BULL. AUSTRAL. MATH. SOC. 60G42, 60F25, 60G50
VOL. 43 (1991) [273–277]

ON MOMENT CONDITIONS FOR SUPREMUM OF NORMED SUMS OF MARTINGALE DIFFERENCES

WI CHONG AHN, BONG DAE CHOI AND SOO HAK SUNG

Let $\{S_n, n \geqslant 1\}$ denote the partial sum of sequence (X_n) of identically distributed martingale differences. It is shown that $E|X_1|^q (\lg|X_1|)^r < \infty$ implies $E\left(\sup\left((\lg n)^{pr/q}/n^{p/q}\right)|S_n|^p\right) < \infty$, where $1 < p < 2$, $p < q$, $r \in R$ and $\lg x = \max\{1, \log^+ x\}$ For the independent identically distributed case, the converse of the above statement holds.

1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $\{c_n, n \geqslant 1\}$ constants such that $0 < c_n \uparrow \infty$. For each $n \geqslant 1$, let $S_n = X_1 + \cdots + X_n$. In this paper, we will investigate the conditions on (X_n) and (c_n) under which

$$E\left(\sup_n |S_n|^p / c_n\right) < \infty. \tag{1.1}$$

For independent identically distributed (i.i.d.) random variables (X_n) with $EX_1 = 0$ and $c_n = n^{p/q}$ $(1 < q < 2, p < q)$, it was shown by Choi and Sung [1] that (1.1) is equivalent to $E|X_1|^q < \infty$. This paper is a continuation of [1], and for the references about related works to the equivalent statements for (1.1), see [1].

In this paper, first we find conditions on (c_n) to guarantee the statement (1.1) when (X_n) is a sequence of identically distributed martingale differences. From this result, it is shown that if (X_n) are independent identically distributed with $EX_1 = 0$ and $c_n = n^{p/q}/(\lg n)^{pr/q}$ $(1 < q < 2, p < q, r \in R)$ then (1.1) is equivalent to $E|X_1|^q (\lg|X_1|)^r < \infty$, where $\lg x = \max\{1, \log^+ x\}$. When $r = 0$, this equivalence is reduced to the one mentioned above.

Throughout this paper, $C > 0$ will always stand for a constant which may be different in various places. $I(A)$ means the indicator function of event A.

Received 20th April 1990
This research was supported by Korea Science and Engineering Foundation 1989.

2. Main results

The following theorem [1] is essential for our main result and gives a sufficient condition of (1.1) for general increasing sequences (c_n) and positive constants α $(0 < \alpha < 2)$.

Theorem 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $\{c_n, n \geqslant 1\}$ constants such that $0 < c_n \uparrow \infty$. If $\sum_{n=1}^{\infty} E|X_n|^{\alpha\beta}/c_n^{\beta} < \infty$ for some $\beta > 1$ and $0 < \alpha\beta \leqslant 2$, then*

$$E\left(\sup_n \frac{|\sum_{k=1}^{n}(X_k - \alpha_k)|^{\alpha}}{c_n}\right) < \infty,$$

where $\alpha_k = 0$ if $0 < \alpha\beta \leqslant 1$ and $\alpha_k = E(X_k \mid X_1, \cdots, X_{k-1})$ if $1 < \alpha\beta \leqslant 2$.

The next result gives conditions on (c_n) to guarantee the statement (1.1).

Theorem 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed martingale differences and $\{c_n, n \geqslant 1\}$ constants such that $0 < c_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} P(|X_1|^p > c_n) < \infty.$$

If $c_n^{2/p} \sum_{i=n}^{\infty} 1/c_i^{2/p} = O(n)$ and $c_n^{\beta} \sum_{i=1}^{n} 1/c_i^{\beta} = O(n)$ for some β with $1 \leqslant p\beta \leqslant 2$ and $1 < \beta$, then

$$E\left(\sup_n \frac{|S_n|^p}{c_n}\right) < \infty.$$

Proof: Define $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}, Y_n = X_n I(|X_n|^p \leqslant c_n) - E(X_n I(|X_n|^p \leqslant c_n) \mid \mathcal{F}_{n-1})$ and $Z_n = X_n I(|X_n|^p > c_n) - E(X_n I(|X_n|^p > c_n) \mid \mathcal{F}_{n-1})$. Then $X_n = Y_n + Z_n$. The proof will be completed by showing that

$$E\left(\sup_n \frac{|\sum_{k=1}^{n} Y_k|^p}{c_n}\right) < \infty \tag{2.1}$$

and

$$E\left(\sup_n \frac{|\sum_{k=1}^{n} Z_k|^p}{c_n}\right) < \infty. \tag{2.2}$$

Result (2.1) is proved by applying Theorem 1 to the case $\alpha = p$ and $\beta = 2/p$, if we show that

$$\sum_{n=1}^{\infty} \frac{E|Y_n|^2}{c_n^{2/p}} < \infty. \tag{2.3}$$

https://doi.org/10.1017/S0004972700029051 Published online by Cambridge University Press

Since $E|Y_n|^2 \leqslant E|X_1|^2 I(|X_1|^p \leqslant c_n)$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{E|Y_n|^2}{c_n^{2/p}} &\leqslant \sum_{n=1}^{\infty} \frac{1}{c_n^{2/p}} E|X_1|^2 I(|X_1|^p \leqslant c_n) \\
&= \sum_{n=1}^{\infty} \frac{1}{c_n^{2/p}} \sum_{i=1}^{n} E|X_1|^2 I(c_{i-1} < |X_1|^p \leqslant c_i) \qquad (c_0 \equiv 0) \\
&= \sum_{i=1}^{\infty} E|X_1|^2 I(c_{i-1} < |X_1|^p \leqslant c_i) \sum_{n=i}^{\infty} \frac{1}{c_n^{2/p}} \\
&\leqslant \sum_{i=1}^{\infty} P(c_{i-1} < |X_1|^p \leqslant c_i) c_i^{2/p} \sum_{n=i}^{\infty} \frac{1}{c_n^{2/p}} \\
&\leqslant C \sum_{i=1}^{\infty} P(c_{i-1} < |X_1|^p \leqslant c_i) i \\
&= C \sum_{i=0}^{\infty} P(|X_1|^p > c_i) < \infty.
\end{aligned}$$

To prove (2.2), by Theorem 1, it is enough to show that

$$\sum_{n=1}^{\infty} \frac{E|Z_n|^{p\beta}}{c_n^{\beta}} < \infty.$$

Since $E|Z_n|^{p\beta} \leqslant 2^{p\beta} E|X_1|^{p\beta} I(|X_1|^p > c_n)$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{E|Z_n|^{p\beta}}{c_n^{\beta}} &\leqslant 2^{p\beta} \sum_{n=1}^{\infty} \frac{1}{c_n^{\beta}} E|X_1|^{p\beta} I(|X_1|^p > c_n) \\
&= 2^{p\beta} \sum_{n=1}^{\infty} \frac{1}{c_n^{\beta}} \sum_{i=n}^{\infty} E|X_1|^{p\beta} I(c_i < |X_1|^p \leqslant c_{i+1}) \\
&= 2^{p\beta} \sum_{i=1}^{\infty} E|X_1|^{p\beta} I(c_i < |X_1|^p \leqslant c_{i+1}) \sum_{n=1}^{i} \frac{1}{c_n^{\beta}} \\
&\leqslant 2^{p\beta} \sum_{i=1}^{\infty} P(c_i < |X_1|^p \leqslant c_{i+1}) c_{i+1}^{\beta} \sum_{n=1}^{i+1} \frac{1}{c_n^{\beta}} \\
&\leqslant C 2^{p\beta} \sum_{i=1}^{\infty} P(c_i < |X_1|^p \leqslant c_{i+1})(i+1) \\
&\leqslant C 2^{p\beta} \sum_{i=0}^{\infty} P(|X_1|^p > c_i) < \infty. \qquad \square
\end{aligned}$$

https://doi.org/10.1017/S0004972700029051 Published online by Cambridge University Press

LEMMA 3. ([2], p.155) *Let X be a random variable and $\{c_n, n \geqslant 1\}$ constants such that $0 < c_n \uparrow \infty$. Let ϕ be any even nondecreasing function satisfying $\phi(c_n) = n$ for all $n \geqslant 1$. Then*

$$E\phi(X) < \infty \text{ if and only if } \sum_{n=1}^{\infty} P(|X| > c_n) < \infty.$$

Let $\phi(x) = q^r x^q (\lg x)^r$ on $[0, \infty)$ for $1 < q < 2$ and $r \in R$. Since $\phi'(x)$ is positive for large x and $\phi(x) \to \infty$ as $x \to \infty$, we can choose an increasing sequence (c_n) such that $\phi(c_n) = n$ for $n \geqslant n_0$ and $c_n \to \infty$. Thus we obtain a nondecreasing sequence $\{c_n, n \geqslant 1\}$ by letting $c_n = c_{n_0}$ for $1 \leqslant n < n_0$. Then we have $c_n \sim n^{1/q}/(\lg n)^{r/q}$ by the following calculation: from the identity $\phi(c_n) = n$, that is, $q^r c_n^q (\lg c_n)^r = n$,

$$\begin{aligned} \frac{n^{1/q}}{c_n (\lg n)^{r/q}} &= \frac{q^{r/q} c_n (\lg c_n)^{r/q}}{c_n (r \log q + q \lg c_n + r \lg(\lg c_n))^{r/q}} \\ &= \frac{(q \lg c_n)^{r/q}}{(r \log q + q \lg c_n + r \lg(\lg c_n))^{r/q}} \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus there exists an integer N such that

$$1 - \varepsilon < n^{1/q}/c_n (\lg n)^{r/q} < 1 + \varepsilon$$

for $n \geqslant N$. Hence we have

$$\sum_{n=N}^{\infty} P(|X| > (1+\varepsilon)c_n) < \sum_{n=N}^{\infty} P\left(|X| > \frac{n^{1/q}}{(\lg n)^{r/q}}\right) < \sum_{n=N}^{\infty} P(|X| > (1-\varepsilon)c_n).$$

Since $E|X|^q (\lg |X|)^r < \infty$ if and only if $E(C|X|)^q (\lg C|X|)^r < \infty$, we have by Lemma 3 that

$$E|X|^q (\lg |X|)^r < \infty \text{ if and only if } \sum_{n=1}^{\infty} P(|X| > Cc_n) < \infty.$$

Thus we have that

$$E|X|^q (\lg |X|)^r < \infty \text{ if and only if } \sum_{n=1}^{\infty} P\left(|X| > \frac{n^{1/q}}{(\lg n)^{r/q}}\right) < \infty. \tag{2.4}$$

THEOREM 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed martingale differences with $E|X_1|^q (\lg |X_1|)^r < \infty$ for $1 < q < 2$ and $r \in R$. Then for $p < q$*

$$E\left(\sup_n \left(\frac{(\lg n)^{r/q}}{n^{1/q}}\right)^p |S_n|^p\right) < \infty.$$

https://doi.org/10.1017/S0004972700029051 Published online by Cambridge University Press

PROOF: Let $c_n = \left(n^{1/q}/(\lg n)^{r/q}\right)^p$. Choose a constant β with $\beta = s/p$ ($p < s < q$, $1 \leqslant s < q$). Then we have by (2.4) that $\sum_{n=1}^{\infty} P(|X_1|^p > c_n) < \infty$. Some computation shows that

$$\int_1^n \frac{(\lg x)^{sr/q}}{x^{s/q}} dx \leqslant C \frac{(\lg n)^{sr/q}}{n^{(s/q)-1}}.$$

Thus we have

$$c_n^\beta \sum_{i=1}^n \frac{1}{c_i^\beta} \leqslant C \frac{n^{s/q}}{(\lg n)^{sr/q}} \int_1^n \frac{(\lg x)^{sr/q}}{x^{s/q}} dx \leqslant Cn.$$

Similarly we have

$$c_n^{2/p} \sum_{i=n}^{\infty} \frac{1}{c_i^{2/p}} \leqslant C \frac{n^{2/q}}{(\lg n)^{2r/q}} \int_n^{\infty} \frac{(\lg x)^{2r/q}}{x^{2/q}} dx \leqslant Cn.$$

Thus the result follows from Theorem 2. □

COROLLARY 5. *Let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with mean zero. Then the followings are equivalent: for $1 < q < 2$, $p < q$ and $r \in R$*

(a) $E\left(\sup_n \left(\frac{(\lg n)^{r/q}}{n^{1/q}}\right)^p |S_n|^p\right) < \infty$;

(b) $E\left(\sup_n \left(\frac{(\lg n)^{r/q}}{n^{1/q}}\right)^p |X_n|^p\right) < \infty$;

(c) $E|X|^q (\lg |X|)^r < \infty$.

PROOF: The proof is similar to [1] and is omitted. □

REMARK. The result in [1] is a special case of Corollary 5 in the case $r = 0$.

REFERENCES

[1] B. D. Choi and S. H. Sung, 'On moment conditions for supremum of normed sums', *Stochastic Process. Appl.* **26** (1987), 99-106.

[2] W. F. Stout, *Almost Sure Convergence* (Academic Press, New York, 1974).

Department of Mathematics Education
Koomin University
Seoul 136-702
Korea

Department of Applied Mathematics
Korea Advanced Institute of Science and Technology
Seoul 130-650
Korea

Coding Technology Section I
Electronics and Telecommunications Research Institute
Taejeon 305-606
Korea

https://doi.org/10.1017/S0004972700029051 Published online by Cambridge University Press