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ON CHARACTERS IN THE PRINCIPAL 2-BLOCK, II

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Abstract

Let k be a non-zero complex number and let u and v be elements of a finite group G. Suppose that at most one of u and v belongs to $O(G)$, the maximal normal subgroup of G of odd order. It is shown that G satisfies $X(v)-X(u)=k$ for every complex nonprincipal irreducible character X in the principal 2-block of G, if and only if $G/O(G)$ is isomorphic to one of the following groups: C_2, $PSL(2, 2^n)$ or $P\Sigma L(2, 5^{2a+1})$, where $n \geqslant 2$ and $a \geqslant 1$.

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1. Introduction

Let G be a finite group. It was shown by Berger and Herzog (1978) that if $u \in G$ and $k \in \mathbf{C}$ satisfy:

$$X(1)-X(u)=k$$

for every complex non-principal irreducible character in the principal 2-block of G, then either $u \in O(G)$ or $G/O(G)$ is isomorphic to one of the following simple groups: C_2, $PSL(2, 2^n)$, $n \geqslant 2$. The converse also holds.

The aim of this paper is to consider the more general equality

$$X(v) - X(u) = k, \tag{1}$$

where k is a non-zero complex number, $v, u \in G$ and (1) holds for every complex non-principal irreducible character in the principal 2-block of G. In this case we obtain new candidates for $G/O(G)$, namely $P\Sigma L(2, 5^{2a+1})$, $a \geqslant 1$, the extension of $PSL(2, 5^{2a+1})$ by the group of automorphisms of the Galois field with 5^{2a+1} elements.

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Our main result is

THEOREM 1. *Let G be a finite group, u and v be elements of G and k be a non-zero complex number. Suppose that* (1) *is satisfied by every complex non-principal irreducible character of G belonging to B, the principal 2-block of G. Then, either* $\{u,v\}\cap O(G)| = 1$ *or* $k = \pm 4$ *and $G/O(G)$ is isomorphic to* $P\Sigma L(2, 5^{2a+1})$, $a \geqslant 1$.

We also prove the following

PROPOSITION. *Let* $G = P\Sigma L(2, 5^{2a+1})$, $u \in G$ *be an involution and* $v \in G$ *be of order* $2a+1$ *such that* $G = \langle PSL(2, 5^{2a+1}), v\rangle$. *Then* (1) *holds for every complex non-principal irreducible character in the principal* 2*-block of G, with* $k = 4$.

The authors are grateful to the referee for providing the proof of the Proposition.
Combining these results with the Theorem of Berger and Herzog (1978), we get

THEOREM 2. *G satisfies the assumptions of Theorem* 1, *if and only if $G/O(G)$ is isomorphic to one of the following groups:* C_2, $PSL(2, 2^n)$, $n \geqslant 2$ *and* $P\Sigma L(2, 5^{2a+1})$, $a \geqslant 1$.

In this paper G denotes a finite group. The order of G is g and if $v \in G$, $o(v)$ denotes the order of v. The principal 2-block of G is denoted by B, and the number of irreducible characters in B is b. The letter X will always denote an irreducible character in B ($X \in B$). A fixed Sylow 2-subgroup of G will be denoted by S. If H is a subgroup of G and $v \in G$, then $o(v \bmod H)$ is the least positive integer n satisfying $v^n \in H$, and $\exp H$ is the least positive integer m satisfying: $h^m = 1$ for every $h \in H$. The group of outer automorphisms of H will be denoted by $\operatorname{Out} H$. We denote by $\sum$ or $\sum^{\#}$ the summation over all $X \in B$ or $X \in B \setminus 1_G$, respectively. The expression 'the orthogonality relations in blocks' will be abbreviated by O.R.B. Finally, C_2 will denote the cyclic group of order 2.

2. Proof of Theorem 1

It is well known that $O(G) = \bigcap \{\ker X \mid X \in B\}$. As $k \neq 0$, it follows that not both u and v belong to $O(G)$. So assume that $u, v \notin O(G)$ and it suffices to prove the theorem under the assumption that $O(G) = 1$.

It is well known that if $y \in G$, then $\sum^{\#} X(y)$ is a rational integer. Thus, by (1), $(b-1)k \in Z$ and since $X(v) - X(u)$ is an algebraic integer, we conclude that

(2) $$k \in Z - \{0\}.$$

Suppose that $y \in G$ does not belong to the 2-sections of either v or u in G. Then, by (1) and the O.R.B.,

$$0 = \sum X(y)(X(v) - X(u)) = k \sum^{\#} X(y)$$

yielding

(3) $$\sum^{\#} X(y) = 0.$$

It follows that $y \neq 1$ and consequently we may assume without loss of generality that

(4) $$v \text{ has odd order}, \quad o(v) > 1.$$

Let w be a 2-element of G of maximal order, and let z be the involution in $\langle w \rangle$. Then, by the O.R.B., $\sum X(1) X(z) = 0$, and since as in Berger and Herzog (1978)

(5) $$X(w) \equiv X(z) \equiv X(1) \pmod{\mathscr{P}},$$

where $\mathscr{P}$ is the prime ideal lying over 2 in $\mathscr{O}$, the integers in $Q(\sqrt[g]{1})$, it follows that

$$\sum^{\#} X(1) \equiv \sum^{\#} X^2(1) \equiv \sum^{\#} X(1) X(z) \equiv 1 \pmod{2}.$$

Hence

(6) $$\sum^{\#} X(w) \equiv \sum^{\#} X(z) \equiv \sum^{\#} X(1) \equiv 1 \pmod{2}.$$

Thus, in view of (3) and (4), $w = z$ and we get

(7) $\exp S = 2$, where S is a Sylow 2-subgroup of G,

(8) G has one class of involutions, and

(9) $o(u) = 2f$, where f is an odd integer.

In particular, G has exactly two 2-sections.

Choose H, a minimal normal subgroup in G. As $O(G) = 1$, it follows by (8) that G/H is of odd order and as in Berger and Herzog (1978), either $H = S$ or H is isomorphic to one of the following simple groups: $PSL(2,q)$, $q > 3$, $q \equiv 0$, 3 or 5 (mod 8), J (Janko's smallest group) or $\mathrm{Re}(q)$ (a group of Ree type). Since none of the above-mentioned groups satisfies (1) for a v satisfying (4), it follows that

(10) G/H is a non-trivial soluble group of odd order.

Let Y be a non-principal linear character of G/H and suppose that $Y \in B$. Clearly, by (1) and (2), $k = \pm 1$ or ± 2. If $k = \pm 2$, then by (1) $\{Y(v), Y(u)\} = \{1, -1\}$, which is impossible since G/H is of odd order. If $k = \pm 1$, then by (1)

$$\{Y(v), Y(u)\} = \{\exp(\tfrac{1}{3}\pi i), \exp(\tfrac{2}{3}\pi i)\} \quad \text{or} \quad \{\exp(\tfrac{4}{3}\pi i), \exp(\tfrac{5}{3}\pi i)\},$$

again in contradiction to (10). Thus:

(11) No non-principal linear character of G/H belongs to B.

Proceeding exactly as in Berger and Herzog (1978), we get

(12) $G = C_G(S)\,H$,

(13) H is non-abelian simple,

(14) G/H is isomorphic to a subgroup of Out H,

(15) $H \not\simeq J$, $PSL(2, 2^n)$, $n \geqslant 2$, and

(16) If Y is an irreducible character of G/H belonging to B, then $Y = 1$.

Suppose that $H \simeq \mathrm{Re}(q)$. As in Berger and Herzog (1978), B consists of 8 characters X_i, $i = 1, \ldots, 8$, such that $X_i|_H = \xi_i$, $i = 1, \ldots, 8$. We use here the notation of Ward (1966) for the irreducible characters and elements of H. By the O.R.B., (1), (4) and (9) we get

$$0 = \sum \bar{X}(v)\,X(u) = 1 + k\sum\nolimits^{\#} X(u) + \sum\nolimits^{\#} |X(u)|^2$$

whence

(17) $$0 = k\sum\nolimits^{\#} X(u) + \sum |X(u)|^2.$$

In addition, the O.R.B. yield:

(18) $$0 = \sum X(u)\,X(R) = X_1(u) + X_2(u) + X_3(u) + X_4(u)$$

and

(19) $$0 = \sum X(u)\,(3X(R) + X(S) + X(V) + X(W)) = 6X_1(u) + 6X_2(u).$$

As $X_1(u) = 1$, (18) and (19) yield:

(20) $$X_2(u) = -X_1(u) = -1, \quad X_4(u) = -X_3(u).$$

The O.R.B. also yield:

$$0 = \sum X(u)\,X(Y) = m(X_5(u) + X_6(u) + X_7(u) + X_8(u))$$

whence

(21) $$X_5(u) + X_6(u) + X_7(u) + X_8(u) = 0.$$

It follows from (17), (18) and (21) that

(22) $$k = \sum |X(u)|^2.$$

Applying the O.R.B. to v we get

$$0 = \sum X(v)\, X(JR) = X_1(v) - X_2(v) + X_3(v) - X_4(v),$$

which implies in view of (1) and (20)

$$X_3(u) = -X_4(u) = (k-2)/2. \tag{23}$$

Thus k is even, and by (20), (22) and (23):

$$k \geqslant 1 + 1 + (k-2)^2/2.$$

It follows that one of the following holds:

$$k = 4, \quad X_i(u) = 0 \quad \text{for } i = 5, 6, 7, 8,$$

or

$$k = 2, \quad X_i(u) = 0 \quad \text{for } i = 3, 4, 5, 6, 7, 8.$$

Another application of the O.R.B. yields, in view of (1), (20) and (23),

$$0 = \sum X(v)\, X(JS) = 1 - (k-1) - (3k/2 - 1) + (k/2 + 1)$$

so that $k = 2$.

A final application of the O.R.B., together with (20), yields:

$$0 = \sum X(u)\, X(1) = 1 + (-1)(q^2 - q + 1) = q(1-q),$$

a contradiction.

Finally, suppose that $H \cong PSL(2,q)$, $q > 5$ and $q \equiv 3$ or $5 \pmod 8$. As in Berger and Herzog (1978), B consists of 4 characters X_i, $i = 1, \ldots, 4$, such that $X_i|_H = \theta_i$, $i = 1, \ldots, 4$. We use here the notation of Ward (1966), pp. 62–65, for the irreducible characters and elements of H. By the O.R.B. we have

$$0 = \sum X(u)\, X(R) = X_1(u) - eX_4(u),$$

where $e = \pm 1$ satisfying $q \equiv 4 + e \pmod 8$, as defined in Ward's paper. Hence,

$$X_4(u) = e. \tag{24}$$

Thus, again by the O.R.B.,

$$0 = \sum X(u)\, X(1) = 1 + (q+e)(X_2(u) + X_3(u))/2 + eq$$

yielding

$$X_2(u) + X_3(u) = -2e. \tag{25}$$

A final application of the O.R.B., together with (1), (24) and (25), yields

$$0 = \sum X(v)\, X(S_0^{(q-e)/4}) = 1 - 2ke + 2 + ek + 1,$$

whence $k = 4e$ and $X_4(v) = 5e$.

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Now by (10) and (14)

$$(26) \qquad PSL(2,q) \subset G \subseteq P\Sigma L(2,q).$$

Thus G has a 2-transitive permutation representation of degree $q+1$, the restriction of which to H is also 2-transitive. Let Y be the irreducible character of G of degree q corresponding to this representation. Then $Y|_H$ is irreducible, and since $Y(1) = q$, $Y|_H = \theta_4 = X_4|_H$ and consequently $X_4 = Y \cdot \xi$, where ξ is a linear character of the cyclic group G/H (see Isaacs (1976), (6.17)). Thus $5e = X_4(v) = Y(v)\,\xi(v)$, where $Y(v)$ is an integer $\geqslant -1$ and $\xi(v)$ is an odd root of 1. We conclude that $e = 1$ and $Y(v) = 5$. So v fixes exactly $5+1 = 6$ elements in the permutation representation of G. Let $q = p^c$ and let $o(v \bmod H)$ be d. Then d divides c and

$$6 = \text{fix}(v) = 1 + p^{c/d}.$$

Consequently $p = 5$ and $d = c$; as $5^c = q \equiv 4 + e = 5 \pmod 8$, $c = 2a+1$ for some $a \geqslant 1$. Since $o(v \bmod H) = c = 2a+1$, by (26) $G = P\Sigma L(2, 5^{2a+1})$, and the proof of Theorem 1 is complete.

3. Proof of the Proposition

Let $2a+1 = r$, $q = 5^r$ and let $H \lhd G$, $H \simeq PSL(2,q)$. Since $|G:H| = r$, then $u \in H$. It follows from the arguments of Section 2 that the principal 2-block B of G consists of 4 irreducible characters: X_i, $i = 1, \ldots, 4$, such that $X_i|_H = \theta_i$, $i = 1, \ldots, 4$. For the irreducible characters and elements of H we use again the notation of Ward (1966), pp. 62–65.

As in Section 2, G has an irreducible character Y of degree q corresponding to the 2-transitive permutation representation of G of degree $1+q$ on Ω, and again $Y|_H$ is irreducible, whence $Y|_H = \theta_4$. Being the unique extension of θ_4 which is rational, $Y \in B$ forcing $Y = X_4$. Moreover, $Y(u) = 1$ and $Y(v) = 5$ since v fixes exactly 6 elements of Ω. Thus X_4 satisfies (1) with $k = 4$, and it suffices to show that also X_2 and X_3 do so. By the O.R.B. we have:

$$0 = \sum X(u)\,X(v) = 1 - X_2(v) - X_3(v) + 5$$

whence $X_2(v) + X_3(v) = 6$. As $X_2(u) = X_3(u) = -1$, it suffices to show that $X_2(v) = 3$. In particular, it suffices to show that $\psi = \theta_2$ has an extension $\hat{\psi}$ to G with $\hat{\psi}(v) = 3$, since being the unique extension of ψ which is rational on v, $\hat{\psi} \in B$ whence $\hat{\psi} = X_2$.

Let $R = \langle v \rangle$ and choose $Q \in \text{Syl}_5(H)$ and a cyclic subgroup C of H of order $(q-1)/2$, such that $N \equiv N_H(Q) = QC$ and $R \subseteq N_G(Q) \cap N_G(C)$. It follows from the character table of H that $\psi|_N = \theta + \lambda$, where θ is irreducible of degree $(q-1)/2$ and $\lambda^2 = 1_N$. Now θ has a unique extension $\hat{\theta}$ to NR such that $\hat{\theta}$ is real (see Isaacs

(1976), Theorems 11.22 and 6.17, remembering that N has 2 irreducible characters of degree $(q-1)/2$ and $|NR:N|$ is odd). It can be shown similarly, that ψ has a unique extension $\hat{\psi}$ to G such that $\hat{\psi}|_{NR}$ contains $\hat{\theta}$ as a component. Thus $\hat{\psi}|_{NR} = \hat{\theta}+\hat{\lambda}$, where $\hat{\lambda}$ is an extension of λ. Since $\hat{\psi}$ is unique and $\hat{\theta}$ is real, so also $\hat{\psi}$ is real, forcing $\hat{\lambda}(v)$ to be real. Consequently $\hat{\lambda}(v) = 1$ and it suffices to show that $\hat{\theta}(v) = 2$.

Since θ is a character of N induced from Q, $\theta|_C$ is the regular character of C. Write the representation which affords $\hat{\theta}|_{CR}$ with reference to a basis consisting of eigenvectors for a generator c of C. As the eigenvectors correspond to distinct eigenvalues, and as v normalizes C, the matrix representing v must be monomial, and has precisely two zero entries on the diagonal (namely, in the positions corresponding to the eigenvalues $+1$ and -1 of c; no other $\frac{1}{2}(q-1)$th root of 1 is invariant under the fifth powering action of v). Thus $\hat{\theta}|_{CR} = \mu+\nu+\tau$, where μ and ν are linear characters and τ is a character vanishing on v and on uv, where u denotes an involution in C. Since $\hat{\theta}$ is real, $\mu+\nu$ is real on v and on uv. Choose the notation so that $\mu(u) = 1 = -\nu(u)$. Then both $(\mu+\nu)(v) = \mu(v)+\nu(v)$ and $(\mu+\nu)(uv) = \mu(v)-\nu(v)$ are real, forcing $\mu(v)$ and $\nu(v)$ to be real. Consequently $\mu(v) = \nu(v) = 1$ and $\hat{\theta}(v) = 2$, as required.

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