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SOLUTIONS TO EXTREMAL PROBLEMS IN E^p SPACE

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1. Introduction.

Let Ω be a bounded domain (in the complex plane) whose boundary, C, consists of finitely many disjoint, rectifiable, closed Jordan curves.

By definition, $F \in E^p(\Omega)$ ($p \in (0, \infty)$) if F is holomorphic on Ω and if there exists a sequence, $\{\Omega_j\}_{j=1}^{\infty}$, of domains such that $\bar{\Omega}_j \subset \Omega_{j+1} \subset \Omega, \bigcup_{j=1}^{\infty} \Omega_j = \Omega$, $\partial\Omega_j$ consists of rectifiable curves homologous to C, and $\sup_j \int_{\partial\Omega_j} |F(z)|^p |dz| < \infty$.

If $F \in E^p(\Omega)$, then F has boundary values for nontangential approach at almost every point of C. We denote the boundary function of F by F^+, and the collection of all such boundary functions by $E^p_+(C)$. $E^p_+(C)$ is a subspace of $L^p(C)$ (the p^{th} Lebesgue space with respect to arc length). (For proofs of the above assertions, see [9] and [2], Chapter 10.)

The following theorem is the basis of much of our work.

THEOREM 1.1. *Let* $p \in (1, \infty), q = p/(p-1), f \in L^p(C), g \in L^\infty(C), \dfrac{1}{g} \in L^\infty(C)$. *Then*:

i) *There exists a unique* $H_0^+ \in E^p_+(C)$ *for which*

$$\|f - gH_0^+\|_p = \inf\{\|f - gF^+\|_p : F^+ \in E^p_+(C)\} = d\ .$$

ii) $$d = \sup\left\{\mathrm{Re}\left(\int_C \frac{f(\zeta)}{g(\zeta)} G^+(\zeta)d\zeta\right) : G^+ \in E^q_+(C) \text{ and } \left\|\frac{G^+}{g}\right\|_q \le 1\right\}.$$

iii) *If* $d \neq 0$, *then there exists a unique* $G_0^+ \in E^q_+(C)$ *for which*

$$\left\|\frac{G_0^+}{g}\right\|_q \le 1 \quad \text{and} \quad d = \mathrm{Re}\int_C \frac{f(\zeta)}{g(\zeta)} G_0^+(\zeta)d\zeta\ .$$

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iv) *There is a unique* $H^+ \in E_+^p(C)$ *and a unique* $R^+ \in E_+^q(C)$ *such that*

$$f = gH^+ + \left|\frac{\zeta'}{g}R^+\right|^q \Big/ \left(\frac{\zeta'}{g}R^+\right).$$

(ζ' denotes the derivative of any arc length parametrization of C which leaves Ω to the left of C).

v) $H^+ = H_0^+$ *and (if* $d \neq 0$*)* $[R^+/\|R^+/g\|_q] = G_0^+$.

Proof. See Tumarkin and Havinson [8], pp. 209, 210. (The present formulation of the result is taken from [7].)

In this paper we assume ζ' is Hölder continuous in order to derive an operator equation which the extremal difference $f - gH^+$ satisfies. For $p = 2$, the operator equation is used to obtain a sequence of $L^2(C)$ functions converging at a geometrical rate in the $L^2(C)$ norm to H^+. (The Rayleigh-Ritz method may also be used to compute H^+, but the rate of convergence is not necessarily geometrical unless C is analytic, [7].) For the case that $p = 2$ and g is Hölder continuous, we transform the operator equation into a Fredholm integral equation in order to obtain a sequence of functions coverging uniformly to H^+.

2. The Operator Equation.

We say $\varphi \in \mathrm{Lip}\,(C, \beta)$ if φ is a (complex-valued) Hölder continuous function on C, whose exponent of Hölder continuity is β $(\in (0, 1])$. Similarly, $\psi \in \mathrm{Lip}\,(C \times C)$ if ψ is Hölder continuous on $C \times C$. (Whenever convenient, the exponent of Hölder continuity will be suppressed.)

LEMMA 2.1. *Let* $\zeta' \in \mathrm{Lip}\,(C)$, *and* $p \in (1, \infty)$. *Then for* $k \in L^p(C)$ *(and* $x \in C$*)*

$$\int_C' \frac{k(\zeta)}{\zeta - x} d\zeta$$

defines a bounded linear operator from $L^p(C)$ *to* $L^p(C)$. *(The symbol* $\int'$ *denotes the Cauchy-Lebesgue principal value integral.)*

Proof. See [4], pp. 19–21.

THEOREM 2.1. *Let the conditions and notation be as in Theorem* 1.1 *with the further assumption that* $\zeta' \in \mathrm{Lip}\,(C)$. *Then for almost every* $x \in C$

$$
(1)\quad
\begin{aligned}
&f(x) - g(x)H^+(x)\\
&+ \frac{1}{\pi i}\int_C' \left\{ \frac{\left| \frac{1}{\pi i}\int_C' \left(\frac{|f(\xi) - g(\xi)H^+(\xi)|^p}{f(\xi) - g(\xi)H^+(\xi)}\right)\frac{g(\xi)}{g(\zeta)}\frac{|d\xi|}{(\zeta - \xi)} \right|^q}{\frac{1}{\pi i}\int_C' \left(\frac{|f(\eta) - g(\eta)H^+(\eta)|^p}{f(\eta) - g(\eta)H^+(\eta)}\right)\frac{g(\eta)}{g(\zeta)}\frac{|d\eta|}{(\zeta - \eta)}} \right\} \frac{g(x)}{g(\zeta)}\frac{|d\zeta|}{(\zeta - x)}\\
&= f(x) - g(x)\frac{1}{\pi i}\int_C' \frac{f(\zeta)}{g(\zeta)}\frac{d\zeta}{(\zeta - x)} .
\end{aligned}
$$

Proof. From Theorem 1.1 (iv) it is clear that

$$
(2)\qquad R^+ = \left(\frac{|f - gH^+|^p}{f - gH^+}\right)\frac{g}{\zeta'}
$$

and

$$
(3)\qquad H^+ = \frac{f}{g} - \frac{1}{g}\left(\left|\frac{\zeta' R^+}{g}\right|^q \Big/ \left(\frac{\zeta' R^+}{g}\right)\right) .
$$

Since $R^+ \in E^q_+(C)$ and $q > 1$, the values of R may be recovered by applying the Cauchy integral formula to R^+ (see [2], Chapter 10). Hence it is clear from the Plemelj-Privalov formulas ([3], p. 431) that for almost every $x \in C$

$$
(4)\qquad R^+(x) = \frac{1}{\pi i}\int_C' \frac{R^+(\zeta)}{\zeta - x} d\zeta .
$$

Similarly,

$$
(5)\qquad H^+(x) = \frac{1}{\pi i}\int_C' \frac{H^+(\zeta)}{\zeta - x} d\zeta .
$$

Formally Theorem 2.1 may be obtained as follows: Substitute the right side of (2) for R^+ in the right side of (4). Substitute the resulting expression for R^+ in the right side of (3). Substitute this new expression for H^+ in the right side of (5). Routine manipulation then produces the desired conclusion. The application of Lemma 2.1 makes this argument rigorous.

3. The Solution when $p = 2$.

DEFINITION 3.1. Let $\zeta' \in \mathrm{Lip}\,(C)$ and let both g and $1/g$ be in $L^\infty(C)$. We then say that:

i) $I : L^2(C) \to L^2(C)$ is the identity operator.

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ii) $T: L^2(C) \to L^2(C)$ is defined for each $h \in L^2(C)$ by

$$T(h)(x) = \frac{1}{\pi i} \int_C' h(\zeta) \frac{g(x)}{g(\zeta)} \frac{|d\zeta|}{(\zeta - x)} .$$

(From Lemma 2.1, we see that T is a bounded linear operator.)

iii) $\tilde{T}: L^2(C) \to L^2(C)$ is defined for each $h \in L^2(C)$ by

$$\tilde{T}(h)(x) = -\frac{1}{\pi i} \int_C' h(\zeta) \frac{\overline{g(\zeta)}}{\overline{g(x)}} \frac{|d\zeta|}{(\bar{x} - \bar{\zeta})} .$$

($\tilde{T}$ is also a bounded linear operator.)

If $p = 2$, then (1) is a linear operator equation, from which we obtain

$$(I + T\tilde{T})(gH^+) = u \tag{6}$$

where $u(x) = g(x)\frac{1}{\pi i} \int_C' \frac{f(\zeta)}{g(\zeta)} \frac{d\zeta}{(\zeta - x)} + T\tilde{T}(f)(x)$, a known $L^2(C)$ function.

Finding H^+ (when $p = 2$) is now reduced to the problem of inverting the bounded linear operator $I + T\tilde{T}$.

LEMMA 3.1. *Let* $\zeta' \in \mathrm{Lip}\,(C)$, $p \in (1, \infty)$, $q = p/(p-1)$, $h \in L^p(C)$, $k \in L^q(C)$. *Then*

$$\int_C \left(\int_C' h(\zeta) k(\xi) \frac{d\zeta}{\zeta - \xi} \right) d\xi = \int_C \left(\int_C' h(\zeta) k(\xi) \frac{d\xi}{\zeta - \xi} \right) d\zeta .$$

Proof. See [4], p. 27.

LEMMA 3.2. *T and $\tilde{T}$ are adjoint operators.*

Proof. Let h and k be $L^2(C)$ functions. Formally then:

$$\langle Th, k \rangle = \int_C \left(\frac{1}{\pi i} \int_C' h(\zeta) \frac{g(\xi)}{g(\zeta)} \frac{|d\zeta|}{(\zeta - \xi)} \right) \overline{k(\xi)} \, |d\xi|$$
$$= \int_C h(\zeta) \overline{\left(-\frac{1}{\pi i} \int_C' k(\xi) \frac{\overline{g(\xi)}}{\overline{g(\zeta)}} \frac{|d\xi|}{(\bar{\zeta} - \bar{\xi})} \right)} |d\zeta| = \langle h, \tilde{T}k \rangle .$$

Lemma 3.1 justifies this formal manipulation.

THEOREM 3.1. *Let the conditions and notation be as in Theorem 1.1 with the further assumptions that $p = 2$ and $\zeta' \in \mathrm{Lip}\,(C)$. Let $c \in \left(0, \frac{1}{\|I + T\tilde{T}\|}\right)$. Then:*

i) $\|I - c(I + T\tilde{T})\| \leq 1 - c < 1$.

ii) $c \sum_{j=0}^{\infty} (I - c(I + T\tilde{T}))^j = (I + T\tilde{T})^{-1}$ *(convergence in the operator norm.)*

iii) $\frac{1}{g}\left(c \sum_{j=0}^{m} (I - c(I + T\tilde{T}))^j u\right)$ *is a sequence of* $L^2(C)$ *functions converging to* H^+ *in the* $L^2(C)$ *norm as* $m \to \infty$.

Proof. Since T is adjoint to $\tilde{T}$ we have that $I + T\tilde{T}$ is a self-adjoint operator. Thus, if $\|h\|_2 = 1$,

$$\langle (I - c(I + T\tilde{T}))h, h\rangle = 1 - c\langle (I + T\tilde{T})h, h\rangle \geq 1 - c\|I + T\tilde{T}\| > 0 . \tag{7}$$

Furthermore,

$$\begin{aligned} \langle (I - c(I + T\tilde{T}))h, h\rangle &= 1 - c\langle (I + T\tilde{T})h, h\rangle \\ &= 1 - c(1 + \|\tilde{T}h\|_2^2) \leq 1 - c < 1 . \end{aligned} \tag{8}$$

Since $I - c(I + T\tilde{T})$ is also self-adjoint, assertion (i) follows from (7) and (8). Assertion (ii) is an immediate consequence of (i), while (iii) may be obtained by applying (ii) to equation (6).

4. The Solution when $p = 2$ and $g \in \mathrm{Lip}\,(C)$.

LEMMA 4.1. *Let* ζ' *be continuous and* $\varphi \in \mathrm{Lip}\,(C \times C, \beta)$. *Then*

$$\omega(\xi, x) = \int_C' \frac{\varphi(\xi, \zeta)}{\zeta - x} d\zeta$$

is in $\mathrm{Lip}\,(C \times C, \delta)$, *where* δ *is any number on* $(0, \beta)$.

Proof. See [5], pp. 45–51.

Throughout the rest of this section we take the conditions and notation to be as in Theorem 1.1, with the further assumptions that $p = 2, \zeta' \in \mathrm{Lip}\,(C, \beta)$, and $g \in \mathrm{Lip}\,(C, \beta)$.

LEMMA 4.2. *For* $h \in L^2(C)$

i) $K(h)(x) = \int_C \left(\frac{1}{2\pi^2} \int_C' \frac{g(x)\overline{g(\xi)}}{|g(\zeta)|^2 (\zeta - x)(\bar{\xi} - \bar{\zeta})} |d\zeta|\right) h(\xi)\, |d\xi|$

determines a bounded linear operator, K, *from* $L^2(C)$ *to* $L^2(C)$.

ii) $K = \frac{1}{2}(I - T\tilde{T})$. *(See Definition* 3.1.)

Proof. From [5], p. 19, it may be seen that $\left(\frac{\xi - \zeta}{\bar{\xi} - \bar{\zeta}}\right)$ is in

Lip $(C \times C, \beta)$ (if the ratio is defined to be $(\zeta')^2$ when $\xi = \zeta$). Thus, as a function of ξ and ζ,

$$\varphi(\xi, \zeta) = \frac{1}{2\pi^2} \frac{\overline{g(\xi)}}{|g(\zeta)|^2} \left(\frac{\xi - \zeta}{\bar{\xi} - \bar{\zeta}} \right) \frac{1}{\zeta'} \tag{9}$$

is in Lip $(C \times C, \beta)$. If we define

$$\kappa(x, \xi) = \frac{1}{2\pi^2} \int_C' \frac{g(x)\overline{g(\xi)}}{|g(\zeta)|^2(\zeta - x)(\bar{\xi} - \bar{\zeta})} |d\zeta| ,$$

routine manipulation shows that

$$\kappa(x, \xi) = \left(\frac{\omega(\xi, x) - \omega(\xi, \xi)}{\xi - x} \right) g(x)$$

where ω is as in Lemma 4.1, and φ is defined by (9). Clearly, $\omega \in$ Lip $(C \times C, \delta)$ (for every $\delta \in (0, \beta)$) so that for $x \neq \xi$, κ is continuous and

$$|\kappa(x, \xi)| \leq \frac{M_\delta}{|\xi - x|^{1-\delta}}$$

(M_δ a positive constant independent of x and ξ). Thus κ is a Fredholm kernel with a weak singularity, and since

$$K(h)(x) = \int_C \kappa(x, \xi) h(\xi) |d\xi| ,$$

K must be a bounded linear operator from $L^2(C)$ to $L^2(C)$ (see, for example, [4], pp. 13–14). This proves (i).

If $h \in$ Lip (C), then $Kh = \frac{1}{2}(I - T\tilde{T})h$ follows from the Poincaré-Bertrand formula ([5], p. 57). But Lip (C) is dense in $L^2(C)$, and K and $\frac{1}{2}(I - T\tilde{T})$ are bounded linear operators, so that assertion (ii) must be true.

From Lemma 4.2 and (6) we have that

$$(I - K)(gH^+) = u_1 \tag{10}$$

where $u_1 = \dfrac{u}{2} \in L^2(C)$. (An integral equation similar to (10) was presented without proof and without solution in the paper of Rosenbloom and Warschawski [7].) Hence

$$(I - K^N)(gH^+) = u_N \tag{11}$$

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where

(12) $$u_N = \left(\sum_{\ell=0}^{N-1} K^\ell\right)u_1 \qquad (N = 1, 2, 3, \cdots) .$$

LEMMA 4.3. *Let v be continuous on C. Let W be a Fredholm integral operator (on $L^2(C)$) with a continuous kernel. Suppose there is a number c such that for every eigenvalue, λ, of W, $\left|(1-c)+\frac{c}{\lambda}\right| < 1$ and $|1-c| < 1$. Then:*

The integral equation $(I - W)\varphi = v$ has exactly one solution in $L^2(C)$, and

$$c \sum_{j=0}^{m} (I - c(I - W))^j v$$

is a sequence of continuous functions converging uniformly to that solution as $m \to \infty$.

Proof. See Bückner [1], pp. 63–65. (Bückner states his result in terms of an iteration scheme, from which the above sequence may be easily obtained.)

THEOREM 4.1. *Let u_N be defined by* (12). *Let N be an odd integer greater than $\frac{1}{\beta}$ and let $c \in \left(0, \frac{2}{1+\|K^N\|}\right)$. Then:*

i) $\frac{c}{g}\sum_{j=0}^{m}(I - c(I - K^N))^j(K^N u_N)$ *is a sequence of continuous functions converging uniformly to $H^+ - (u_N/g)$ as $m \to \infty$.*

ii) *If $f \in \mathrm{Lip}\,(C)$, $\frac{c}{g}\sum_{j=0}^{m}(I - c(I - K^N))^j(u_N)$ is a sequence of continuous functions converging uniformly to H^+ as $m \to \infty$.*

Proof. We know $\kappa(x, \xi)$ is continuous except when $x = \xi$, and has a weak singularity of order $1 - \delta$, where δ is any number on $(0, \beta)$. Thus if $N > \frac{1}{\beta}$, K^N has a continuous kernel (see, for example, [6], pp. 29–38). Since $K = \frac{1}{2}(I - T\tilde{T})$ is self-adjoint, any eigenvalue of K must be real. Furthermore, K has no eigenvalues on $[0, 2)$. (If λ is an eigenvalue with eigenfunction h, then

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$$\frac{1}{\lambda} = \frac{\langle Kh, h\rangle}{\langle h, h\rangle} = \frac{\langle \frac{1}{2}(I - T\tilde{T})h, h\rangle}{\langle h, h\rangle} = \frac{1}{2} - \frac{1}{2}\frac{\langle T\tilde{T}h, h\rangle}{\langle h, h\rangle}$$

$$= \frac{1}{2} - \frac{1}{2}\frac{\langle \tilde{T}h, \tilde{T}h\rangle}{\langle h, h\rangle} \leq \frac{1}{2}. \quad \text{Thus when } \lambda \text{ is positive, } \lambda \geq 2.\Big)$$

Hence, the eigenvalues of K^N are real, and since N is odd, no eigenvalue of K_N lies on $[0, 2^N)$.

If λ is a negative eigenvalue of $K^N, 1 > (1 - c) + \frac{c}{\lambda} \geq 1 - c(1 + \|K^N\|) > -1$. If λ is a positive eigenvalue of $K^N, -1 < (1 - c) + \frac{c}{\lambda} \leq 1 - c + \frac{c}{2^N} < 1$. Hence for every eigenvalue, λ, of K^N, $\left|(1 - c) + \frac{c}{\lambda}\right| < 1$. From our choice of c, it is obvious that $|1 - c| < 1$.

Suppose $f \in \mathrm{Lip}\,(C)$. Then Lemma 4.1 may be used to show that $u_N \in \mathrm{Lip}\,(C)$. Hence assertion (ii) follows from Lemma 4.3 and (11) if we take W to be K^N and v to be u_N.

Lemma 4.3 also yields (i), if we take W to be K^N and v to be $K^N u_N$. ($K^N u_N$ is continuous since $u_N \in L^2(C)$ and K^N has a continuous kernel.)

Given the conditions in §3 and §4 it is clear that the results of these sections may be used to find the extremal function R^+ (which is expressed in terms of H^+, f, g in Theorem 1.1 (iv) and (v)).

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