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CHARACTER DEGREES AND DERIVED LENGTH OF A SOLVABLE GROUP

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Let G be a finite group. (All groups considered here are finite). There exist several results which control the structure of G in terms of $\mathrm{cd}(G)$, the set of degrees of the irreducible complex characters of G. Here, we are concerned with the situation where only the cardinality of $\mathrm{cd}(G)$ is given. If $|\mathrm{cd}(G)| \leqq 3$, then it is known [**9**; **7**] that G is solvable and the derived length $\mathrm{dl}(G) \leqq |\mathrm{cd}(G)|$. If $|\mathrm{cd}(G)| = 4$, then G need not be solvable (e.g., $G = \mathrm{PSL}(2, 2^n)$); however [**5**], if G is solvable then $\mathrm{dl}(G) \leqq 4$. It is conjectured that for all solvable G, $\mathrm{dl}(G) \leqq |\mathrm{cd}(G)|$. In this paper we prove for solvable groups that

$$\mathrm{dl}(G) \leqq 3|\mathrm{cd}(G)| - 2$$

and that if G is nonabelian of odd order, then

$$\mathrm{dl}(G) \leqq 2|\mathrm{cd}(G)| - 2.$$

How can a hypothesis on $|\mathrm{cd}(G)|$ be used? One way is to show that if $\chi \in \mathrm{Irr}(G)$ and if only r different degrees $f \in \mathrm{cd}(G)$ satisfy $f \leqq \chi(1)$, then $G/\ker \chi$ is under control. (For instance if $r = 1$, then $G/\ker \chi$ is abelian.) Typical of this method, is Taketa's proof that M-groups are solvable. (See, [**4**, Satz V. 18.6].) This shows that if in the above situation, G is an M-group, then $\mathrm{dl}(G/\ker \chi) \leqq r$ and thus in particular, $\mathrm{dl}(G) \leqq |\mathrm{cd}(G)|$.

Let $\mathrm{cd}(G) = \{f_1, f_2, \ldots, f_n\}$ with $1 = f_1 < f_2 < \ldots < f_n$, and let $\alpha_G(r)$ denote

$$\max \{\mathrm{dl}(G/\ker \chi) | \chi \in \mathrm{Irr}(G), \chi(1) \leqq f_r\}.$$

(If $r > n$, write $\alpha_G(r) = \mathrm{dl}(G)$.) In this notation, we have $\alpha_G(r) \leqq r$ whenever G is an M-group. Our main result here is that $\alpha_G(r) \leqq 3r - 2$ for solvable groups and that $\alpha_G(r) \leqq 2r - 2$ if $r > 1$ and $|G|$ is odd. If $r = 2$, these bounds are best possible, but it seems highly unlikely that this is true for larger values of r.

1. The result of this section is just a corollary of the Fong-Swan Theorem (see [**2**, Theorem 72.1]).

THEOREM 1. *Let G be solvable and suppose that G acts faithfully and completely reducibly on the abelian group, A. Then G has a faithful (possibly reducible)*

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complex character, χ, *with* $\chi(1) \leqq \log_p(|A|)$, *where* p *is the smallest prime divisor of* $|A|$.

Proof. If $A = A_1 \dot{+} A_2$ where A_i is a proper G-invariant subgroup, then G has characters χ_1 and χ_2 with $\ker \chi_i = \mathbf{C}_G(A_i)$ and $\chi_i(1) \leqq \log_{p_i}(|A_i|) \leqq \log_p(|A|_i)$ where p_i is the smallest prime divisor of $|A_i|$, $p_i \geqq p$. Then $\chi = \chi_1 + \chi_2$ has the desired properties.

Since A is completely reducible, we may now assume that A is irreducible under G. Let $F = \mathrm{Hom}_G(A, A)$ so that F is a finite field and A is an irreducible $F[G]$-module. Let $\mathscr{Y}$ be the corresponding F-representation of G. Since $\mathrm{Hom}_{F[G]}(A, A) = F$, we may conclude that $\mathscr{Y}$ is absolutely irreducible and hence by the Fong-Swan Theorem, there exists a ring, $R \subseteq \mathbf{C}$ and an R-representation, $\mathscr{X}$, of G such that $(\mathscr{X}(g))\theta = \mathscr{Y}(g)$ for $g \in G$, where θ is a homomorphism of R onto an extension field of F.

Let χ be the (complex) character afforded by $\mathscr{X}$. Then $\ker \chi = \ker \mathscr{X} \subseteq \ker \mathscr{Y} = 1$ and $\chi(1) = \log_q(|A|)$ where $q = |F|$. The result now follows.

2. The result of this section is more or less known. (Compare [**1**, Theorems 4.4 and 4.5].)

THEOREM 2. *Let* $Z = \mathbf{Z}(G)$ *be cyclic and contain every abelian normal subgroup of* G. *Let* $F = \mathbf{F}(G)$, *the Fitting subgroup. Then* F/Z *is abelian. Suppose* $Z \subseteq A \subseteq F$ *with* $A \lhd G$ *and let* $C = \mathbf{C}_G(A/Z)$ *and* $B = \mathbf{C}_G(A)$. *Then* $AB = C$ *and* $A \cap B = Z$. *Furthermore,* F/Z *is a completely reducible* (G/F)*-module and if* G *is solvable, it is a faithful module.*

Proof. If $F' \not\subseteq Z$, we can choose $K \subseteq F'$, $K \lhd G$ minimal such that $K \not\subseteq Z$. Since F is nilpotent, $K > [K, F] \lhd G$ and hence $[K, F] \subseteq Z$. Thus $[K, F, F] = 1$ and therefore $[F', K] = 1$ by the Three Subgroups Lemma. Thus K is abelian, and since $K \lhd G$ and $K \not\subseteq Z$, this is a contradiction. Thus F/Z is abelian.

Now $A \cap B = \mathbf{Z}(A)$ is abelian. Since $Z \subseteq \mathbf{Z}(A) \lhd G$, we conclude that $A \cap B = Z$. Since A/Z is abelian and Z is cyclic, it follows from the fundamental theorem of abelian groups that $|\mathrm{Hom}(A/Z, Z)| \leqq |A/Z|$. If $x \in C$, we can define $\theta_x \in \mathrm{Hom}(A/Z, Z)$ by $\theta_x(\bar{a}) = [a, x]$. Note that if $\theta_x = \theta_y$, it follows that $yx^{-1} \in \mathbf{C}(A) = B$ and thus there are at least $|C : B|$ distinct θ_x and hence

$$|C|/|B| \leqq |\mathrm{Hom}(A/Z, Z)| \leqq |A|/|Z|$$

and $|AB| = |A||B|/|Z| \geqq |C|$. It follows that $AB = C$.

In particular, $F/Z = (A/Z) \times ((F \cap B)/Z)$ and thus F/Z is completely reducible. Also, $F/Z = \mathbf{F}(G/Z)$ and hence if G is solvable, then $F = \mathbf{C}_G(F/Z)$ and F/Z is a faithful G/F module and the proof is complete.

3. We need the following lemma. (See [**1**, Theorem 4.3] or [**8**, Proposition 4.1].)

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LEMMA 3. *Let* $\chi \in \mathrm{Irr}(G)$ *be faithful and suppose* $G' \subseteq \mathbf{Z}(G)$. *Then* $|G : \mathbf{Z}(G)| = \chi(1)^2$.

The next two results serve only to decrease the bound on $\alpha_G(r)$ from $2r - 1$ to $2r - 2$ when $2 \nmid |G|$ and $r > 1$. Nevertheless, it seems worthwhile to do this since the result $\alpha_G(2) \leqq 2$ is best possible. Theorem 4 is known and has appeared in numerous versions (see, [**4**, Satz V. 17.13] or [**6**, Proposition 5.2].) We include a proof here which seems shorter than most.

THEOREM 4. *Let* $N \lhd H$ *and suppose that a cyclic group*, C, *acts on* H, *stabilizes* N *and is semi-regular on* $(H/N)^\#$. *Let* $\theta \in \mathrm{Irr}(H)$ *be invariant under* C *and suppose* $\theta_N = e\varphi$ *with* $\varphi \in \mathrm{Irr}(N)$ *and* $e^2 = |H : N|$. *Then* $e \equiv \pm 1 \pmod{|C|}$.

Proof. Work in the semi-direct product $G = H \times C$. Since G/H is cyclic, [**4**, Satz V. 17.12 (2)] yields that θ is extendible to $\chi \in \mathrm{Irr}(G)$ and similarly φ is extendible to $\xi \in \mathrm{Irr}(NC)$. Every irreducible constituent of χ_{NC} is an extension of φ and hence has the form $\lambda\xi$ for some $\lambda \in \mathrm{Irr}(NC/N)$. Write $\chi_{NC} = (\sum_\lambda a_\lambda \lambda)\xi$ where λ runs over $\mathrm{Irr}(NC/N)$ and a_λ is a non-negative integer. Clearly

$$(1) \quad \sum_\lambda a_\lambda = e.$$

Now G/N is a Frobenius group. Let $\mathscr{C}$ be the set of conjugates of NC in G so that $|\mathscr{C}| = |H : N| = e^2$ and $G = H \cup \bigcup \mathscr{C}$. If $A, B \in \mathscr{C}$ are distinct, we have $A \cap B = N = A \cap H$. Therefore

$$|G|[\chi, \chi] = |H|[\chi_H, \chi_H] + |\mathscr{C}||NC|[\chi_{NC}, \chi_{NC}] - |\mathscr{C}||N|[\chi_N, \chi_N].$$

Since $[\chi, \chi] = 1 = [\chi_H, \chi_H]$, $[\chi_N, \chi_N] = e^2 = |\mathscr{C}|$, $|H| = e^2|N|$ and $[\chi_{NC}, \chi_{NC}] = \sum_\lambda a_\lambda^2$, this yields

$$|C|\,|N|e^2 = e^2|N| + e^2|N|\,|C|\sum a_\lambda^2 - e^2|N|e^2$$

and

$$(2) \quad |C| \sum_\lambda a_\lambda^2 = |C| - 1 + e^2.$$

Since $|\mathrm{Irr}(NC/N)| = |C|$, equations (1) and (2) yield

$$(3) \quad \sum_{\lambda,\mu} (a_\lambda - a_\mu)^2 = 2|C| \sum_\lambda a_\lambda^2 - 2\Big(\sum_\lambda a_\lambda\Big)^2 = 2(|C| - 1).$$

In particular, not all a_λ are equal.

It follows from (3) that for some $\lambda \in \mathrm{Irr}(NC/N)$, we have

$$\sum_\mu (a_\lambda - a_\mu)^2 \leqq 2(|C| - 1)/|C| < 2$$

and hence a_μ can differ from a_λ for at most one μ, and there $a_\mu = a_\lambda + \epsilon$ where $\epsilon = \pm 1$. Now equation (1) yields $e = (|C| - 1)a_\lambda + (a_\lambda + \epsilon)$ so that $a_\lambda = (e - \epsilon)/|C|$ and the proof is complete.

Note. The above proof actually shows that $\chi_C = ((e - \epsilon)/|C|)\rho_C + \epsilon\mu_C$ where ρ_C is the regular character of C. We will not need this, however.

COROLLARY 5. *Suppose $\chi \in \text{Irr}(G)$ is nonlinear and primitive and that $G/\mathbf{F}(G)$ is abelian. Then there exists nonlinear $\psi \in \text{Irr}(G)$ with $\psi(1)|(e + 1)$ for some $e|\chi(1)$.*

Proof. We may assume that χ is faithful. Then G satisfies the hypotheses of Theorem 2. Let $F = \mathbf{F}(G)$ and $Z = \mathbf{Z}(G)$. Since χ is nonlinear and primitive, G is not an M-group and so $G > F$. Since $F/Z = \mathbf{F}(G/Z)$, we conclude that G acts nontrivially on F/Z. Since F/Z is completely reducible, we can choose $A \lhd G$ with $Z < A \subseteq F$ and A/Z a chief factor of G with $C = \mathbf{C}_G(A/Z) < G$. Since $C \supseteq F$, G/C is abelian and acts irreducibly on A/Z. It follows that G/C is cyclic. Say $|G : C| = m$. Also, G/C acts semi-regularly on $(A/Z)^\#$.

We have $\chi_A = a\theta$ for some $\theta \in \text{Irr}(A)$. Let $\theta(1) = e$. Lemma 3 yields $e^2 = |A : Z|$. It follows from Theorem 4 that $e \equiv \pm 1 \pmod m$ and since the action of G/C on A/Z is irreducible and G/C is cyclic of order m, it follows from [**4**, Satz II. 3.10] that $e \not\equiv 1 \pmod m$. Thus $m|(e + 1)$.

Now let $\lambda \in \text{Irr}(A/Z)$ with $\lambda \neq 1_A$. Since $[A, G]Z = A$, λ is not invariant in G. By Theorem 2, we have $C/Z = (A/Z) \times (B/Z)$ where $B = \mathbf{C}_G(A)$ and it follows that λ is extendible to $\nu \in \text{Irr}(C)$. Now let ψ be any irreducible constituent of ν^G. Since λ is not invariant, $\psi(1) > 1$. On the other hand, $\psi(1) = \psi(1)/\nu(1)$ divides $|G : C| = m$ and the result follows.

4. We can now prove our main results. The method of proof is closely related to Huppert's derivation of a bound on $\text{dl}(G)$ when the degree of a faithful representation of G is given [**3**].

THEOREM 6. *Suppose G is solvable. Let $\chi \in \text{Irr}(G)$ and $M \lhd G$ such that $M \subseteq \ker \psi$ whenever $\psi \in \text{Irr}(G)$ with $\psi(1) < \chi(1)$. Then*
(a) $M''' \subseteq \ker \chi$,
(b) $M'' \subseteq \ker \chi$ *if* $2 \nmid \chi(1)$ *and*
(c) $M' \subseteq \ker \chi$ *if* $M = G'$ *and* $2 \nmid |G|$.

Proof. Use induction on $|G|$. In the group $G/\ker \chi$, the hypotheses are satisfied with respect to $M \ker \chi/\ker \chi$ and hence we may assume that χ is faithful. We may also assume that $\chi(1) > 1$.

Suppose χ is imprimitive so that $\chi = \theta^G$ with $\theta \in \text{Irr}(H)$ and $H < G$. Then all irreducible constituents of $(1_H)^G$ have degree $< |G : H| \leqq \chi(1)$ and thus $M \subseteq \ker ((1_H)^G) \subseteq H$. If $\varphi \in \text{Irr}(H)$ and $\varphi(1) < \theta(1)$, then all irreducible constituents of φ^G have degree $\leqq \varphi(1)\,|G : H| < \chi(1)$ and thus $M \subseteq \ker (\varphi^G) \subseteq \ker \varphi$.

Now, the hypotheses are satisfied by θ on H and by the inductive hypothesis, $M''' \subseteq \ker \theta$. Since $M''' \lhd G$, (a) follows. If $2 \nmid \chi(1)$, then $2 \nmid \theta(1)$ and $M'' \subseteq \ker \theta$. Since $M'' \lhd G$, (b) follows.

Suppose $M = G'$. If $\theta(1) = 1$, then $G'' \subseteq H' \subseteq \ker \theta$ and (c) follows in that case. If $\theta(1) > 1$, then $M \subseteq \ker \varphi$ for all linear $\varphi \in \operatorname{Irr}(H)$ and hence $G' = M \subseteq H' \subseteq G'$ and $M = H'$. Now the inductive hypothesis yields $G'' = H'' \subseteq \ker \theta$ and (c) follows here too.

Now suppose χ is primitive. Since χ is faithful, the hypotheses of Theorem 2 are satisfied and we let $F = \mathbf{F}(G)$ and $Z = \mathbf{Z}(G)$ so that G/F acts faithfully and completely reducibly on the abelian group, F/Z.

Suppose $M \subseteq F$. Then $M'' \subseteq F'' = 1$ and (a) and (b) follow. If $M = G'$, then G/F is abelian and Corollary 5 applies. Thus there exists $\psi \in \operatorname{Irr}(G)$ with $\psi(1) > 1$ and $\psi(1)|(e + 1)$ for some $e|\chi(1)$. Assume $|G|$ is odd. Then $\psi(1)$ is odd and $e + 1$ is even and we have $1 < \psi(1) \leqq (e + 1)/2 < e \leqq \chi(1)$ and thus $G' = M \subseteq \ker \psi$, a contradiction since ψ is nonlinear. Thus the hypotheses of (c) cannot hold in this case.

We may now assume that $M \nsubseteq F$. By Theorem 1, G has a character ξ with $F = \ker \xi$ and $\xi(1) \leqq \log_2(|F/Z|)$. If $2 \nmid |F/Z|$, then $\xi(1) \leqq \log_3 (|F/Z|)$. Now $\chi_F = a\theta$ for some $\theta \in \operatorname{Irr}(F)$ and $|F/Z| = \theta(1)^2 \leqq \chi(1)^2$ by Lemma 3. Thus $\xi(1) \leqq 2 \log_2 (\chi(1))$. Also, if $2 \nmid \chi(1)$, then $2 \nmid \theta(1)$, $2 \nmid |F/Z|$ and $\xi(1) \leqq 2 \log_3 (\chi(1))$.

Since $M \nsubseteq F = \ker \xi$, it follows that for some irreducible constituent, φ, of ξ, we have $\varphi(1) \geqq \chi(1)$. This yields $\chi(1) \leqq 2 \log_3 (\chi(1))$ if $2 \nmid \chi(1)$ and hence $\chi(1) \leqq 2$. Since $\chi(1) > 1$, this is a contradiction and the proofs of (b) and (c) are complete.

Thus $2|\chi(1)$ and $\chi(1) \leqq \varphi(1) \leqq \log_2 (|F/Z|) \leqq 2 \log_2 (\chi(1))$ forces $\chi(1) \leqq 4$ and thus $\chi(1) = \xi(1) = \varphi(1) = |F/Z|^{1/2} = 2$ or 4. Since $F/Z = \mathbf{F}(G/Z)$ is a 2-group, we have $\mathbf{O}_2(G/F) = 1$. Let $K/F = \mathbf{O}_{2'}(G/F)$ so that ξ_K is a sum of linear constituents (since $\xi = \varphi$ is irreducible) and K/F is abelian. Since $K/F = \mathbf{F}(G/F)$, we have $K = \mathbf{C}_G(K/F)$ and thus G/K faithfully permutes the linear constituents of ξ_K, and G/K is isomorphic to a subgroup of the symmetric group on $\xi(1) \leqq 4$ symbols. It follows that every $\psi \in \operatorname{Irr}(G/K)$ satisfies $\psi(1) < \xi(1) = \chi(1)$ and thus $M \subseteq K$. Therefore $M''' \subseteq K''' = 1$ and the proof is complete.

COROLLARY 7. *Let G be solvable. We have*
(a) $\alpha_G(r) \leqq 3r - 2$, *and*
(b) *if* $2 \nmid |G|$ *and* $r > 1$, *then* $\alpha_G(r) \leqq 2r - 2$.

Proof. Use induction on r. Suppose $\chi \in \operatorname{Irr}(G)$ with $\chi(1) \leqq f_r$ so that

$$G^{(\alpha_G(r-1))} \subseteq \ker \psi$$

for all $\psi \in \operatorname{Irr}(G)$ with $\psi(1) < \chi(1)$. By Theorem 6 (a), we have $G^{(\alpha_G(r-1)+3)} \subseteq \ker \chi$ and thus $\alpha_G(r) \leqq \alpha_G(r - 1) + 3$. Since $\alpha_G(1) = 1$, (a) now follows by induction.

Suppose $2 \nmid |G|$ so that $2 \nmid \chi(1)$. Using Theorem 6 (b), it follows that $\alpha_G(r) \leqq \alpha_G(r - 1) + 2$. Now suppose $r = 2$ so that $G' \subseteq \ker \psi$ whenever

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$\psi(1) < \chi(1)$. Theorem 6 (c) yields $G'' \subseteq \ker \chi$ and $\alpha_G(2) \leqq 2$. Now (b) follows by induction.

Note that if G is any nonabelian solvable group, then $\alpha_G(2) \geqq 2$ and hence if $2 \nmid |G|$ we have $\alpha_G(2) = 2$. On the other hand, if $G = GL(2, 3)$, we have $\alpha_G(2) = 4$ so that when $r = 2$, (a) and (b) are both best possible. We know of no examples where $\alpha_G(r) > r + 2$; however, for any prime, p, and positive integer, r, it is possible to construct a p-group, G, with $\alpha_G(r) = r$.

Since $\mathrm{dl}(G) = \alpha_G(|\mathrm{cd}(G)|)$, all of the results stated in the introduction have now been proved.

Added in proof. Using an inductive argument related to that of this paper, T. R. Berger has recently proved that $\mathrm{dl}(G) \leqq |\mathrm{cd}(G)|$ when $2 \nmid |G|$.

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