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On Extensions of Stably Finite C^*-Algebras (II)

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Abstract. For any C^*-algebra A with an approximate unit of projections, there is a smallest ideal I of A such that the quotient A/I is stably finite. In this paper a sufficient and necessary condition for an ideal of a C^*-algebra with real rank zero to be this smallest ideal is obtained by using K-theory.

1 Introduction and Main Results

Let A be a C^*-algebra. Denote by A_+ the set of all positive elements in A. We will also use $K_0(A)_+$ for the positive cone of the K_0-group, $K_0(A)$, of A, *i.e.*, $K_0(A)_+ = \{[p]_0 \in K_0(A) : p \text{ is a projection in } A \otimes \mathcal{K}\}$. Throughout this paper, by an ideal of an arbitrary C^*-algebra we will, unless otherwise specified, mean a closed two-sided ideal. A C^*-algebra A is called finite if it admits an approximate unit of projections and all projections in A are finite. If $A \otimes \mathcal{K}$ is finite, then A is called stably finite. Concerning extensions of stably finite C^*-algebras, J. S. Spielberg [4, 1.5] obtained the following important result.

Theorem 1.1 *Let A be a C^*-algebra, let I be an ideal in A, and suppose that I and A/I are stably finite. Then A is stably finite if and only if $\partial(K_1(A/I)) \cap K_0(I)_+ = 0$.*

Let A be a C^*-algebra with an approximate unit of projections, and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of ideals of A. We proved [5] that if A/I_λ is a stably finite C^*-algebra for each $\lambda \in \Lambda$, then $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ is a stably finite C^*-algebra. Thus, there is a smallest ideal I of A such that the quotient A/I is stably finite. Throughout this paper, we denote this smallest ideal of A by $I(A)$. It is easy to see that for any stably finite quotient Q of A there is a canonical surjective *-homomorphism from $A/I(A)$ to Q.

Theorem 1.2 (1.3 [5]) *Let A be a C^*-algebra with an approximate unit of projections and let I be an ideal of A which has real rank zero. If A/I is stably finite and for any $x \in K_0(I)_+$ there is an element y in $\partial(K_1(A/I)) \cap K_0(I)_+$ such that $x \le y$, then $I = I(A)$.*

At the end of [5], we left a question concerning the converse direction as follows: let A be a C^*-algebra which has real rank zero; for any $x \in K_0(I(A))_+$ is there an element y in $\partial(K_1(A/I(A))) \cap K_0(I(A))_+$ such that $x \le y$? The main purpose of this

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paper is to give a positive answer to this question. We will show the following main result.

Theorem 1.3 *Let A be a C^*-algebra with real rank zero and let I be an ideal of A. Then $I = I(A)$ if and only if A/I is stably finite and for any $x \in K_0(I)_+$ there is an element y in $\partial(K_1(A/I)) \cap K_0(I)_+$ such that $x \le y$.*

I do not know if the hypothesis of real rank zero is necessary in the above theorem. The next result is an immediate corollary of Theorem 1.3.

Corollary 1.4 *If A is a C^*-algebra with real rank zero, then $I(A) = A$ if and only if $K_0(A)_+$ is a group. Furthermore, if A is also unital, then $I(A) = A$ if and only if $K_0(A)_+ = K_0(A)$.*

Let A and B be C^*-algebras. If ϕ is a *-homomorphism from A to B, then $\phi(I(A)) \subset I(B)$. In fact, the image $\pi \circ \phi(A)$ is stably finite where π is the canonical map from B to $B/I(B)$. Hence $I(A) \subset \ker(\pi \circ \phi)$ and so $\phi(I(A)) \subset \ker \pi = I(B)$. It is easy to show that the following statement holds.

Corollary 1.5 *For each sequence $A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots$ of C^*-algebras, if $\varinjlim A_n$ has real rank zero, then $I(\varinjlim A_n) = \varinjlim I(A_n)$.*

2 Proofs

Before we prove the main result, let us introduce the following several lemmas. The first lemma is a generalization of Lemma 3.3.6 of [3].

Lemma 2.1 (2.5 [5]) *If $B \subset A_+$ is a subset of a C^*-algebra A and p is a projection in the ideal generated by B, then there are $x_1, \dots, x_k$ in A, and $a_1, \dots, a_k$ in B such that*

$$p = \sum_{i=1}^{k} x_i a_i x_i^*.$$

Let A be a C^*-algebra and let $M_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. For any $a \in M_n(A)$ and $b \in M_m(A)$, $a \oplus b$ refers to the matrix $\operatorname{diag}(a, b)$ in $M_{n+m}(A)$. Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A), \phi_n)$, where $\phi_n \colon M_n(A) \to M_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

We will also use $M_\infty(A)_+$ to denote the set of all positive elements in $M_\infty(A)$. Given $a, b \in M_\infty(A)_+$, we say that a is Cuntz subequivalent to b, written $a \precsim b$, if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of $M_\infty(A)$ such that $\lim_{n\to\infty} \|x_n b x_n^* - a\| = 0$. We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. It is easy to see that if p and q are projections, the definition of $p \precsim q$ is equivalent to there being a partial isometry $u \in M_\infty(A)$ with $u^* u = p$ and $uu^* \le q$.

Lemma 2.2 (2.4 [5]) *Let A be a C^*-algebra, $a, b \in A_+$. Then $a + b \precsim a \oplus b$. If A has real rank zero and $a \perp b$ (i.e., $ab = 0$), then $a + b \sim a \oplus b$.*

Lemma 2.3 *Let A be a C^*-algebra with an approximate unit of projections. Let J be an ideal of A generated by*

$$\big\{ q \in A : \text{there is a hyponormal partial isometry } v \in A \text{ such that } v^*v - vv^* = q \big\}.$$

Then for any $x = [p]_0$ in $K_0(J)_+$ where p is a projection in J, there is an element y in $\partial(K_1(A/J)) \cap K_0(J)_+$ such that $x \le y$.

Proof Note that J is the ideal of A generated by

$$C = \{q \in A : \text{there is a hyponormal partial isometry } v \in A \text{ such that } v^*v - vv^* = q\}.$$

For any projection p in J, by Lemma 2.1, there are $x_1, \dots, x_k$ in A and there are projections $q_1, \dots, q_k$ in C such that $p = \sum_{i=1}^k x_i q_i x_i^*$. By Lemma 2.2,

$$p \precsim \bigoplus_{i=1}^k x_i q_i x_i^* \precsim \bigoplus_{i=1}^k q_i.$$

So $[p]_0 \le \sum_{i=1}^k [q_i]_0$. Note that by the construction of C, $\sum_{i=1}^k [q_i]_0$ belongs to

$$\partial(K_1(A/J)) \cap K_0(J)_+. \qquad \blacksquare$$

Lemma 2.4 (2.2 [5]) *Let A be a C^*-algebra with an approximate unit of projections.*

(i) *If B is an ideal of A, with an approximate unit of projections, then $I(B) \subset I(A)$.*
(ii) *$I(\widetilde{A}) = I(A)$ where $\widetilde{A}$ is the unitization of A.*
(iii) *$I(M_n(A)) = M_n(I(A))$, $I(A \otimes \mathcal{K}) = I(A) \otimes \mathcal{K}$.*

Proof of Theorem 1.3 It suffices to show the "only if" part of the statement. By Lemma 2.4(iii), without any loss of generality we may assume that I, A, and A/I are stable. Let $\mathcal{S}$ be the set of all ideals J in A that satisfy that $J \subset I(A)$ and for any $x \in K_0(J)_+$ there is an element y in $\partial(K_1(A/J)) \cap K_0(J)_+$ such that $x \le y$. Then $(\mathcal{S}, \subset)$ is a partially ordered set. The theorem will be proved by showing that $I(A)$ belongs to $\mathcal{S}$.

Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be a chain in S and set $K = \overline{\bigcup_\lambda J_\lambda}$. For each λ the diagram

$$\begin{array}{ccc} K_1(A/J_\lambda) & \xrightarrow{\partial_\lambda} & K_0(J_\lambda) \\ {\scriptstyle \eta_\lambda}\downarrow & & \downarrow{\scriptstyle \iota_\lambda} \\ K_1(A/K) & \xrightarrow[\partial]{} & K_0(K) \end{array}$$

commutes. For any $x \in K_0(K)_+$, there are λ in Λ and x' in $K_0(J_\lambda)_+$ such that $\iota_\lambda(x') = x$. According to the definition of $\mathcal{S}$, there is an element y' in $K_1(A/J_\lambda)$ such that $\partial_\lambda(y') \ge x'$. Put $y = \eta_\lambda(y')$. Then $\partial(y) = \iota_\lambda(\partial_\lambda(y')) \ge \iota_\lambda(x') = x$. Hence K is an upper bound of the chain $\{J_\lambda\}_{\lambda \in \Lambda}$. Therefore by Zorn's lemma there is a maximal element M of $\mathcal{S}$.

We claim that $M = I(A)$. Otherwise, $M \subsetneq I(A)$ and there is a hyponormal partial isometry v in A/M. Let M_0 be the ideal in A/M generated by $v^*v - vv^*$, and let π be the

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canonical mapping from C^*-algebra A to A/M. Putting $M_1 = \{a \in A : \pi(a) \in M_0\}$, it is easy to see that $M_1 \subset I(A)$. We get a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & A & \longrightarrow & A/M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & A & \longrightarrow & A/M_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1/M & \longrightarrow & A/M & \longrightarrow & A/M_1 & \longrightarrow & 0 \end{array}$$

where each row is exact. Therefore we have the following commutative diagram.

$$\begin{array}{ccc} K_1(A/M) & \xrightarrow{\partial''} & K_0(M) \\ \pi_0 \downarrow & & \downarrow \phi_0 \\ K_1(A/M_1) & \xrightarrow{\partial} & K_0(M_1) \\ \equiv \downarrow & & \downarrow \psi_0 \\ K_1(A/M_1) & \xrightarrow[\partial']{} & K_0(M_1/M) \end{array}$$

For any $x \in K_0(M_1)_+$, let $x' = \psi_0(x)$. Note that x' is in $K_0(M_1/M)_+$. By Lemma 2.3, there are $a' \in K_0(M_1/M)_+$ and $b \in K_1(A/M_1)$ such that $a' \geq x'$ and $\partial'(b) = a'$. Set $a = \partial(b)$. Since A has real rank zero, by [6], there is $c \in K_0(M_1)_+$ such that $\psi_0(c) = a' - x'$. Set $d = c + x$. We then have $d \geq x$ and $\psi_0(d) = a'$. Since

$$\psi_0(d - a) = a' - \partial'(b) = 0,$$

there is $d'' \in K_0(M)$ such that $\phi_0(d'') = d - a$. Note that M have real rank zero, and so $K_0(M)_+ - K_0(M)_+ = K_0(M)$. Hence there are e'' and f'' in $K_0(M)_+$ such that $d'' = e'' - f''$. According to the definition of $\mathcal{S}$, there is $g'' \in K_1(A/M)$ such that $\partial''(g'') \geq e''$. Set $g = \pi_0(g'')$. We obtain that

$$\partial(b + g) = a + \phi_0(\partial''(g'')) \geq a + \phi_0(e'') \geq a + \phi_0(d'') = a + (d - a) = d = c + x \geq x.$$

Consequently, $M_1 \in \mathcal{S}$ which contradicts the maximality of M. Therefore, $M = I(A) \in \mathcal{S}$. This completes the proof of Theorem 1.3. ∎

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