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ON PRIME IMMERSIONS OF S^1 INTO R^2

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1. Introduction. A C^1-mapping f from the oriented circle S^1 into the oriented plane R^2 such that $f'(t) \neq 0$ for all t is called a *regular immersion.* We call a point p in $\operatorname{Im} f$ a *double point* if $f^{-1}(p)$ is a two element set with the corresponding tangent vectors being linearly independent. A regular immersion which is one-to-one except at a finite number of points whose images are double points is called a *normal immersion.* The work of Whitney [**7**], Titus [**3**] and Verhey [**6**] shows that the normal immersions form a dense open subset in the space of regular immersions with the usual C^1-topology, and can be characterized up to diffeomorphic equivalence by a combinatorial invariant called the *intersection sequence.* It follows that any invariant which produces the intersection sequence characterizes a normal immersion up to an orientation preserving diffeomorphism of R^2. In [**2**] it is shown that the Marx-Blank invariant has this property.

In this note, with every normal immersion we associate a "word" and an integer ± 1, both of which may be read from the diagram (oriented image) of the immersion. A *prime immersion* is defined and it is shown that two prime immersions are diffeomorphically equivalent if and only if they possess equivalent signed words. It is shown that a word of a normal immersion can be uniquely factored into prime words. Using this fact, we obtain necessary and sufficient conditions for two normal immersions to be diffeomorphically equivalent.

2. The word of a normal immersion. Let f be a normal immersion of S^1 into R^2. We shall call the directed arc between two successive double points in $\operatorname{Im} f$ a *boundary arc.* With each boundary arc A lying in the boundary of the unbounded region of $R^2 - \operatorname{Im} f$, we shall associate a point in $\operatorname{Int} A$ and call this point a *boundary point.*

Let $x_1, x_2, \ldots, x_n$ denote the preimages of double points and boundary points in their natural cyclic order in S^1 such that $f(x_1)$ is a boundary point. Then we shall call $f(x_1)f(x_2)\ldots f(x_n)$ the *word* of f with respect to $f(x_1)$. By omitting the boundary points from the word of f, we obtain the associated *reduced word* of f. A nonempty reduced word is called *prime* if it contains no proper segment S such that each letter in the alphabet of S occurs precisely twice. For instance, *abcabc* is prime while *dabcabcd* is not. A word is called

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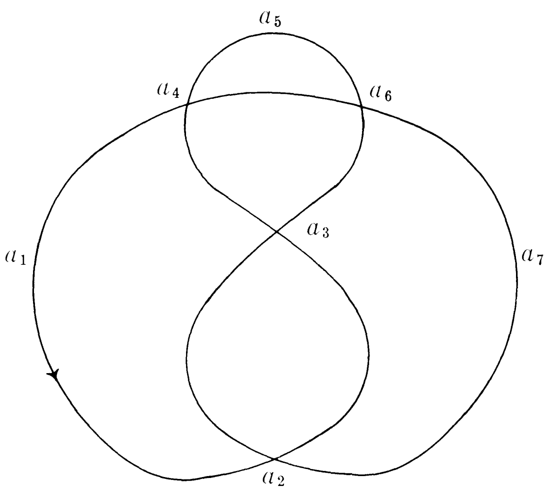
prime if its associated reduced word is prime, and we shall call a normal immersion *prime* if it possesses a prime word.

With each of the points $x_1, x_2, \ldots, x_n$ we associate a sign, denoted by $\epsilon(x_i)$, as follows:
If $i \neq j$ and $f(x_i) = f(x_j)$, then $\epsilon(x_j) = \text{sgn}\det[f'(x_i), f'(x_j)]$. If $f(x_j)$ is a boundary point, then $\epsilon(x_j)$ is the sign of the determinant of the outward normal at $f(x_j)$ with $f'(x_j)$. We shall define the sign of a boundary point to be $\epsilon(f(x_j)) = \epsilon(x_j)$.

Two words $W_1 = a_1a_2 \ldots a_m$ and $W_2 = b_1b_2 \ldots b_n$ are *equivalent* if and only if $m = n$ and $a_i = a_j$ if and only if $b_i = b_j$. If in addition $\epsilon(a_1) = \epsilon(b_1)$, we shall call W_1 and W_2 *equivalent signed words.*

As an example of a signed word associated with a normal immersion, consider the normal immersion represented in Figure 1.



$a_1a_2a_3a_4a_5a_6a_3a_2a_7a_6a_4, \quad \epsilon(a_1) = 1$

FIGURE 1

We note that a word together with all the signs associated with the pre-images of the letters in its alphabet determine an intersection sequence [**6**, p. 48] for a normal immersion f.

THEOREM 1. *Two prime immersions f, g are diffeomorphically equivalent if and only if they possess equivalent signed words.*

Proof. (Necessity) If f and g are diffeomorphically equivalent normal immersions, then by Theorem 2.1 of [**6**] the set of distinct intersection sequences of f is equal to the set of distinct intersection sequences of g. It then follows that f and g must possess equivalent signed words.

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(Sufficiency) Suppose f, g are prime immersions with equivalent signed words $W_1 = f(x_1)f(x_2)\ldots f(x_n)$, $W_2 = g(y_1)g(y_2)\ldots g(y_n)$ respectively. To complete the proof of the theorem it suffices to show that $\epsilon(x_i) = \epsilon(y_i)$ for $i = 2, \ldots, n$. There are two cases to consider.

Case 1. There is exactly one double point in $\operatorname{Im} f$, $\operatorname{Im} g$. In this case there are just two distinct words to consider. One word must be of the form $a_1a_2a_2$ with signs $\epsilon(x_1) = 1$, $\epsilon(x_2) = 1$, $\epsilon(x_3) = -1$ or $\epsilon(x_1) = -1$, $\epsilon(x_2) = -1$, $\epsilon(x_3) = 1$. The second possibility is a word of the form $a_1a_2a_3a_2$ with signs $\epsilon(x_1) = 1$, $\epsilon(x_2) = -1$, $\epsilon(x_3) = -1$, $\epsilon(x_4) = 1$ or $\epsilon(x_1) = -1$, $\epsilon(x_2) = 1$, $\epsilon(x_3) = 1$, $\epsilon(x_4) = -1$. In either event, the theorem follows.

Case 2. Now suppose $\operatorname{Im} f$ and $\operatorname{Im} g$ have more than one double point. Then the mappings f, g are prime mappings in the sense defined by Treybig in [**5**, p. 248]. Since W_1, W_2 are equivalent prime words, it follows from Theorem 3 of [**5**] that there is an autohomeomorphism h of R^2 which maps the boundary arcs of $\operatorname{Im} f$ onto the corresponding boundary arcs of $\operatorname{Im} g$.

Let D_1, D_2 denote the unbounded regions of $R^2 - \operatorname{Im} f$, $R^2 - \operatorname{Im} g$ respectively. Since f, g are prime mappings, it follows from Theorem 9 of [**4**] that the boundaries $\operatorname{Bd} D_1$, $\operatorname{Bd} D_2$ are simple closed curves. Without loss of generality we may assume that $\operatorname{Bd} D_i$ $(i = 1, 2)$ lies on a circle C_i except in a neighborhood of double points.

Let $A_1, A_2, \ldots, A_k$ denote the boundary arcs lying on $\operatorname{Bd} D_1$ in a counterclockwise order such that $f(x_1) \epsilon A_1$ and $A_i \cap A_{i+1} \neq \phi$ for $i = 1, 2, \ldots, k$, where $A_{k+1} = A_1$. Let $h(A_i) = B_i$ for $i = 1, 2, \ldots, k$. Since $\epsilon(x_1) = \epsilon(y_1)$, it follows that A_1 induces a counterclockwise orientation on C_1 if and only if B_1 induces a counterclockwise orientation on C_2. Then, since $h(A_i) = B_i$ with the initial (terminal) point of A_i being mapped onto the initial (terminal) point of B_i for $i = 1, 2, \ldots, k$, an inductive argument can be used to show that A_i induces a counterclockwise orientation on C_1 if and only if B_i induces a counterclockwise orientation on C_2. Consequently, we may suppose that h is the identity on $R^2 - \operatorname{Int} C$ where C denotes a circle in R^2 whose interior contains $\operatorname{Im} f$ and $\operatorname{Im} g$. Hence h must be an orientation preserving homeomorphism.

Let $\alpha : S^1 \to S^1$ be an orientation preserving diffeomorphism such that $\alpha(y_i) = x_i$ for $i = 1, 2, \ldots, n$. Then $g = hf\alpha$ and it follows that $\epsilon(x_i) = \epsilon(y_i)$ for $i = 1, 2, \ldots, n$.

We remark that the class of prime immersions is reasonably substantial. In fact, it follows from the proof of Theorem 5.1 in [**1**] that every oriented tame knot type has a representative whose projected diagram is the image of a prime immersion.

3. The prime factorization of a normal immersion. Let f be a normal immersion possessing a nonempty reduced word W which is not prime. Then W can be written in the form ABC where AC, B are nonempty words such that each letter in the alphabet of B occurs precisely twice. Clearly AC and B themselves are the reduced words for some immersion. Moreover, W can be written in the above form where B is prime. If AC is not prime the process may be repeated and so on. In this manner every nonempty reduced word W may be factored into a finite number of prime reduced words, called the *prime factors* of W. Furthermore, it is easy to see that this factorization is unique.

We shall say that two normal immersions f, g have *equivalent prime factorizations* if they possess equivalent signed words W_1, W_2 such that for every pair of corresponding prime factors $U_1 = f(x_{i_1}) \ldots f(x_{i_{2k}})$, $U_2 = g(y_{i_1}) \ldots g(y_{i_{2k}})$ associated with the reduced words of W_1, W_2 respectively, $\epsilon(x_{i_1}) = \epsilon(y_{i_1})$.

THEOREM 2. *Two normal immersions f, g are diffeomorphically equivalent if and only if they possess equivalent prime factorizations.*

Proof. (Necessity) Since two diffeomorphically equivalent normal immersions possess identical sets of distinct intersection sequences [**6**, p. 48], it follows that they must possess equivalent prime factorizations.

(Sufficiency) It suffices to show that if f, g are prime immersions with equivalent reduced words $f(x_1)f(x_2) \ldots f(x_{2n})$, $g(y_1)g(y_2) \ldots g(y_{2n})$ such that $\epsilon(x_1) = \epsilon(y_1)$, then $\epsilon(x_i) = \epsilon(y_i)$ for $i = 2, \ldots, 2n$. For then it would follow that two normal immersions having equivalent prime factorizations would possess identical sets of intersection sequences.

Let f, g be as above and we may suppose $n > 1$, for otherwise we are finished. Then f, g are prime mappings in the sense defined by Treybig in [**5**, p. 248]. Let $\alpha : S^1 \to S^1$ be an orientation preserving diffeomorphism such that $\alpha(y_i) = x_i$ for $i = 1, 2, \ldots, 2n$. By Theorem 2 in [**5**] there is a natural one-to-one correspondence between the complementary regions of Im f and those of Im g according to the equation $\alpha^{-1}f^{-1}(\text{Bd } U) = g^{-1}(\text{Bd } V)$, where U and V are corresponding complementary regions of Im f and Im g, respectively. Unfortunately, since the reduced word of a normal immersion does not determine the boundary arcs of the unbounded complementary regions of the immersion, the unbounded regions of $R^2 -$ Im f and $R^2 -$ Im g may not be corresponding regions.

Let D denote the complementary region of Im g which corresponds to the unbounded complementary region of Im f and let E denote the unbounded region of $R^2 -$ Im g. First suppose D has a boundary arc B in common with E. Consider a ray L with initial point in D such that $L \cap$ Im g is a single point b in Int B, and whose open end tends to ∞. Let abc be a directed arc in Int B containing b in its interior, and let A be a smooth arc whose interior lies in $E-L$ and is such that $A \cap$ Im $g = \{a, c\}$. Then, by replacing the arc abc by A and smoothing the resulting curve at the points a and b, we obtain the image of an immersion whose reduced word and associated signs are identical to those of g, and whose unbounded complementary region corresponds to the unbounded complementary region of Im f.

If D is not adjacent to the unbounded region of $R^2 - \text{Im } g$, then the above argument can be successively applied a finite number of times to obtain the same result. Hence we may assume that the unbounded complementary regions of $\text{Im} f$ and $\text{Im } g$ correspond.

By Theorem 3 of [**5**] there is an autohomeomorphism h of R^2 such that $g = hf\alpha$. Since f, g are prime mappings, by Theorem 9 of [**4**] the boundaries of the unbounded complementary regions of $\text{Im} f$, $\text{Im } g$ are simple closed curves, and without loss of generality we may assume that except for a neighborhood of double points these curves lie on circles C_1, C_2 respectively. Let $A_1(B_1)$ denote the boundary arc $[f(x_{2n}), f(x_1)]([g(y_{2n}), g(y_1]])$. Since $\epsilon(x_1) = \epsilon(y_1)$, it follows that A_1 induces a counterclockwise orientation on C_1 if and only if B_1 induces a counterclockwise orientation on C_2. Then, as in the proof of Theorem 1, it follows that h is orientation preserving and $\epsilon(x_i) = \epsilon(y_i)$ for $i = 1, 2, \ldots, 2n$.

We note that any two reduced words for a normal immersion f differ by a cyclic permutation and thus have the same number of prime factors. Consequently, we have the following corollary to Theorem 2.

COROLLARY. *The number of prime factors of a normal immersion is a numerical invariant.*

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