros of g and f. Since $S^{-1} \cap S = \varnothing$, we conclude $R = S^{-1}$ and all its elements lie strictly outside the unit circle. Therefore f cannot have a zero on the unit circle. ■

Lemma 4.2. *Let* $f \in \mathbb{Z}[x]$ *be the polynomial defined at the beginning of the section. That is,* $M(f) = 1 + \sqrt{17}$ *and f is irreducible over* $\mathbb{Q}$ *then*

(1) $\sigma(S) = R$ *and* $\sigma(R) = S$ *for any* $\sigma \in G\backslash H$,
(2) $a_n = -4$.

Proof (1) By the previous lemma R has no elements on the unit circle, so $|\gamma_i| < 1$ for all elements of R. We have

$$\lambda = \varepsilon a_0 \alpha_1 \dots \alpha_s \text{ with suitable } \varepsilon \in \{-1, +1\} \tag{4.1}$$

and

$$a_n = (-1)^{2d} a_0 \alpha_1 \dots \alpha_s \gamma_1 \dots \gamma_s.$$

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Since $a_0 = |a_n| = 4$ and $\lambda\lambda' = -16$ we get

$$|\alpha_1 \dots \alpha_s| = \lambda/4 \text{ and } |\gamma_1 \dots \gamma_s| = |4/\lambda| = |\lambda'/4|.$$

Then for any $\sigma \in G\backslash H$ we get $|\sigma(\alpha_1 \dots \alpha_s)| = |\lambda'/4|$. Hence

$$|\sigma(\alpha_1)||\sigma(\alpha_2)| \dots |\sigma(\alpha_s)| = |\lambda'/4| = |\gamma_1||\gamma_2| \dots |\gamma_s|.$$

Since the right-hand side has the smallest value among the absolute value of the products of s distinct zeros of f, its conjugates are uniquely determined and we conclude that $\sigma(S) = R$. Since σ is injective and $\sigma(S) = R$ then $\sigma(R) \cap R = \varnothing$, so also $\sigma(R) = S$.

(2) From (1), we conclude that $\sigma(\alpha_1 \dots \alpha_s) = \gamma_1 \dots \gamma_s$. Also by (4.1) $\alpha_1 \dots \alpha_s = \varepsilon\lambda/4$. Hence

$$a_n = (-1)^{2d} a_0 \alpha_1 \dots \alpha_s \gamma_1 \dots \gamma_s = a_0 \varepsilon\lambda/4 \sigma(\varepsilon\lambda/4) = a_0 \frac{\lambda\lambda'}{16} = -a_0 = -4. \qquad \blacksquare$$

Now we proceed to the conclusion of the proof of Theorem 1.5. The previous lemmas show that

$$f(x) = 4x^{2d} + a_1 x^{2d-1} + \dots + a_{2d-}x - 4, \text{ while}$$

$$g(x) = 4x^{2d} - a_{2d-1}x^{2d-1} - \dots - a_1 x - 4.$$

It is convenient to introduce four polynomials

$$\hat{f}(x) = 4\prod_{i=1}^{d}(x - \alpha_i), \quad \check{f}(x) = 4\prod_{i=1}^{d}(x - \gamma_i)$$

and

$$\hat{g}(x) = 4\prod_{i=1}^{d}(x - \delta_i), \quad \check{g}(x) = 4\prod_{i=1}^{d}(x - \kappa_i),$$

where $\delta_i = \gamma_i^{-1}$ and $\kappa_i = \alpha_i^{-1}$ for $i = 1 \dots d$. We note that all zeros of $\hat{f}$ and $\hat{g}$ lie outside the unit circle, while all zeros of $\check{f}$ and $\check{g}$ lie inside the unit circle. By (3.1) and Lemma 2.1, all polynomials are in $\mathcal{O}_K[x]$. Further

$$4f = \hat{f}\check{f} \text{ and } 4g = \hat{g}\check{g}. \tag{4.2}$$

We claim that

$$\frac{1}{2}\hat{f} \text{ and } \frac{1}{2}\check{f} \text{ are in } \mathcal{O}_K[x] \tag{4.3}$$

or

$$\frac{1}{2}\hat{g} \text{ and } \frac{1}{2}\check{g} \text{ are in } \mathcal{O}_K[x]. \tag{4.4}$$

For this note that $K = \mathbb{Q}(\sqrt{17})$ has class number 1, so $\mathcal{O}_K[x]$ is a unique factorization ring and the content of polynomials is well defined up to a unit factor. By (4.2), we have

$$4 = c(\hat{f})c(\check{f}) \text{ and } 4 = c(\hat{g})c(\check{g}). \tag{4.5}$$

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To proceed further we need to list basic arithmetic facts about $\mathcal{O}_K[x]$. We have:

(a) $u = 4 + \sqrt{17}$ is the fundamental unit in $\mathcal{O}_K$. The group of unit of $\mathcal{O}_K$ is $U = \{\pm u^n : n \in \mathbb{Z}\}$,
(b) $\pi_1 = \frac{-3+\sqrt{17}}{2}$ and $\pi_2 = \frac{-3-\sqrt{17}}{2}$ are primes in $\mathcal{O}_K$,
(c) $\pi_1\pi_2 = -2$,
(d) $\frac{1+\sqrt{17}}{2} = u\pi_1^2$,
(e) $\frac{1-\sqrt{17}}{2} = -u^{-1}\pi_2^2$.

Further, we also have:

(1) $4\alpha_1 \dots \alpha_d = \varepsilon\lambda, \quad 4\gamma_1 \dots \gamma_d = \varepsilon\lambda'$,
(2) $4\delta_1 \dots \delta_d = -\varepsilon\lambda, \quad 4\kappa_1 \dots \kappa_d = -\varepsilon\lambda'$,
(3) $4f(x) = \hat{f}(x)\check{f}(x)$ and $4g(x) = \hat{g}(x)\check{g}(x)$,
(4) $\hat{f}(0) = (-1)^d\varepsilon\lambda = (-1)^d\varepsilon u 2\pi_1^2$,
(5) $\check{f}(0) = (-1)^d\varepsilon\lambda' = -(-1)^d\varepsilon u^{-1}2\pi_2^2$,
(6) $\hat{g}(0) = -(-1)^d\varepsilon\lambda = -(-1)^d\varepsilon u 2\pi_1^2$,
(7) $\check{g}(0) = -(-1)^d\varepsilon\lambda' = (-1)^d\varepsilon u^{-1}2\pi_2^2$.

In particular, $4 = \pi_1^2\pi_2^2$ and $\pi_1\pi_2 = -2$. By Lemma 4.2, for any $\sigma \in G\backslash H$ we have $\check{f} = \sigma(\hat{f})$ and $\check{g} = \sigma(\hat{g})$, also $\sigma(\pi_1) = \pi_2$. The item (4) on the list above shows that $\pi_2^2 \nmid c(\hat{f})$. This together with (4.2) leaves us with two possibilities:

(1) $2 \mid c(\hat{f})$ and $2 \mid c(\check{f})$ or
(2) $\pi_1^2 \mid c(\hat{f})$ and $\pi_2^2 \mid c(\check{f})$.

We shall show that if the possibility (2) occurs then

$$2 \mid c(\hat{g}) \text{ so also } 2 \mid c(\check{g}) \text{ because } \check{g} = \sigma(\hat{g}).$$

Indeed

$$\hat{g}(x) = \frac{4}{\check{f}(0)}x^d\check{f}(x^{-1}) = \frac{4\varepsilon u}{-(-1)^d 2\pi_2^2}x^d\check{f}(x^{-1}).$$

Hence $c(\hat{g}) = c(\pm\frac{2u}{\pi_2^2})c(x^d\check{f}(x^{-1})) = c(\pm\frac{2u}{\pi_2^2})c(\check{f})$, we deduce that $2|c(\hat{g})$ because $\pi_2^2|c(\check{f})$. Consequently also $2|c(\check{g})$. Therefore if the second case occurs we can work with polynomial g instead of f, so without loss of generality we assume that the first case occurs.

We thus conclude that

$$\hat{f}_1(x) = \frac{1}{2}\hat{f}(x) = 2x^d + \sum_{i=1}^{d-1} A_i x^{d-i} + (-1)^d\varepsilon u\pi_1^2$$

and

$$\check{f}_1(x) = \frac{1}{2}\check{f}(x) = 2x^d + \sum_{i=1}^{d-1} \tilde{A}_i x^{d-i} - (-1)^d\frac{\varepsilon}{u}\pi_2^2$$

are in $\mathcal{O}_K[x]$, and $f = \hat{f}_1(x)\check{f}_1(x)$. Here, $\tilde{A}_i$ are algebraic conjugates of A_i, $i = 1 \dots d$. In the last part, we employ Lemma 2.3 to study the coefficients of these polynomials. Put $p(x) = x^d\hat{f}_1(x^{-1}) = (-1)^d\varepsilon u\pi_1^2 x^d + \sum_{i=1}^{d-1} A_i x^i + 2$. Then $p*(x) = \hat{f}_1(x)$ and

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$$Tp(x) = \sum_{i=1}^{d-1}(2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i})x^i + 4 - \varepsilon^2 u^2 \pi_1^4.$$

Hence

$$\delta = 4 - \varepsilon^2 u^2 \pi_1^4 \approx -2.56 < 0.$$

The polynomial $p*$ has no roots inside the unit circle, therefore the same is true about Tp. The degree of Tp is less than d. Suppose that $\deg Tp = i$ for some $i, 1 \leq i \leq d-1$. Then the leading coefficient of Tp is $2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i}$. Since all roots of Tp lie outside of the unit circle, we must have

$$|2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i}| < |4 - \varepsilon^2 u^2 \pi_1^4| = |Tp(0)|.$$

Now we apply the same argument to

$$p(x) = \check{f}_1 = 2x^d + \sum_{i=1}^{d-1} \tilde{A}_i x^{d-i} - (-1)^d \varepsilon \frac{1}{u} \pi_2^2$$

whose roots lie inside the unit circle. Then

$$p*(x) = -(-1)^d \varepsilon \frac{1}{u} \pi_2^2 x^d + \sum_{i=1}^{d-1} \tilde{A}_i x^i + 2.$$

$$Tp(x) = \sum_{i=1}^{d-1}(-(-1)^d \varepsilon u^{-1} \pi_2^2 \tilde{A}_{d-i} - 2\tilde{A}_i)x^i + \varepsilon^2 u^{-2} \pi_2^4 - 4.$$

Hence

$$\delta = \varepsilon^2 \frac{1}{u^2} \pi_2^4 - 4 \approx -1.56 < 0.$$

We conclude as in the previous case that

$$\left|\frac{-\varepsilon}{u}(-1)^d \pi_2^2 \tilde{A}_{d-i} - 2\tilde{A}_i\right| = \left|2\tilde{A}_i + \frac{\varepsilon}{u}(-1)^d \pi_2^2 \tilde{A}_{d-i}\right| < \left|\frac{1}{u^2}\pi_2^4 - 4\right|.$$

From both inequalities, we get

$$\begin{aligned} \left|2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i}\right| \left|2\tilde{A}_i + \frac{\varepsilon}{u}(-1)^d \pi_2^2 \tilde{A}_{d-i}\right| &= |N(2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i})| \\ &< \left|4 - \varepsilon^2 u^2 \pi_1^4\right| \left|\frac{1}{u^2}\pi_2^4 - 4\right| = 4, \end{aligned}$$

where N is the norm from K to $\mathbb{Q}$. Further

$$2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i} = -\pi_1(\pi_2 A_i - (-1)^d \varepsilon u \pi_1 A_{d-i})$$

and

$$2\tilde{A}_i + \frac{\varepsilon}{u}(-1)^d \pi_2^2 \tilde{A}_{d-i} = -\pi_2(\pi_1 \tilde{A}_i - (-1)^d \frac{\varepsilon}{u} \pi_2 \tilde{A}_{d-i}).$$

Hence

$$|N(\pi_2 A_i + (-1)^d \varepsilon u \pi_1 A_{d-i})| = \frac{1}{2}|N(2A_i - (-1)^d \varepsilon u \pi_1^2 A_{d-i}| < 2.$$

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We conclude that $\pi_2 A_i - (-1)^d \varepsilon u \pi_1 A_{d-i}$ is a unit. However, we have

$$|\pi_2 A_i + (-1)^d \varepsilon u \pi_1 A_{d-i}| < \left| \frac{4 - \varepsilon^2 u^2 \pi_1^4}{\pi_1} \right| < 4.562$$

and

$$\left| \pi_1 \tilde{A}_i - (-1)^d \frac{\varepsilon}{u} \pi_2 \tilde{A}_{d-i} \right| < \left| \frac{\frac{1}{u^2}\pi_2^4 - 4}{\pi_2} \right| < 0.4385.$$

The last inequality excludes the possibility $\pi_2 A_i + (-1)^d \varepsilon u \pi_1 A_{s-i} = \pm 1$. It remains the possibility that $\pi_2 A_i + (-1)^d \varepsilon u \pi_1 A_{d-i} = \pm u^k$ with $k \neq 0$. However, then $\pi_1 \tilde{A}_i - (-1)^d \frac{\varepsilon}{u} \pi_2 \tilde{A}_{s-i} = \pm u^{-k}$, but

$$\max(|u^k|, |u^{-k}|) \geq u = 4 + \sqrt{17} > 4.562,$$

hence this possibility is also excluded. Finally, we have proved that Tp has degree 0, so that

$$\pi_2 A_i + (-1)^d \varepsilon u \pi_1 A_{d-i} = 0 \text{ for } i = 1 \ldots d-1.$$

This implies that π_1 and π_2 divide each A_i for $i = 1 \ldots d-1$, so also divide each $\tilde{A}_i$. Thus $2|A_i$, so also $2|\tilde{A}_i$. Hence $A_i = 2B_i$ with $B_i \in \mathcal{O}_K$ for all i. and we get

$$\pi_2 B_i + (-1)^d \varepsilon u \pi_1 B_{d-i} = 0 \text{ for } i = 1 \ldots d-1.$$

We can repeat the same argument again and conclude that $2|B_i$ for all i. After several repetitions, we get

$$2^k | A_i \text{ for every positive integer } k \text{ and all } i.$$

Hence all coefficients A_i are zero. We have

$$\hat{f}_1 = 2x^d + (-1)^d \varepsilon u \pi_1^2 \text{ and } \check{f}_1 = 2x^d - (-1)^d \varepsilon u^{-1} \pi_2^2.$$

Finally, we get

$$f(x) = \frac{1}{4} \hat{f} \check{f} = \hat{f}_1 \check{f}_1 = 4x^{2d} \pm 2x^d - 4.$$

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