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K-stable smooth Fano threefolds of Picard rank two

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Abstract

We prove that all smooth Fano threefolds in the families №2.1, №2.2, №2.3, №2.4, №2.6 and №2.7 are K-stable, and we also prove that smooth Fano threefolds in the family №2.5 that satisfy one very explicit generality condition are K-stable.

Contents

1. Introduction

Recently, we witnessed huge progress on K-stability of Fano varieties [1, 15, 18, 22], which resulted in an immense breakthrough in the solution of the following problem:

Problem. Find all K-polystable smooth Fano threefolds.

Let us describe what is done in this direction. To do this, fix a smooth Fano threefold X. Then X belongs to one of the 105 deformation families found by Iskovskikh, Mori, Mukai. These families are often labeled as №1.1, №1.2, №1.3, №1.4, ..., №7.1, №8.1, №9.1, №10.1, where the first digit stands for the rank of the Picard group of the threefolds in the family. For the detailed descriptions of the 105 families, we refer the reader to [3].

The valuative criterion for K-stability [15, 18] allows us to explain when X is K-stable in relatively simple terms. To do this, let us remind what are β-invariant and δ-invariant. Recall that E is a prime divisor over X if there is a birational morphism $f: \widetilde{X} \to X$ with normal $\widetilde{X}$ and a prime divisor $E \subset \widetilde{X}$. We set $\beta(E) = A_X(E) - S_X(E)$, where

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Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

$$S_X(E) = \frac{1}{(-K_X)^3} \int_0^\infty \text{vol}\big(f^*(-K_X) - uE\big)du,$$

and $A_X(E)$ is the log discrepancy of the divisor E. Then, for every point $P \in X$, we let

$$\delta_P(X) = \inf_{\substack{F/X \\ P \in C_X(F)}} \frac{A_X(F)}{S_X(F)},$$

where the infimum is taken over all prime divisors over X whose centers on X contain P. Finally, we set

$$\delta(X) = \inf_{P \in X} \delta_P(X).$$

The valuative criterion says that X is K-stable (respectively, K-semistable) if $\beta(F) > 0$ (respectively, $\beta(F) \geqslant 0$) for any prime divisor F over X. Hence, X is K-stable if $\delta(X) > 1$. We say that X is K-unstable if it is not K-semistable.

The simplest way to apply the valuative criterion is to check whether $\beta(S) < 0$ or not for some irreducible surface S in the threefold X. This has been done in [15]. As a result, we know that X is K-unstable if it belongs to any of the following 26 families:

№2.23, №2.28, №2.30, №2.31, №2.33, №2.35, №2.36, №3.14,

№3.16, №3.18, №3.21, №3.22, №3.23, №3.24, №3.26, №3.28, №3.29,

№3.30, №3.31, №4.5, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2.

To be precise, if X is contained in one of these 26 families, then it contains an irreducible surface S such that $\beta(S) < 0$, so that X is K-unstable, which implies that X does not admit a Kähler–Einstein metric. Almost in every case, such surface S is not hard to find. For instance, if X is the unique smooth Fano threefold in the deformation family №2.35, then X is a blow up of $\mathbb{P}^3$ at a point, and S is the exceptional surface of the blow up.

We listed 26 families of smooth Fano threefolds that contain no K-semistable members. There is another family that does not contain K-polystable members – the family №2.26. This family is quite special; see [3, §5.10] for its detailed description, and it contains exactly two smooth Fano threefolds. One of them is K-semistable and not K-polystable, while another one is K-unstable. Moreover, it follows from [3, Main Theorem] that general members of the remaining 78 families of smooth Fano threefolds are K-polystable.

Further, all K-polystable smooth Fano threefolds in 53 families among these 78 families are described in [2, 3, 4, 9, 13, 17, 19, 20, 23, 7, 8]. The remaining 25 families are:

№1.9, №1.10, №2.1, №2.2, №2.3, №2.4, №2.5, №2.6, №2.7, №2.9, №2.10, №2.11, №2.12,

№2.13, №2.14, №2.16, №2.17, №2.19, №2.20, №3.2, №3.5, №3.6, №3.7, №3.8, №3.11.

The deformation families №1.10, №2.20, №3.5, №3.8 contain non-K-polystable members, and [3, §7] provides conjectures that describe K-polystable smooth Fano threefolds in these four families. On the other hand, all smooth Fano threefolds in the 21 families

№1.9, №2.1, №2.2, №2.3, №2.4, №2.5, №2.6,

№2.7, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14,

№2.16, №2.17, №2.19, №3.2, №3.6, №3.7, №3.11

are conjectured to be K-stable [3]. The goal of this paper is to prove this for six families:

Main Theorem. *Let X be a smooth Fano threefold contained in one of the following deformation families: №2.1, №2.2, №2.3, №2.4, №2.6, №2.7. Then X is K-stable.*

Therefore, to solve the problem posed above, it remains to find all K-polystable smooth Fano threefolds in the following 19 families:

№1.9, №1.10, №2.5, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14,

№2.16, №2.17, №2.19, №2.20, №3.2, №3.5, №3.6, №3.7, №3.8, №3.11.

We hope that this will be done in a nearest future.

To prove Main Theorem, we use Abban–Zhuang theory [1] and its applications [3, 16]. A similar approach also works for almost all smooth Fano threefolds in the family №2.5. Namely, if X is a smooth Fano threefold in the family №2.5, we show that $\delta_P(X) > 1$ for every point $P \in X$ that satisfy certain geometric conditions. As a result, we obtained

Auxiliary Theorem. Let X be a smooth Fano threefold in the deformation family №2.5. Recall that there exists the following Sarkisov link:

$$
\begin{array}{ccc}
 & X & \\
{\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \phi} \\
V & & \mathbb{P}^1,
\end{array}
$$
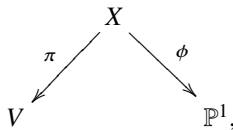
where V is a smooth cubic threefold in $\mathbb{P}^4$, the morphism π is a blow up of a smooth plane cubic curve, and ϕ is a morphism whose fibers are normal cubic surfaces. Suppose that

$$\text{no fiber of the morphism } \phi \text{ has a Du Val singular point of type } \mathbb{D}_5 \text{ or } \mathbb{E}_6. \qquad (\bigstar)$$

Then X is K-stable.

Let us describe the structure of this paper. In Section 2, we prove auxiliary theorem, and we prove that all smooth Fano threefolds in the families №2.1 and №2.3 are K-stable. In Sections 3, 4, 5, 6, we prove that all smooth Fano threefolds in the families №2.2, №2.4, №2.6, №2.7 are K-stable, respectively. Note that Section 6 is very technical and long.

As we already mentioned, we use applications of Abban–Zhuang theory [1] which have been discovered in [3, 16]. For the background material, we refer the reader to [3, 16, 22].

2. Families №2.1, №2.3, №2.5

Fix $d \in \{1, 2, 3\}$. Let V be one of the following smooth Fano threefolds:

$\boxed{d = 1}$ a smooth sextic hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$;
$\boxed{d = 2}$ a smooth quartic hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$;
$\boxed{d = 3}$ a smooth cubic threefold in $\mathbb{P}^4$.

Then $-K_V \sim 2H$ for an ample divisor $H \in \mathrm{Pic}(V)$ such that $H^3 = d$ and $\mathrm{Pic}(V) = \mathbb{Z}[H]$. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathcal{C} = S_1 \cap S_2$. Suppose that the curve $\mathcal{C}$ is smooth. Then $\mathcal{C}$ is an elliptic curve by the adjunction formula. Let $\pi \colon X \to V$ be the blow up of the curve $\mathcal{C}$, and let E be the π-exceptional surface.

○ If $d = 1$, then X is a smooth Fano threefold in the deformation family №2.1.
○ If $d = 2$, then X is a smooth Fano threefold in the deformation family №2.3.
○ If $d = 3$, then X is a smooth Fano threefold in the deformation family №2.5.

Moreover, all smooth Fano threefolds in these families can be obtained in this way.

Note that $(-K_X)^3 = 4d$. Moreover, we have the following commutative diagram:

$$
\begin{array}{ccc}
 & X & \\
\pi \swarrow & & \searrow \phi \\
V \dashrightarrow & & \mathbb{P}^1,
\end{array}
$$
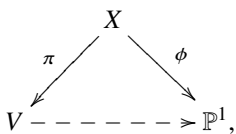
where $V \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a morphism whose general fiber is a smooth del Pezzo surface of degree d.

The goal of this section is to show that X is K-stable in the case when $d = 1$ or $d = 2$, and to show that X is K-stable in the case when $d = 3$ and X satisfies the condition ($\bigstar$). To show that X is K-stable, it is enough to show that $\delta_O(X) > 1$ for every point $O \in X$. This follows from the valuative criterion for K-stability [15, 18].

Lemma 2.1. *Let O be a point in X; let A be the fiber of the morphism ϕ such that $O \in A$. Suppose that A has at most Du Val singularities at the point O. Then*

$$
\delta_O(X) \geqslant
\begin{cases}
\min\left\{ \dfrac{16}{11}, \dfrac{16}{15}\delta_O(A) \right\} & \text{if } O \notin E, \\[2ex]
\min\left\{ \dfrac{16}{11}, \dfrac{16\delta_O(A)}{\delta_O(A) + 15} \right\} & \text{if } O \in E.
\end{cases}
$$

Proof. Let u be a nonnegative real number. Then $-K_X - uA \sim_{\mathbb{R}} (2-u)A + E$, which implies that divisor $-K_X - uA$ is pseudoeffective if and only if $u \leqslant 2$. For every $u \in [0, 2]$, let us denote by $P(u)$ the positive part of Zariski decomposition of the divisor $-K_X - uA$, and let us denote by $N(u)$ its negative part. Then

$$
P(u) =
\begin{cases}
(2-u)A + E & \text{if } 0 \leqslant u \leqslant 1, \\
(2-u)H & \text{if } 1 \leqslant u \leqslant 2,
\end{cases}
$$

and

$$
N(u) =
\begin{cases}
0 & \text{if } 0 \leqslant u \leqslant 1, \\
(u-1)E & \text{if } 1 \leqslant u \leqslant 2.
\end{cases}
$$

Integrating, we get $S_X(A) = \frac{11}{16}$. Using [1, Theorem 3.3] and [3, Corollary 1.102], we get

$$
\delta_O(X) \geqslant \min\left\{ \frac{1}{S_X(A)}, \inf_{\substack{F/A \\ O \in C_A(F)}} \frac{A_A(F)}{S(W^A_{\bullet,\bullet}; F)} \right\} = \min\left\{ \frac{16}{11}, \inf_{\substack{F/A \\ O \in C_A(F)}} \frac{A_A(F)}{S(W^A_{\bullet,\bullet}; F)} \right\}, \tag{2.1}
$$

where the infimum is taken by all prime divisors F over the surface A with $O \in C_A(F)$. The value $S(W^A_{\bullet,\bullet}; F)$ can be computed using [3, Corollary 1.108] as follows:

$$
S(W^A_{\bullet,\bullet}; F) = \frac{3}{4d} \int_1^2 \left(P(u)\big|_A \right)^2 (u-1)\mathrm{ord}_F\left(E\big|_A \right) du + \frac{3}{4d} \int_0^2 \int_0^\infty \mathrm{vol}\left(P(u)\big|_A - vF \right) dv\, du.
$$

Now, let F be any prime divisor over the surface A such that $O \in C_A(F)$. Since

$$
P(u)\big|_A =
\begin{cases}
-K_A & \text{if } 0 \leqslant u \leqslant 1, \\
(2-u)(-K_A) & \text{if } 1 \leqslant u \leqslant 2,
\end{cases}
$$

we have

$$S(W_{\bullet,\bullet}^A; F) = \frac{3}{4d} \int_1^2 d(2-u)^2(u-1)\mathrm{ord}_F\left(E\big|_A\right)du +$$

$$+ \frac{3}{4d} \int_0^1 \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dvdu + \frac{3}{4d} \int_1^2 \int_0^\infty \mathrm{vol}\left((2-u)(-K_A) - vF\right)dvdu =$$

$$= \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{3}{4d} \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dv + \frac{3}{4d} \int_1^2 (2-u)^3 \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dvdu =$$

$$= \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{3}{4d} \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dv + \frac{3}{16d} \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dv =$$

$$= \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{15}{16d} \int_0^\infty \mathrm{vol}\left(-K_A - vF\right)dv =$$

$$= \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{15}{16}S_A(F) \leqslant \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{15A_A(F)}{16\delta_O(A)}.$$

Therefore, if $O \notin E$, then $\mathrm{ord}_F(E|_A) = 0$, which implies that

$$S\left(W_{\bullet,\bullet}^A; F\right) \leqslant \frac{15A_A(F)}{16\delta_O(A)}.$$

Similarly, if $O \in E$, then $\mathrm{ord}_F(E|_A) \leqslant A_A(F)$ because $(A, E|_A)$ is log canonical, so that

$$S\left(W_{\bullet,\bullet}^A; F\right) = \frac{\mathrm{ord}_F\left(E\big|_A\right)}{16} + \frac{15}{16}S_A(F) \leqslant \frac{A_A(F)}{16} + \frac{15A_A(F)}{16\delta_O(A)} = \frac{\delta_O(A) + 15}{16\delta_O(A)}A_A(F).$$

Now, using Equation (2.1), we obtain the required inequality. □

Suppose X is not K-stable. Let us seek for a contradiction. Using the valuative criterion for K-stability [15, 18], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0,$$

where $A_X(\mathbf{F})$ is a log discrepancy of the divisor $\mathbf{F}$, and $S_X(\mathbf{F})$ is defined in [15] or [3, §1.2]. Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface [3, Theorem 3.17]. We see that Z is an irreducible curve or a point. Let P be a point in Z. Then $\delta_P(X) \leqslant 1$.

Lemma 2.2. *One has $P \notin E$.*

Proof. Let us compute $S_X(E)$. Note that $S_X(E) < 1$ by [3, Theorem 3.17]. Fix $u \in \mathbb{R}_{\geqslant 0}$. Then $-K_X - uE$ is pseudoeffective $\iff$ $-K_X - uE$ is nef $\iff$ $u \leqslant 1$. Thus, we have

$$S_X(E) = \frac{1}{4d} \int_0^1 \left(-K_X - uE\right)^3 du = \frac{1}{4d} \int_0^1 d(2u^3 - 6u + 4)du = \frac{3}{8}.$$

Suppose that $P \in E$. Let us seek for a contradiction.

Note that $E \cong \mathcal{C} \times \mathbb{P}^1$. Let $\mathbf{s}$ be a fiber of the projection $\phi|_E : E \to \mathbb{P}^1$ that contains P, and let $\mathbf{f}$ be a fiber of the projection $\pi|_E : E \to \mathcal{C}$. Fix $u \in [0,1]$, and take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uE\big|_E - v\mathbf{s} \equiv (1 + u - v)\mathbf{s} + d(1-u)\mathbf{f}.$$

This implies that

$$(-K_X - uE)\big|_E - v\mathbf{s} \text{ is pseudoeffective} \iff (-K_X - uE)\big|_E - v\mathbf{s} \text{ is nef} \iff v \leqslant 1 + v.$$

Therefore, using [3, Corollary 1.109], we get

$$S\left(W_{\bullet,\bullet}^{E}; \mathbf{s}\right) = \frac{3}{4d} \int_0^1 \int_0^{1+u} 2d(1-u)(1+u-v)dv\,du = \frac{11}{16}.$$

Similarly, using [3, Theorem 1.112], we get

$$S\left(W_{\bullet,\bullet,\bullet}^{E,s}; P\right) = \frac{3}{4d} \int_0^1 \int_0^{1+u} \left(d(1-u)\right)^2 dv\,du = \frac{5d}{16}.$$

Therefore, it follows from [3, Theorem 1.112] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(E)}, \frac{1}{S(W_{\bullet,\bullet}^E, \mathbf{s})}, \frac{1}{S\left(W_{\bullet,\bullet,\bullet}^{E,s}; P\right)}\right\} = \min\left\{\frac{8}{3}, \frac{16}{11}, \frac{16}{5d}\right\} \geqslant \frac{16}{15} > 1,$$

which is a contradiction.   □

Let A be the fiber of the del Pezzo fibration ϕ such that A passes through the point P. Then A is a del Pezzo surface of degree $d \in \{1, 2, 3\}$ that has at most isolated singularities. In particular, we see that A is normal. Applying Lemmas 2.1 and 2.2, we obtain

Corollary 2.3. *One has* $\delta_P(A) \leqslant \frac{15}{16}$.

Proof. Since $1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \delta_P(X)$, we get $\delta_P(A) \leqslant \frac{15}{16}$ by Lemmas 2.1 and 2.2.   □

Corollary 2.4. *The surface A is singular.*

Proof. If A is smooth, then $\delta_P(A) \geqslant \delta(A) \geqslant \frac{3}{2}$ [3, §2], which contradicts Corollary 2.3.   □

Let $\overline{S}$ be a general surface in $|H|$ that passes through $\pi(P)$, and let S be the proper transform on X of the surface $\overline{S}$. Then

○ the surface $\overline{S}$ is a smooth del Pezzo surface of degree d,
○ the surface $\overline{S}$ intersects the curve $\mathcal{C}$ transversally at d points,
○ the induced morphism $\pi|_S \colon S \to \overline{S}$ is a blow up of the points $\overline{S} \cap \mathcal{C}$.

Observe that $\phi|_S \colon S \to \mathbb{P}^1$ is an elliptic fibration given by the pencil $|-K_S|$. Set $C = A\big|_S$. Then C is a reduced curve of arithmetic genus 1 in $|-K_S|$ that has at most d components. In particular, if $d = 1$, then C is irreducible. Therefore, the following cases may happen:

(1) the curve C is irreducible, and C is smooth at P,
(2) the curve C is irreducible, and C has an ordinary node at P,
(3) the curve C is irreducible, and C has an ordinary cusp at P,
(4) the curve C is reducible.

Fix $u \in \mathbb{R}_{\geqslant 0}$. Then $-K_X - uS$ is nef $\iff u \leqslant 1 \iff -K_X - uS$ is pseudoeffective. Using this, we see that

$$S_X(S) = \frac{1}{4d} \int_0^1 (-K_X - uS)^3 du = \frac{1}{4d} \int_0^1 d(4-u)(1-u)^2 du = \frac{5}{16} < 1,$$

which also follows from [3, Theorem 3.17]. Moreover, if $u \in [0, 1]$, then

$$(-K_X - uS)|_S \sim_{\mathbb{R}} (1-u)(\pi|_S)^*(-K_{\overline{S}}) - K_S \sim_{\mathbb{R}} (1-u)\sum_{i=1}^d \mathbf{e}_i + (2-u)(-K_S),$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are exceptional curves of the blow up $\pi|_S \colon S \to \overline{S}$.

Lemma 2.5. *Suppose that C is irreducible. Then C is singular at the point P.*

Proof. As in the proof of Lemma 2.2, it follows from [3, Theorem 1.112] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{ \frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; C)}, \frac{1}{S(W^{S,C}_{\bullet,\bullet,\bullet}; P)} \right\},$$

where $S(W^S_{\bullet,\bullet}; C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet}; P)$ are defined in [3, §1.7]. Since we know that $S_X(S) < 1$, we see that $S(W^S_{\bullet,\bullet}; C) \geqslant 1$ or $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \geqslant 1$. Let us compute these numbers.

Let $P(u, v)$ be the positive part of the Zariski decomposition of $(-K_X - uS)|_S - vC$, and let $N(u, v)$ be its negative part, where $u \in [0, 1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Since

$$(-K_X - uS)|_S - vC \sim_\mathbb{R} (1 - u) \sum_{i=1}^{d} \mathbf{e}_i + (2 - u - v)C,$$

we see that $(-K_X - uS)|_S - vC$ is pseudoeffective $\iff v \leqslant 2 - u$. Moreover, we have

$$P(u, v) = \begin{cases} (1 - u) \displaystyle\sum_{i=1}^{d} \mathbf{e}_i + (2 - u - v)C \text{ if } 0 \leqslant v \leqslant 1, \\[2mm] (2 - u - v)\Big(C + \displaystyle\sum_{i=1}^{d} \mathbf{e}_i\Big) \text{ if } 1 \leqslant v \leqslant 2 - u, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\[2mm] (v - 1) \displaystyle\sum_{i=1}^{d} \mathbf{e}_i \text{ if } 1 \leqslant v \leqslant 2 - u. \end{cases}$$

Thus, it follows from [3, Corollary 1.109] that

$$S(W^S_{\bullet,\bullet}; C) = \frac{3}{4d} \int_0^1 \int_0^{2-u} P(u, v)^2 dv\,du =$$

$$= \frac{3}{4d} \int_0^1 \int_0^1 d(1 - u)(3 - u - 2v)dv\,du + \frac{3}{4d} \int_0^1 \int_1^{2-u} d(2 - u - v)^2 dv\,du = \frac{11}{16} < 1.$$

Thus, we conclude that $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \geqslant 1$.

Since $P \notin \mathbf{e}_1 \cup \cdots \cup \mathbf{e}_d$ by Lemma 2.2, it follows from [3, Theorem 1.112] that

$$S(W^{S,C}_{\bullet,\bullet,\bullet}; P) = \frac{3}{4d} \int_0^1 \int_0^{2-u} \big(P(u, v) \cdot C\big)^2 dv\,du =$$

$$= \frac{3}{4d} \int_0^1 \int_0^1 d^2(u - 1)^2 dv\,du + \frac{3}{4d} \int_0^1 \int_1^{2-u} d^2(2 - u - v)^2 dv\,du = \frac{5d}{16} < 1,$$

which is a contradiction. $\qquad\square$

Now, let us show that C is reducible for $d \in \{1, 2\}$.

Lemma 2.6. *Suppose that C is irreducible. Then $d = 3$ and C has a cusp at P.*

Proof. By Lemma 2.5, the curve C is singular at the point P.

Now, let $\sigma \colon \widetilde{S} \to S$ be the blow up of the point P; let $\mathbf{f}$ be the σ-exceptional curve, and let $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_d, \widetilde{C}$ be the proper transforms on $\widetilde{S}$ of the curves $\mathbf{e}_1, \ldots, \mathbf{e}_d, C$, respectively. Then the curve $\widetilde{C}$ is smooth, and it follows from [3, Remark 1.113] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\inf_{O \in \mathbf{f}} \frac{1}{S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)}, \frac{2}{S(V^S_{\bullet,\bullet}; \mathbf{f})}, \frac{1}{S_X(S)}\right\},$$

where $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)$ and $S(V^S_{\bullet,\bullet}; \mathbf{f})$ are defined in [3, §1.7]. Since we know that $S_X(S) < 1$, we see that $S(V^S_{\bullet,\bullet}; \mathbf{f}) \geqslant 2$ or there exists a point $O \in \mathbf{f}$ such that $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet}; O) \geqslant 1$.

Let us compute $S(V^S_{\bullet,\bullet}; \mathbf{f})$. Fix $u \in [0, 1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Since $\sigma^*(C) \sim \widetilde{C} + 2\mathbf{f}$, we get

$$\sigma^*\big((-K_X - uS)|_S\big) - v\mathbf{f} \sim_{\mathbb{R}} (2 - u)\widetilde{C} + (4 - 2u - v)\mathbf{f} + (1 - u)\sum_{i=1}^{d} \widetilde{\mathbf{e}}_i.$$

Then the divisor $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$ is pseudoeffective $\iff v \leqslant 4 - 2u$.

Let $P(u, v)$ be the positive part of the Zariski decomposition of $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$, and let $N(u, v)$ be its negative part. Then

$$P(u, v) = \begin{cases} (2 - u)\widetilde{C} + (4 - 2u - v)\mathbf{f} + (1 - u)\sum_{i=1}^{d} \widetilde{\mathbf{e}}_i \text{ if } 0 \leqslant v \leqslant \dfrac{d - du}{2}, \\[2mm] \dfrac{8 + d - 4u - du - 2v}{4}\widetilde{C} + (4 - 2u - v)\mathbf{f} + (1 - u)\sum_{i=1}^{d} \widetilde{\mathbf{e}}_i \text{ if } \dfrac{d - du}{2} \leqslant v \leqslant \dfrac{4 + d - du}{2}, \\[2mm] \dfrac{4 - 2u - v}{4 - d}\left(2\widetilde{C} + (4 - d)\mathbf{f} + 2\sum_{i=1}^{d} \widetilde{\mathbf{e}}_i\right) \text{ if } \dfrac{4 + d - du}{2} \leqslant v \leqslant 4 - 2u, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{d - du}{2}, \\[2mm] \dfrac{2v + du - d}{4}\widetilde{C} \text{ if } \dfrac{d - du}{2} \leqslant v \leqslant \dfrac{4 + d - du}{2}, \\[2mm] \dfrac{2v + du - 2d}{4 - d}\widetilde{C} + \dfrac{2v + du - 4 - d}{4 - d}\sum_{i=1}^{d} \widetilde{\mathbf{e}}_i \text{ if } \dfrac{4 + d - du}{2} \leqslant v \leqslant 4 - 2u. \end{cases}$$

Thus, using [3, Corollary 1.109], we get

$$S\big(W^S_{\bullet,\bullet}; \mathbf{f}\big) = \frac{3}{4d} \int_0^1 \int_0^{\frac{d - du}{2}} \big(du^2 - 4du - v^2 + 3d\big)\, dv\, du +$$
$$+ \frac{3}{4d} \int_0^1 \int_{\frac{d - du}{2}}^{\frac{4 + d - du}{2}} \frac{d(1 - u)(12 - du + d - 4u - 4v)}{4}\, dv\, du +$$
$$+ \frac{3}{4d} \int_0^1 \int_{\frac{4 + d - du}{2}}^{4 - 2u} \frac{d(4 - 2u - v)^2}{4 - d}\, dv\, du = \frac{44 + 5d}{32} < 2.$$

Therefore, there exists a point $O \in \mathbf{f}$ such that $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet}; O) \geqslant 1$.

Let us compute $S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O)$. Observe that

$$
P(u,v)\cdot\mathbf{f} = \begin{cases} v \text{ if } 0 \leqslant v \leqslant \dfrac{d-du}{2}, \\[2mm] \dfrac{d-du}{2} \text{ if } \dfrac{d-du}{2} \leqslant v \leqslant \dfrac{4+d-du}{2}, \\[2mm] \dfrac{d(4-2u-v)}{4-d} \text{ if } \dfrac{4+d-du}{2} \leqslant v \leqslant 4-2u. \end{cases}
$$

Hence, it follows from [3, Remark 1.113] that

$$
\begin{aligned}
S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O) &= \frac{3}{4d}\int_0^1\int_0^{4-2u}\left((P(u,v)\cdot\mathbf{f})\right)^2 dv\,du+ \\
&\quad + \frac{6}{4d}\int_0^1\int_0^{4-2u}(P(u,v)\cdot\mathbf{f})\operatorname{ord}_O\left(N(u,v)\big|_{\mathbf{f}}\right)dv\,du = \\
&= \frac{3}{4d}\int_0^1\int_0^{\frac{d-du}{2}} v^2 dv\,du+ \\
&\quad + \frac{3}{4d}\int_{\frac{d-du}{2}}^{\frac{4+d-du}{2}}\left(\frac{d-du}{2}\right)^2 dv\,du + \frac{3}{4d}\int_0^1\int_{\frac{4+d-du}{2}}^{4-2u}\left(\frac{d(4-2u-v)}{4-d}\right)^2 dv\,du+ \\
&\quad + \frac{6}{4d}\int_0^1\int_0^{4-2u}(P(u,v)\cdot\mathbf{f})\operatorname{ord}_O\left(N(u,v)\big|_{\mathbf{f}}\right)dv\,du = \\
&= \frac{5d}{32} + \frac{3}{2}\int_0^1\int_0^{4-2u}(P(u,v)\cdot\mathbf{f})\operatorname{ord}_O\left(N(u,v)\big|_{\mathbf{f}}\right)dv\,du.
\end{aligned}
$$

Therefore, if $O \notin \widetilde{C}$, we obtain $S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O) = \frac{5d}{32}$, which contradicts to $S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O) \geqslant 1$. Similarly, if $O \in \widetilde{C}$ and $\widetilde{C}$ intersects the curve $\mathbf{f}$ transversally at the point O, then

$$
S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O) = \frac{5d}{32} + \frac{6}{4d}\int_0^1\int_0^{4-2u}(P(u,v)\cdot\mathbf{f})\operatorname{ord}_O\left(N(u,v)\big|_{\mathbf{f}}\right)dv\,du = \frac{44+5d}{64} < 1,
$$

which again contradicts $S(W_{\bullet,\bullet,\bullet}^{\widetilde{S},\mathbf{f}};O) \geqslant 1$. Therefore, the curves $\widetilde{C}$ and $\mathbf{f}$ are tangent at O, which implies that C has a cusp at the point P.

Thus, to proceed, we may assume that $d = 1$ or $d = 2$.

Now, let us consider the following commutative diagram:

$$
\begin{array}{ccccc}
\widetilde{S} & \xleftarrow{\;\rho\;} & \widehat{S} & \xleftarrow{\;\eta\;} & \check{S} \\
{\scriptstyle\sigma}\downarrow & & & & \downarrow{\scriptstyle\psi} \\
S & & \xleftarrow{\;\;\;\upsilon\;\;\;} & & \mathcal{S},
\end{array}
$$

where

○ ρ is the blow up of the point $\widetilde{C} \cap \mathbf{f}$,
○ η is the blow up of the intersection point of the ρ-exceptional curve and the proper transform of the curve $\widetilde{C}$,
○ ψ is the contraction of the proper transforms of both $(\sigma \circ \rho)$-exceptional curves,
○ υ is the birational contraction of the proper transform of the η-exceptional curve.

Let $\mathscr{F}$ be the v-exceptional curve; let $\mathscr{C}$ be the proper transform on $\mathcal{S}$ of the curve C. Then $\mathscr{F}$ and $\mathscr{C}$ are smooth, $\mathscr{C}^2 = -6$, $\mathscr{F}^2 = -\frac{1}{6}$, $\mathscr{C} \cdot \mathscr{F} = 1$, and $v^*(C) = \mathscr{C} + 6\mathscr{F}$.

Observe that $\mathscr{F}$ contains two singular points of the surfaces $\mathcal{S}$, which are quotient singular points of type $\frac{1}{2}(1,1)$ and $\frac{1}{3}(1,1)$. Denote these points by Q_2 and Q_3, respectively. Note that $\mathscr{C}$ does not contain Q_2 and Q_3. Write $\Delta_{\mathscr{F}} = \frac{1}{2}Q_2 + \frac{2}{3}Q_3$. Then, since $A_S(\mathscr{F}) = 5$, it follows from [3, Remark 1.113] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{ \inf_{Q \in \mathscr{F}} \frac{1 - \mathrm{ord}_Q(\Delta_{\mathscr{F}})}{S(W^{\mathcal{S},\mathscr{F}}_{\bullet,\bullet,\bullet}; Q)}, \frac{5}{S(V^S_{\bullet,\bullet}; \mathscr{F})}, \frac{1}{S_X(S)} \right\}.$$

But we already proved that $S_X(S) < 1$. Hence, we conclude that $S(V^S_{\bullet,\bullet}; \mathscr{F}) \geqslant 5$ or there exists a point $Q \in \mathscr{F}$ such that $S(W^{\mathcal{S},\mathscr{F}}_{\bullet,\bullet,\bullet}; Q) \geqslant 1 - \mathrm{ord}_Q(\Delta_{\mathscr{F}})$.

Let us compute $S(V^S_{\bullet,\bullet}; \mathscr{F})$. Take $v \in \mathbb{R}_{\geqslant 0}$. Set $P(u) = -K_X - uS$. Then

$$v^*\big(P(u)|_S\big) - v\mathscr{F} \sim_{\mathbb{R}} (2-u)\mathscr{C} + (1-u)\sum_{i=1}^{d} \mathscr{E}_i + (12 - 6u - v)\mathscr{F},$$

where $\mathscr{E}_i$ is the proper transform on $\mathcal{S}$ of the (-1)-curve $\mathbf{e}_i$. Using this, we conclude that the divisor $v^*(P(u)|_S) - v\mathscr{F}$ is pseudoeffective $\iff v \leqslant 12 - 6u$.

Let $\mathscr{P}(u,v)$ be the positive part of the Zariski decomposition of $v^*(P(u)|_S) - v\mathscr{F}$, and let $\mathscr{N}(u,v)$ be its negative part. Then

$$\mathscr{P}(u,v) = \begin{cases} (2-u)\mathscr{C} + (1-u)\sum_{i=1}^{d} \mathscr{E}_i + (12 - 6u - v)\mathscr{F} & \text{if } 0 \leqslant v \leqslant d(1-u), \\[2mm] \dfrac{12 + d - (6+d)u - v}{6}\mathscr{C} + (1-u)\sum_{i=1}^{d} \mathscr{E}_i + (12 - 6u - v)\mathscr{F} & \text{if } d(1-u) \leqslant v \leqslant 6 + d - du, \\[2mm] \dfrac{12 - 6u - v}{6 - d}\left(\mathscr{C} + \sum_{i=1}^{d} \mathscr{E}_i + (6-d)\mathscr{F}\right) & \text{if } 6 + d - du \leqslant v \leqslant 12 - 6u, \end{cases}$$

and

$$\mathscr{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant d(1-u), \\[2mm] \dfrac{v + d(u-1)}{6}\mathscr{C} & \text{if } d(1-u) \leqslant v \leqslant 6 + d - du, \\[2mm] \dfrac{v + du - 2d}{6 - d}\mathscr{C} + \dfrac{v + du - 6 - d}{6 - d}\sum_{i=1}^{d} \mathscr{E}_i & \text{if } 6 + d - du \leqslant v \leqslant 12 - 6u. \end{cases}$$

This gives

$$\mathscr{P}(u,v)^2 = \begin{cases} \dfrac{6du^2 - 24du - v^2 + 18d}{6} & \text{if } 0 \leqslant v \leqslant d(1-u), \\[2mm] \dfrac{d(1-u)(18 - (d+6)u + d - 2v)}{6} & \text{if } d(1-u) \leqslant v \leqslant 6 + d - du, \\[2mm] \dfrac{d(12 - 6u - v)^2}{6(6-d)} & \text{if } 6 + d - du \leqslant v \leqslant 12 - 6u. \end{cases}$$

Thus, using [3, Corollary 1.109] and integrating, we get

$$S(W_{\bullet,\bullet}^{\mathcal{S}};\mathcal{F}) = \frac{3}{4d}\int_0^1\int_0^{12-6u}\mathscr{P}(u,v)^2\,dvdu = \frac{66+5d}{16} \leqslant \frac{19}{4} < 5 = A_S(\mathcal{F})$$

since $d = 1$ or $d = 2$. Thus, there is a point $Q \in \mathcal{F}$ such that $S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) \geqslant 1 - \mathrm{ord}_Q(\Delta_{\mathcal{F}})$.

Now, using [3, Remark 1.103] again, we see that

$$S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) = \frac{3}{4d}\int_0^1\int_0^{12-6u}\left((\mathscr{P}(u,v)\cdot\mathcal{F})\right)^2 dvdu+$$

$$+ \frac{6}{4d}\int_0^1\int_0^{12-6u}(P(u,v)\cdot\mathcal{F})\mathrm{ord}_Q(\mathcal{N}(u,v)|_{\mathcal{F}})dvdu.$$

On the other hand, we have

$$\mathscr{P}(u,v)\cdot\mathcal{F} = \begin{cases} \dfrac{v}{6} & \text{if } 0 \leqslant v \leqslant d(1-u), \\ \dfrac{d(1-u)}{6} & \text{if } d(1-u) \leqslant v \leqslant 6+d-du, \\ \dfrac{d(12-6u-v)}{6(6-d)} & \text{if } 6+d-du \leqslant v \leqslant 12-6u, \end{cases}$$

and

$$\mathcal{N}(u,v)\cdot\mathcal{F} = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant d(1-u), \\ \dfrac{v-d(1-u)}{6} & \text{if } d(1-u) \leqslant v \leqslant 6+d-du, \\ \dfrac{v+du-2d}{6-d} & \text{if } 6+d-du \leqslant v \leqslant 12-6u. \end{cases}$$

In particular, we have

$$S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) = \frac{5d}{96} + \frac{6}{4d}\int_0^1\int_0^{12-6u}(P(u,v)\cdot\mathcal{F})\mathrm{ord}_Q(\mathcal{N}(u,v)|_{\mathcal{F}})dvdu.$$

Hence, if $Q \notin \mathscr{C}$, then $\frac{1}{3} \leqslant 1 - \mathrm{ord}_Q(\Delta_{\mathcal{F}}) \leqslant S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) = \frac{5d}{96} < \frac{1}{3}$, which is absurd. Thus, we conclude that $Q = \mathscr{C} \cap \mathcal{F}$. Then

$$S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) = \frac{5d}{96} + \frac{6}{4d}\int_0^1\int_0^{12-6u}(P(u,v)\cdot\mathcal{F})\mathrm{ord}_Q(\mathcal{N}(u,v)|_{\mathcal{F}})dvdu \leqslant$$

$$\leqslant \frac{5d}{96} + \frac{6}{4d}\int_0^1\int_0^{12-6u}(P(u,v)\cdot\mathcal{F})(\mathcal{N}(u,v)\cdot\mathcal{F})dvdu = \frac{11}{16} < 1,$$

which is a contradiction, since $S(W_{\bullet,\bullet,\bullet}^{\mathcal{S},\mathcal{F}};Q) \geqslant 1 - \mathrm{ord}_Q(\Delta_{\mathcal{F}}) = 1$. □

In particular, we conclude that either $d = 2$ or $d = 3$.

Corollary 2.7. *All smooth Fano threefolds in the family №2.1 are K-stable.*

Recall that A is the fiber of the del Pezzo fibration $\phi\colon X \to \mathbb{P}^1$ that passes through P. Note also that we have the following possibilities:

∘ $d = 2$, and A is a double cover of $\mathbb{P}^2$ branched over a reduced quartic curve;
∘ $d = 3$, and A is a normal cubic surface in $\mathbb{P}^3$.

Observe that $C = S \cap A$, where S is a general surface in $|\pi^*(H)|$ that contains the point P. Since C is singular at P, the surface A must be singular at P, which confirms Corollary 2.4. Now, using classifications of reduced singular plane quartic curves and singular normal cubic surfaces [6], we see that $P = \mathrm{Sing}(A)$, and one of the following three cases holds:

- $d = 2$, and A is a double cover of $\mathbb{P}^2$ branched over four lines intersecting in a point;
- $d = 3$, and A is a cone in $\mathbb{P}^3$ over a smooth plane cubic curve;
- $d = 3$, and A has Du Val singular point of type $\mathbb{D}_4$, $\mathbb{D}_5$, or $\mathbb{E}_6$.

Let us show that the first case is impossible.

Lemma 2.8. *One has $d = 3$.*

Proof. Suppose that $d = 2$. Then $P = \mathrm{Sing}(A)$, and A is a double cover of $\mathbb{P}^2$ branched over a reduced reducible plane quartic curve that is a union of four distinct lines passing through one point. Let us seek for a contradiction.

Let $\alpha\colon \widetilde{X} \to X$ be the blow up of the point P, let E_P be the α-exceptional divisor and let $\widetilde{A}$ be the proper transform on $\widetilde{X}$ of the surface A. Then $\widetilde{A} \cap E_P$ is a line $L \subset E_P \cong \mathbb{P}^2$, and the surface $\widetilde{A}$ is singular along this line. Let $\beta\colon \overline{X} \to \widetilde{X}$ be the blow up of the line L, let E_L be the β-exceptional divisor, let $\overline{A}$ be the proper transform on $\overline{X}$ of the surface $\widetilde{A}$ and let $\overline{E}_P$ be the proper transforms on $\overline{X}$ of the surface E_P. Then

- $E_L \cong \mathbb{F}_2$,
- the intersection $\overline{E}_P \cap E_L$ is the (-2)-curve in E_L,
- the surface $\overline{A}$ is smooth, and there exists a $\mathbb{P}^1$-bundle $\overline{A} \to \mathcal{C}$,
- $\overline{A} \cap E_L$ is a smooth elliptic curve that is a section of the $\mathbb{P}^1$-bundle $\overline{A} \to \mathcal{C}$,
- the surfaces $\overline{A}$ and $\overline{E}_P$ are disjoint,
- $\overline{E}_P \cong \mathbb{P}^2$ and $\overline{E}_P|_{\overline{E}_P} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.

There is a birational contraction $\gamma\colon \overline{X} \to \widehat{X}$ of the surface $\overline{E}_P$ such that $\widehat{X}$ is a projective threefold that has one singular point $O = \gamma(\overline{E}_P)$, which is a terminal cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$. Thus, there exists the following commutative diagram

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\;\;\beta\;\;} & \widetilde{X} \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\
\widehat{X} & \xrightarrow{\;\;\sigma\;\;} & X,
\end{array}
$$

where σ is a birational morphism that contracts the surface $\gamma(E_L)$ to the point P.

Let $G = \gamma(E_L)$, let $\widehat{A} = \gamma(\overline{A})$ and let $\widehat{E}$ be the proper transform on $\widehat{X}$ of the surface E. Then $A_X(G) = 4$. Moreover, we have

$$
\sigma^*(-K_X) \sim 2\widehat{A} + \widehat{E} + 8G,
$$
$$
\sigma^*(A) \sim \widehat{A} + 4G.
$$

Note that $\widehat{A} \cong \overline{A}$ and $G \cong \mathbb{P}(1, 1, 2)$, so we can identify G with a quadric cone in $\mathbb{P}^3$. Note also that O is the vertex of the cone G. Moreover, by construction, we have $O \notin \widehat{A}$. Furthermore, the exists a $\mathbb{P}^1$-bundle $\widehat{A} \to \mathcal{C}$ such that $G|_{\widehat{A}}$ is its section.

Let $\mathbf{g}$ be a ruling of the quadric cone G, let $\mathbf{l}$ be a fiber of the $\mathbb{P}^1$-bundle $\widehat{A} \to \mathcal{C}$ and let $\mathbf{f}$ be a fiber of the $\mathbb{P}^1$-bundle $\pi \circ \sigma|_{\widehat{E}}\colon \widehat{E} \to \mathcal{C}$. Then $G|_G \sim_{\mathbb{Q}} -\mathbf{g}$ and $\widehat{A}|_G \sim_{\mathbb{Q}} 4\mathbf{g}$. Moreover, the intersections of the surfaces G, $\widehat{A}$, $\widehat{E}$ with the curves $\mathbf{g}$, $\mathbf{l}$, $\mathbf{f}$ are given here:

	G	$\widehat{A}$	$\widehat{E}$
g	$-\frac{1}{2}$	2	0
l	1	-4	1
f	0	1	-1

Fix a nonnegative real number u. We have $\sigma^*(-K_X) - uG \sim_{\mathbb{R}} 2\widehat{A} + \widehat{E} + (8-u)G$, which implies that $\sigma^*(-K_X) - uG$ is pseudoeffective $\iff u \in [0,8]$. Furthermore, if $u \in [0,8]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X) - uG$ can be described as follows:

$$P(u) = \begin{cases} 2\widehat{A} + \widehat{E} + (8-u)G \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{9-u}{4}\widehat{A} + \widehat{E} + (8-u)G \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{8-u}{3}\left(\widehat{A} + \widehat{E} + 3G\right) \text{ if } 5 \leqslant u \leqslant 8, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{u-1}{4}\widehat{A} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{u-2}{3}\widehat{A} + \dfrac{u-5}{3}\widehat{E} \text{ if } 5 \leqslant u \leqslant 8, \end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the decomposition. Then

$$\mathrm{vol}\left(\sigma^*(-K_X) - uG\right) = P(u)^3 = \begin{cases} 8 - \dfrac{u^3}{8} \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{18-3u}{2} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{(8-u)^3}{18} \text{ if } 5 \leqslant u \leqslant 8. \end{cases}$$

Integrating, we get $S_X(G) = \frac{27}{8} < 4 = A_X(G)$. But [16, Corollary 4.18] gives

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{Q \in G} \delta_Q(G, V_{\bullet,\bullet}^G)\right\} = \min\left\{\frac{32}{27}, \inf_{Q \in G} \delta_Q(G, V_{\bullet,\bullet}^G)\right\},$$

where $\delta_Q(G, V_{\bullet,\bullet}^G)$ is defined in [16]. Moreover, if Q is a point in G and Z is a smooth curve in G that passes through Q, then it follows from [16, Corollary 4.18] that

$$\delta_Q(G, V_{\bullet,\bullet}^G) \geqslant \min\left\{\frac{1}{S(V_{\bullet,\bullet}^G; Z)}, \frac{1 - \mathrm{ord}_Q(\Delta_Z)}{S(W_{\bullet,\bullet,\bullet}^{G,Z}; Q)}\right\},$$

where $S(V_{\bullet,\bullet}^G; Z)$ and $S(W_{\bullet,\bullet,\bullet}^{G,Z}; Q)$ are defined in [16], and

$$\Delta_Z = \begin{cases} 0 \text{ if } O \notin Z, \\ \dfrac{1}{2}O \text{ if } O \in Z. \end{cases}$$

Let us show that $\delta_Q(G, V_{\bullet,\bullet}^G) > 1$ for every $Q \in G$, which would imply a contradiction.

Let $\mathscr{C} = \widehat{A}\big|_G$, and let ℓ be the curve in $|\mathbf{f}|$ that contains Q. Then $O \notin \mathscr{C}$ and $O \in \ell$ so that $\Delta_{\mathscr{C}} = 0$ and $\Delta_\ell = \frac{1}{2}O$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G - v\ell \sim_{\mathbb{R}} \begin{cases} (u-v)\mathbf{g} \text{ if } 0 \leqslant u \leqslant 1, \\ (1-v)\mathbf{g} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{8-u-3v}{3}\mathbf{g} \text{ if } 5 \leqslant u \leqslant 8. \end{cases}$$

Now, using [16, Theorem 4.8], we get

$$S(W^G_{\bullet,\bullet};\ell) = \frac{3}{8}\int_0^8\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu = \frac{3}{8}\int_0^1\int_0^u \frac{(u-v)^2}{2}dvdu +$$

$$+ \frac{3}{12}\int_1^5\int_0^1 \frac{(1-v)^2}{4}dvdu + \frac{3}{12}\int_5^8\int_0^{\frac{8-u}{3}} \frac{(8-u-3v)^2}{18}dvdu = \frac{5}{16}.$$

Similarly, if $Q \notin \mathscr{C}$, then it follows from [16, Theorem 4.17] that

$$S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) = \frac{3}{8}\int_0^1\int_0^u \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + \frac{3}{8}\int_1^5\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu +$$

$$+ \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{3}} \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + F_Q = \frac{3}{8}\int_0^1\int_0^u \frac{(u-v)^2}{4}dvdu +$$

$$+ \frac{3}{8}\int_1^5\int_0^1 \frac{(1-v)^2}{4}dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{3}} \frac{(8-u-3v)^2}{36}dvdu = \frac{5}{32}$$

so that $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) = \frac{5}{32}$, which implies that $\delta_Q(G,V^G_{\bullet,\bullet}) \geqslant \frac{16}{5}$. Likewise, if $Q \in \mathscr{C}$, then

$$S(W^G_{\bullet,\bullet};\mathscr{C}) = \frac{3}{8}\int_0^8 \big(P(u)^2\cdot G\big)\cdot\mathrm{ord}_{\mathscr{C}}\Big(N(u)\big|_G\Big)du + \frac{3}{8}\int_0^8\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\mathscr{C}\big)dvdu =$$

$$= \frac{3}{8}\int_1^5 \frac{u-1}{8}du + \frac{3}{8}\int_5^8 \frac{(u-2)(8-u)^2}{54}du + \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} \frac{(u-4v)^2}{2}dvdu +$$

$$+ \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} \frac{(1-4v)^2}{2}dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \frac{(8-u-12v)^2}{18}dvdu = \frac{11}{16}$$

and

$$S(W^{G,\mathscr{C}}_{\bullet,\bullet,\bullet};Q) = \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu +$$

$$+ \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu =$$

$$= \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} (2u-8v)^2 dvdu + \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} (2-8v)^2 dvdu +$$

$$+ \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \big(\frac{(16-2u-24v)^2}{9}\big)dvdu = \frac{5}{8}.$$

This implies that $\delta_Q(G,V^G_{\bullet,\bullet}) \geqslant \min\big\{\frac{16}{11},\frac{8}{5}\big\} = \frac{16}{11}$, which is a contradiction. $\qquad\square$

Corollary 2.9. *All smooth Fano threefolds in the family №2.3 are K-stable.*

We see that $d = 3$ so that A is a singular cubic surface in $\mathbb{P}^3$ such that $P = \text{Sing}(A)$. Let $\sigma \colon \widehat{X} \to X$ be the blow up of the point P, and let G be the σ-exceptional surface. Denote by $\widehat{A}$ and $\widehat{E}$ the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively.

Lemma 2.10. *The surface A has Du Val singularities.*

Proof. Suppose that A is a cone in $\mathbb{P}^3$ with vertex P. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(-K_X) - vG \sim_{\mathbb{R}} 2\widehat{A} + \widehat{E} + (6-u)G.$$

Thus, the divisor $\sigma^*(-K_X) - vG$ is pseudoeffective $\iff u \in [0,6]$. Moreover, if $u \in [0,6]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X) - vG$ can be described as follows:

$$P(u) = \begin{cases} 2\widehat{A} + E + (6-u)G \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{7-u}{3}\widehat{A} + \widehat{E} + (6-u)G \text{ if } 1 \leqslant u \leqslant 4, \\ \dfrac{6-u}{2}(\widehat{A} + \widehat{E} + 2G) \text{ if } 4 \leqslant u \leqslant 6, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{u-1}{3}\widehat{A} \text{ if } 1 \leqslant u \leqslant 4, \\ \dfrac{u-2}{2}\widehat{A} + \dfrac{u-4}{2}\widehat{E} \text{ if } 4 \leqslant u \leqslant 6, \end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the Zariski decomposition, respectively. Using this, we compute

$$S_X(G) = \frac{3}{12}\int_0^1 u^3\,du + \frac{3}{12}\int_1^4 u\,du + \frac{3}{12}\int_4^6 \frac{u(6-u)^2}{4}\,du = \frac{43}{16} < 3 = A_X(G).$$

Let us apply [16, Theorem 4.8], [16, Corollary 4.17], [16, Corollary 4.18] using notations introduced in [16, §4]. To start with, we apply [16, Corollary 4.18] to get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{Q \in G} \delta_Q(G, V^G_{\bullet,\bullet})\right\} = \min\left\{\frac{48}{43}, \inf_{Q \in G} \delta_Q(G, V^G_{\bullet,\bullet})\right\}, \qquad (2.2)$$

where $\delta_Q(G, V^G_{\bullet,\bullet})$ is defined in [16, §4]. Let Q be an arbitrary point in the surface G, and let ℓ is a general line in $G \cong \mathbb{P}^2$ that contains Q. Then [16, Corollary 4.18] gives

$$\delta_Q(G, V^G_{\bullet,\bullet}) \geqslant \min\left\{\frac{1}{S(V^G_{\bullet,\bullet}; \ell)}, \frac{1}{S(W^{G,\ell}_{\bullet,\bullet,\bullet}; Q)}\right\},$$

where $S(V^F_{\bullet,\bullet}; \ell)$ and $S(W^{G,\ell}_{\bullet,\bullet,\bullet}; Q)$ are defined in [16, §4]. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G - v\ell \sim_{\mathbb{R}} \begin{cases} (u-v)\ell \text{ if } 0 \leqslant u \leqslant 1, \\ (1-v)\ell \text{ if } 1 \leqslant u \leqslant 4, \\ \dfrac{6-u-2v}{2}\ell \text{ if } 4 \leqslant u \leqslant 6. \end{cases}$$

Let $\mathscr{C} = \widehat{A}|_G$. Then $\mathscr{C}$ is a smooth cubic curve in $G \cong \mathbb{P}^2$. Let

$$N'(u) = N(u)\big|_G = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{u-1}{3}\mathscr{C} \text{ if } 1 \leqslant u \leqslant 4, \\ \dfrac{u-2}{2}\mathscr{C} \text{ if } 4 \leqslant u \leqslant 6. \end{cases}$$

Now, using [16, Theorem 4.8], we get

$$S(W^G_{\bullet,\bullet};\ell) = \frac{3}{12}\int_0^6\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu = \frac{3}{12}\int_0^1\int_0^u (u-v)^2 dvdu +$$

$$+ \frac{3}{12}\int_1^4\int_0^1 (1-v)^2 dvdu + \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}}\left(\frac{6-u-2v}{2}\right)^2 dvdu = \frac{5}{16}.$$

Similarly, it follows from [16, Theorem 4.17] that

$$S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) = \frac{3}{12}\int_0^1\int_0^u \big((P(u)\big|_G - v\ell)\cdot\ell\big)^2 dvdu + \frac{3}{12}\int_1^4\int_0^1 \big((P(u)\big|_G - v\ell)\cdot\ell\big)^2 dvdu +$$

$$+ \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}}\big((P(u)\big|_G - v\ell)\cdot\ell\big)^2 dvdu + F_Q = \frac{3}{12}\int_0^1\int_0^u (u-v)^2 dvdu +$$

$$+ \frac{3}{12}\int_1^4\int_0^1 (1-v)^2 dvdu + \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}}\left(\frac{6-u-2v}{2}\right)^2 dvdu + F_Q = \frac{5}{16} + F_Q,$$

where

$$F_Q = \frac{6}{12}\int_1^4\int_0^1 \big((P(u)\big|_G - v\ell)\cdot\ell\big)\mathrm{ord}_Q\big(N'(u)\big|_\ell\big)dvdu +$$

$$+ \frac{6}{12}\int_4^6\int_0^{\frac{6-u}{2}}\big((P(u)\big|_G - v\ell)\cdot\ell\big)\mathrm{ord}_Q\big(N'(u)\big|_\ell\big)dvdu \leqslant$$

$$\leqslant \frac{6}{12}\int_1^4\int_0^1 \frac{(1-v)(u-1)}{3}dvdu + \frac{6}{12}\int_4^6\int_0^{\frac{6-u}{2}}\frac{(6-u-2v)(u-2)}{4}dvdu = \frac{7}{12}.$$

So, we have $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) \leqslant \frac{43}{48}$. Then $\delta_Q(G, V^G_{\bullet,\bullet}) > 1$, which contradicts Equation (2.2).   □

Thus, we see that P is a Du Val singular point of the surface A of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{E}_6$. Now, arguing as in the proof of [22, Lemma 9.11], we see that $\beta(\mathbf{F}) > 0$ if

(1) the inequality $\beta(G) > 0$ holds,
(2) and for every prime divisor $\mathbf{E}$ over X such that $C_X(\mathbf{E})$ is a curve containing P, the following inequality holds:

$$\frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \frac{4}{3}.$$

Since $\beta(\mathbf{F}) \leqslant 0$ by our assumption, we see that at least one of these conditions must fail.

Lemma 2.11. *One has $\beta(G) \geqslant \frac{465}{2048}$.*

Proof. Let $\widehat{A}$ and $\widehat{E}$ be the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(-K_X) - uG \sim \sigma^*(2H - E) - uG \sim \sigma^*(2A + E) - uG \sim 2\widehat{A} + \widehat{E} + (4-u)G,$$

which easily implies that the divisor $-K_{\widehat{X}} - G$ is pseudoeffective $\iff u \leqslant 4$ because we can contract the surfaces $\widehat{A}$ and $\widehat{E}$ simultaneously after flops. Then

$$\beta(G) = A_X(G) - S_X(G) = 3 - \frac{1}{12} \int_0^4 \mathrm{vol}\big(\sigma^*(-K_X) - uG\big)\,du.$$

Note that $\sigma^*(-K_X) - uG$ is nef for $u \in [0, 1]$ because the divisor $-K_X$ is very ample. Thus, if $u \in [0, 1]$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) = \big(\sigma^*(-K_X) - uG\big)^3 = 12 - u^3.$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant \mathrm{vol}\big(\sigma^*(-K_X) - G\big) = \big(\sigma^*(-K_X) - G\big)^3 = 11.$$

Finally, let us estimate $\mathrm{vol}(\sigma^*(-K_X) - uG)$ in the case when $4 \geqslant u > \frac{3}{2}$.

Let Z be a general hyperplane section of the cubic surface A that passes through P, and let $\widehat{Z}$ be its proper transform on the threefold $\widehat{X}$. Then Z is an irreducible cuspidal cubic curve, and $\widehat{Z} \subset \widehat{A}$. Observe that $(\sigma^*(-K_X) - uG) \cdot \widehat{Z} = 3 - 2u$ and $\widehat{A} \cdot \widehat{Z} = -4$, so $\widehat{A}$ is contained in the asymptotic base locus of the divisor $\sigma^*(-K_X) - uG$ for $u > \frac{3}{2}$. Moreover, if $\sigma^*(-K_X) - uG \sim_{\mathbb{R}} \widehat{D} + \lambda \widehat{A}$ for $\lambda \in \mathbb{R}_{\geqslant 0}$ and an effective $\mathbb{R}$-divisor $\widehat{D}$ whose support does not contain $\widehat{A}$, then $\widehat{Z} \not\subset \widehat{D}$, which implies that

$$0 \leqslant \widehat{D} \cdot \widehat{Z} = \big(\sigma^*(-K_X) - uG - \lambda\widehat{A}\big) \cdot \widehat{Z} = 3 - 2u - 4\lambda$$

so that $\lambda \geqslant \frac{3-2u}{4}$. Thus, if $4 \geqslant u > \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant \mathrm{vol}\Big(\sigma^*(2H - E) - uG - \frac{2u-3}{4}\widehat{A}\Big).$$

Moreover, if $4 \geqslant u > \frac{3}{2}$, then

$$\sigma^*(2H - E) - uG - \frac{2u-3}{4}\widehat{A} \sim_{\mathbb{R}} \frac{11-2u}{4}\sigma^*(H) - \frac{7-2u}{2}\sigma^*(E) - \frac{3}{2}G.$$

Therefore, if $4 \geqslant u > \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant \mathrm{vol}\Big(\frac{11-2u}{4}\sigma^*(H) - \frac{7-2u}{2}\sigma^*(E)\Big) = \mathrm{vol}\Big(\frac{11-2u}{4}H - \frac{7-2u}{2}E\Big).$$

Furthermore, if $\frac{7}{2} \geqslant u > \frac{3}{2}$, then $\frac{11-2u}{4}H - \frac{7-2u}{2}E$ is nef so that

$$\mathrm{vol}\Big(\frac{11-2u}{4}H - \frac{7-2u}{2}E\Big) = \Big(\frac{11-2u}{4}H - \frac{7-2u}{2}E\Big)^3 = \frac{25-6u}{16}.$$

Similarly, if $4 \geqslant u > \frac{7}{2}$, then

$$\mathrm{vol}\Big(\frac{11-2u}{4}H - \frac{7-2u}{2}E\Big) = \Big(\frac{11-2u}{4}H\Big)^3 = \frac{(11-2u)^3}{256}.$$

Now, we can estimate $\beta(G)$ as follows

$$\beta(G) = 3 - \frac{1}{12}\int_0^4 \mathrm{vol}\big(\sigma^*(-K_X) - uG\big)du \geqslant 3 - \frac{1}{12}\int_0^1 (12 - u^3)du - \frac{1}{12}\int_1^{\frac{3}{2}} 11\,du -$$

$$- \frac{1}{12}\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{25 - 6u}{16}du - \frac{1}{12}\int_{\frac{7}{2}}^4 \frac{(11 - 2u)^3}{256}du = 3 - \frac{5679}{2048} = \frac{465}{2048}$$

as claimed.   □

Therefore, there exists a prime divisor $\mathbf{E}$ over X such that $C_X(\mathbf{E})$ is a curve, $P \in C_X(\mathbf{E})$, and $A_X(\mathbf{E}) < \frac{4}{3}S_X(\mathbf{E})$. Set $Z = C_X(\mathbf{E})$. Then $\delta_O(X) < \frac{4}{3}$ for every point $O \in Z$.

Lemma 2.12. *One has $Z \subset A$, and Z is a line in the cubic surface A.*

Proof. Let O be a general point in Z, and let A_O be the fiber of ϕ that passes through O. If $Z \not\subset A$, then A_O is smooth, so that $\delta_O(A_O) \geqslant \frac{3}{2}$ by [3, Lemma 2.13], which gives

$$\frac{4}{3} > \frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \delta_O(X) \geqslant \min\left\{\frac{16}{11}, \frac{16\delta_O(A_O)}{\delta_O(A_O) + 15}\right\} \geqslant \min\left\{\frac{16}{11}, \frac{16 \times \frac{3}{2}}{\frac{3}{2} + 15}\right\} = \frac{16}{11} > \frac{4}{3}$$

by Lemma 2.1. This shows that $Z \subset A$ and $A_O = A$.

To complete the proof of the lemma, we have to show that Z is a line in the surface A. Suppose that Z is not a line. Then the point O is not contained in a line in the surface A because A contains finitely many lines [6]. Now, arguing as in the proof of [3, Lemma 2.13], we get $\delta_O(A) \geqslant \frac{3}{2}$. So, applying Lemma 2.1 again, we get a contradiction as above.   □

Now, our auxiliary theorem follows from the following lemma:

Lemma 2.13. *The surface A does not have a singular point of type $\mathbb{D}_4$.*

Proof. Suppose A has singularity of type $\mathbb{D}_4$. Then it follows from [6] that, for a suitable choice of coordinates x, y, z, t on the projective space $\mathbb{P}^3$, one of the following cases hold:

(A) $A = \{tx^2 = y^3 - z^3\} \subset \mathbb{P}^3$,
(B) $A = \{tx^2 = y^3 - z^3 + xyz\} \subset \mathbb{P}^3$.

Note that $P = [0:0:0:1]$, and A contains six lines [6]. In case (A), these lines are

$$L_1 = \{x = y - z = 0\},$$
$$L_2 = \{x = y - \omega_3 z = 0\},$$
$$L_3 = \{x = y + \omega_3^2 z = 0\},$$
$$L_4 = \{t = y - z = 0\},$$
$$L_5 = \{t = y + \omega_3 z = 0\},$$
$$L_6 = \{t = y + \omega_3^2 z = 0\},$$

where ω_3 is a primitive cube root of unity. In case (B), these lines are

$$L_1 = \{x = y - z = 0\},$$
$$L_2 = \{x = y - \omega_3 z = 0\},$$
$$L_3 = \{x = y - \omega_3^2 z = 0\},$$
$$L_4 = \{x + 3(y - z) = y - z - 9t = 0\},$$
$$L_5 = \{x + 3\omega_3(y - \omega_3 z) = \omega_3 y - \omega_3^2 z - 9t = 0\},$$
$$L_6 = \{x + 3\omega_3^2(y - \omega_3^2 z) = \omega_3^2 y - \omega_3 z - 9t = 0\}.$$

Note that $P = L_1 \cap L_3 \cap L_3$, $P \notin L_4 \cup L_5 \cup L_6$ and $-K_A \sim 2L_1 + L_4 \sim 2L_2 + L_5 \sim 2L_3 + L_6$.

By Lemma 2.12, we may assume that $Z = L_1$.

Recall that $S_X(A) = \frac{11}{16}$; see the proof of Lemma 2.1. Using [3, Theorem 1.112], we get

$$\frac{4}{3} > \frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \min\left\{\frac{1}{S_X(A)}, \frac{1}{S(W_{\bullet,\bullet}^A; L_1)}\right\} = \min\left\{\frac{16}{11}, \frac{1}{S(W_{\bullet,\bullet}^A; L_1)}\right\},$$

where $S(W_{\bullet,\bullet}^A; L_1)$ is defined in [3, §1.7]. Therefore, we conclude that $S(W_{\bullet,\bullet}^A; L_1) < \frac{4}{3}$. Let us compute $S(W_{\bullet,\bullet}^A; L_1)$ using [3, Corollary 1.109].

To do this, we use notations introduced in the proof of Lemma 2.1 applied to $O = P$. Then using [3, Corollary 1.109] and computations from the proof of Lemma 2.1, we get

$$S\left(W_{\bullet,\bullet}^A; L_1\right) = \frac{1}{4}\int_0^1 \int_0^\infty \mathrm{vol}\left(-K_A - vL_1\right)dvdu + \frac{1}{4}\int_1^2 \int_0^\infty \mathrm{vol}\left((2-u)(-K_A) - vL_1\right)dvdu$$

since $L_1 \not\subset \mathrm{Supp}(N(u))$ since $L_1 \not\subset E$. Let us compute $S(W_{\bullet,\bullet}^A; L_1)$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_A - vL_1 \sim_{\mathbb{R}} (2-v)L_1 + L_4.$$

Thus, the divisor $-K_A - vL_1$ is pseudoeffective $\iff v \leqslant 2$ since $L_4^2 = -1$. Fix $v \in [0, 2]$. Let $P(u, v)$ be the positive part of the Zariski decomposition of the divisor $-K_A - vL_1$, and let $N(u, v)$ be its negative part. Then

$$P(u, v) = \begin{cases} (2-v)L_1 + L_4 & \text{if } 0 \leqslant v \leqslant 1, \\ (2-v)(L_1 + L_4) & \text{if } 1 \leqslant v \leqslant 2, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)L_4 & \text{if } 1 \leqslant v \leqslant 2. \end{cases}$$

Thus, if $0 \leqslant v \leqslant 1$, then $\mathrm{vol}(-K_A - vL_1) = 3 - 2v$ because $L_1^2 = 0$ and $L_1 \cdot L_4 = 0$. Similarly, if $1 \leqslant v \leqslant 2$, then $\mathrm{vol}(-K_A - vL_1) = (v-2)^2$. This gives

$$\frac{1}{4}\int_0^1 \int_0^\infty \mathrm{vol}\left(-K_A - vL_1\right)dvdu = \frac{1}{4}\int_0^1 \int_0^1 (3-2v)dvdu + \frac{1}{4}\int_0^1 \int_1^2 (v-2)^2 dvdu = \frac{7}{12}$$

and

$$\frac{1}{4}\int_1^2\int_0^\infty \mathrm{vol}\big((2-u)(-K_A)-vL_1\big)dvdu = \frac{1}{4}\int_1^2\int_0^\infty (2-u)^3\mathrm{vol}\big((-K_A)-vL_1\big)dvdu =$$

$$= \frac{1}{4}\int_1^2\int_0^1 (2-u)^3(3-2v)dvdu + \frac{1}{4}\int_1^2\int_1^2 (2-u)^3(v-2)^2dvdu = \frac{7}{48}.$$

Combining, we get $S(W^A_{\bullet,\bullet};L_1) = \frac{35}{48} < \frac{3}{4}$. This is a contradiction. $\qquad\square$

3. Family №2.2.

Let R be a smooth surface of degree $(2,4)$ in $\mathbb{P}^1\times\mathbb{P}^2$; let $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^2$ be a double cover ramified over the surface R. Then X is a smooth Fano threefold in the family № 2.2. Moreover, all smooth Fano threefolds in this family can be obtained this way.

Let $\mathrm{pr}_1\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^1$ and $\mathrm{pr}_2\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^2$ be the projections to the first and the second factors, respectively. Set $p_1 = \mathrm{pr}_1\circ\pi$ and $p_2 = \mathrm{pr}_2\circ\pi$. We have the following commutative diagram:
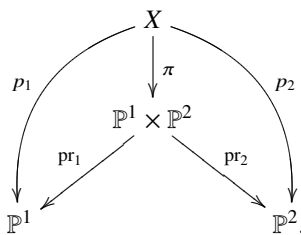
where p_1 is a fibration into del Pezzo surfaces of degree 2, and p_2 is a conic bundle.

Lemma 3.1. *Let S be a fiber of the morphism p_1. Then S is irreducible and normal.*

Proof. Since $p_1\colon X\to\mathbb{P}^1$ is a Mori fiber space, any fiber of p_1 is irreducible and reduced. Moreover, any fiber of $\mathrm{pr}_1|_R$ is reduced by local computations. Thus, the assertion follows. $\qquad\square$

Lemma 3.2. *Let S be a fiber of the morphism p_1, let C be a fiber of the morphism p_2 and let P be a point in $S\cap C$. Then S or C is smooth at P.*

Proof. Local computations. $\qquad\square$

Now, we are ready to prove that X is K-stable. Recall from [10] that $\mathrm{Aut}(X)$ is finite. Thus, the threefold X is K-stable if and only if it is K-polystable [22].

Let τ be the Galois involution of the double cover $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^2$, and let $G = \langle\tau\rangle$. Suppose that X is not K-polystable. Then it follows from [24, Corollary 4.14] that there exists a G-invariant prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of this divisor on X. Then Z is not a surface by [3, Theorem 3.17]. Hence, we see that either Z is a G-invariant irreducible curve, or Z is a G-fixed point. Let us seek for a contradiction.

Let P be a general point in Z, and let S be the fiber of p_1 that passes through P.

Lemma 3.3. *The surface S is singular at P.*

Proof. Suppose that S is smooth at P. Let B be a general curve in $|-K_S|$ that contains P. Then B is a smooth curve in S. Applying [3, Theorem 1.112], we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; B)}, \frac{1}{S(W^{S,B}_{\bullet,\bullet,\bullet}; P)}\right\}.$$

Since $S_X(S) < 1$ by [3, Theorem 3.17], we see that $S(W^S_{\bullet,\bullet}; B) \geqslant 1$ or $S(W^{S,B}_{\bullet,\bullet,\bullet}; P) \geqslant 1$. We refer the reader to [3, §1.7] for definitions of $S_X(S)$, $S(W^S_{\bullet,\bullet}; B)$, $S(W^{S,B}_{\bullet,\bullet,\bullet}; P)$.

Note that [3, Theorem 1.112] requires S to have Du Val singularities, but S may have non-Du Val singularities. Nevertheless, we still can apply [3, Theorem 1.112] here since the proof of [3, Theorem 1.112] remains valid in our case because S is smooth along B.

Let us compute $S(W^S_{\bullet,\bullet}; B)$ and $S(W^{S,B}_{\bullet,\bullet,\bullet}; P)$. Take $u \in \mathbb{R}_{\geqslant 0}$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uS \text{ is nef} \iff -K_X - uS \text{ is pseudoeffective} \iff u \leqslant 1,$$

Similarly, if $u \in [0, 1]$, then $(-K_X - uS)|_S - vB \sim_{\mathbb{R}} (1-v)(-K_S)$, so

$$(-K_X - uS)|_S - vB \text{ is nef} \iff (-K_X - uS)|_S - vB \text{ is pseudoeffective} \iff v \leqslant 1.$$

Now, applying [3, Corollary 1.109], we get

$$S(W^S_{\bullet,\bullet}; B) = \frac{3}{6}\int_0^1 \int_0^1 \left((1-v)(-K_S)\right)^2 dv\,du = \frac{1}{2}\int_0^1 \int_0^1 2(1-v)^2 dv = \frac{1}{3} < 1.$$

Similarly, using [3, Theorem 1.112], we get

$$S(W^{S,B}_{\bullet,\bullet,\bullet}, P) = \frac{3}{6}\int_0^1 \int_0^1 \left((1-v)(-K_S)\cdot B\right)^2 dv\,du = \frac{2}{3} < 1.$$

But we already know that $S(W^S_{\bullet,\bullet}; B) \geqslant 1$ or $S(W^{S,B}_{\bullet,\bullet,\bullet}; P) \geqslant 1$. This is a contradiction. □

If Z is a curve, then S is smooth at P by Lemma 3.1 because P is a general point in Z. Hence, we conclude that $Z = P$ because S is singular at the point P by Lemma 3.3. Recall that Z is G-invariant. This implies that $\tau(P) \in R$.

Let C be the fiber of p_2 that passes through P. Then C is smooth at P by Lemma 3.2 because S is singular at P. Since $\tau(P) \in R$, we see that C is irreducible and smooth.

Let T be a sufficiently general surface in linear system $|p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ that contains C. Since C is smooth, it follows from Bertini's theorem that the surface T is smooth.

As in the proof of Lemma 3.3, it follows from [3, Theorem 1.112] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(T)}, \frac{1}{S(W^T_{\bullet,\bullet}; C)}, \frac{1}{S(W^{T,C}_{\bullet,\bullet,\bullet}; P)}\right\}.$$

Moreover, it follows from [3, Theorem 3.17] that $S_X(T) < 1$. Thus, we conclude that

$$\max\left\{S(W^T_{\bullet,\bullet}; C), S(W^{T,C}_{\bullet,\bullet,\bullet}; P)\right\} \geqslant 1.$$

In fact, since P is the center of the divisor $\mathbf{F}$ on X, [3, Theorem 3.17] gives

$$\max\left\{S(W^T_{\bullet,\bullet}; C), S(W^{T,C}_{\bullet,\bullet,\bullet}; P)\right\} > 1. \tag{3.1}$$

Now, let us compute $S(W^T_{\bullet,\bullet}; C)$ and $S(W^{T,C}_{\bullet,\bullet,\bullet}; P)$ using the results obtained in [3, §1.7].

Take $u \in \mathbb{R}_{\geqslant 0}$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uT \text{ is nef} \iff -K_X - uT \text{ is pseudoeffective} \iff u \leqslant 1,$$

Similarly, if $u \in [0, 1]$, then

$$(-K_X - uT)|_T - vC \text{ is nef} \iff (-K_X - uT)|_T - vC \text{ is pseudoeffective} \iff v \leqslant 1 - u$$

because $(-K_X - uT)|_T - vC \sim_{\mathbb{R}} S|_T + (1 - u - v)C$. So, using [3, Corollary 1.109], we get

$$S(W^T_{\bullet,\bullet}; C) = \frac{3}{6} \int_0^1 \int_0^{1-u} \left(S|_T + (1 - u - v)C \right)^2 dv\,du = \frac{1}{2} \int_0^1 \int_0^{1-u} 4(1 - u - v)\,dv\,du = \frac{1}{3} < 1.$$

Hence, it follows from Equation (3.1) that $S(W^{T;C}_{\bullet,\bullet,\bullet}, P) > 1$. Now, using [3, Theorem 1.112], we get

$$S(W^{T;C}_{\bullet,\bullet,\bullet}, P) = \frac{3}{6} \int_0^1 \int_0^{1-u} \left((S|_T + (1 - u - v)C) \cdot C \right)^2 dv\,du = \frac{3}{6} \int_0^1 \int_0^{1-u} 4\,dv\,du = 1,$$

which is a contradiction. This shows that X is K-stable.

Corollary 3.4. *All smooth Fano threefolds in the family №2.2 are K-stable.*

4. Family №2.4.

Let $\mathcal{S}$ and $\mathcal{S}'$ be smooth cubic surfaces in $\mathbb{P}^3$ such that their intersection is a smooth curve of genus 10. Set $\mathcal{C} = \mathcal{S} \cap \mathcal{S}'$, and let $\pi \colon X \to \mathbb{P}^3$ be the blow up of the curve $\mathcal{C}$. Then X is a smooth Fano threefold in the family №2.4, and every smooth Fano threefold in this family can be obtained in this way. Moreover, there exists a commutative diagram
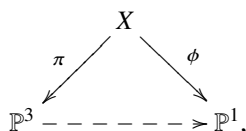
where $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ is a map that is given by the pencil generated by the surfaces $\mathcal{S}$ and $\mathcal{S}'$, and ϕ is a fibration into cubic surfaces. Note that $-K_X^3 = 10$ and $\mathrm{Aut}(X)$ is finite [10].

Let $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$, and let E be the π-exceptional surface. Then $-K_X \sim 4H - E$, the morphism ϕ is given by the linear system $|3H - E|$, and $E \cong \mathcal{S} \times \mathbb{P}^1$.

The goal of this section is to prove that X is K-stable. Suppose that X is not K-stable. Let us seek for a contradiction. First, using the valuative criterion for K-stability [15, 18], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface by [3, Theorem 3.17]. Therefore, either Z is an irreducible curve or Z is a point. Fix a point $P \in Z$.

Let A be the surface in $|3H - E|$ that contains P. Fix $u \in \mathbb{R}_{\geqslant 0}$. Let $\mathcal{P}(u)$ be the positive part of the Zariski decomposition of $-K_X - uA$, and let $\mathcal{N}(u)$ be its negative part. Then

$$-K_X - uA \sim_{\mathbb{R}} (4 - 3u)H - (1 - u)E \sim_{\mathbb{R}} \left(\frac{4}{3} - u \right)A + \frac{1}{3}E.$$

This implies that $-K_X - uA$ is pseudoeffective $\iff u \leqslant \frac{4}{3}$. Moreover, we have

$$\mathcal{P}(u) = \begin{cases} (4-3u)H - (1-u)E \text{ if } 0 \leqslant u \leqslant 1, \\ (4-3u)H \text{ if } 1 \leqslant v \leqslant \frac{4}{3}, \end{cases}$$

and

$$\mathcal{N}(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)E \text{ if } 1 \leqslant u \leqslant \frac{4}{3}. \end{cases}$$

Integrating, we obtain $S_X(A) = \frac{67}{120} < 1$, which also follows from [3, Theorem 3.17].

Note that $\pi(A)$ is a normal cubic surface in $\mathbb{P}^3$, and $\pi(A)$ is smooth along the curve $\mathscr{C}$. In particular, we see that $A \cong \pi(A)$, and A is smooth along the intersection $E \cap A$.

Lemma 4.1. *The surface A is singular at the point P.*

Proof. Suppose that A is smooth at P. Let C be a general curve in $|-K_A|$ that passes through the point P. Then C is a smooth irreducible elliptic curve. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$\mathcal{P}(u)|_S - vC \sim_{\mathbb{R}} \begin{cases} (1-v)C \text{ if } 0 \leqslant u \leqslant 1, \\ (4-3u-v)C \text{ if } 1 \leqslant u \leqslant \frac{4}{3}. \end{cases}$$

Therefore, using [3, Corollary 1.109], we obtain

$$S\big(W^A_{\bullet,\bullet}; C\big) = \frac{3}{10} \int_0^1 \int_0^1 3(1-v)^2 dv du + \frac{3}{10} \int_1^{\frac{4}{3}} \int_0^{4-3u} 3(4-3u-v)^2 dv du = \frac{13}{40}.$$

Similarly, using [3, Theorem 1.112], we obtain

$$S\big(W^{A,C}_{\bullet,\bullet,\bullet}; P\big) \leqslant \frac{3}{10} \int_0^1 \int_0^1 \big(3(1-v)\big)^2 dv du + \frac{3}{10} \int_1^{\frac{4}{3}} \int_0^{4-3u} \big(3(4-3u-v)\big)^2 dv du +$$

$$+ \underbrace{\frac{6}{10} \int_1^{\frac{4}{3}} \int_0^{4-3u} 3(4-3u-v)(u-1) dv du}_{\text{if } P \in E} = \frac{39}{40} + \frac{1}{120} = \frac{59}{60}.$$

Therefore, it follows from [3, Theorem 1.112] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{ \frac{1}{S_X(A)}, \frac{1}{S(W^A_{\bullet,\bullet}; C)}, \frac{1}{S(W^{A,C}_{\bullet,\bullet,\bullet}; P)} \right\} \geqslant \min\left\{ \frac{120}{67}, \frac{40}{13}, \frac{60}{59} \right\} > 1,$$

which is absurd. $\qquad\square$

Corollary 4.2. *The point P is not contained in the surface E.*

Since $A \cong \pi(A)$, we may consider A as a cubic surface in $\mathbb{P}^3$. Then

○ either $\operatorname{mult}_P(A) = 2$ and A has Du Val singularities.
○ or $\operatorname{mult}_P(A) = 3$ and A is a cone over a plane smooth cubic curve with vertex P.

Lemma 4.3. *One has $\operatorname{mult}_P(A) \neq 3$.*

Proof. Let $\sigma\colon \widehat{X} \to X$ be a blow up of the point P, and let G be the σ-exceptional surface. Denote by $\widehat{A}$ and $\widehat{E}$ the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively. Suppose that $\mathrm{mult}_P(A) = 3$. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(-K_X) - vG \sim_{\mathbb{R}} \frac{1}{3}\widehat{E} + \frac{4}{3}\widehat{A} + (4-u)G.$$

Thus, the divisor $\sigma^*(-K_X) - vG$ is pseudoeffective $\iff u \in [0, 4]$. Moreover, if $u \in [0, 4]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X) - vG$ can be described as follows:

$$P(u) = \begin{cases} \dfrac{1}{3}\widehat{E} + \dfrac{4}{3}\widehat{A} + (4-u)G \text{ if } 0 \leqslant u \leqslant 1, \\[2mm] \dfrac{1}{3}\widehat{E} + \dfrac{5-u}{3}\widehat{A} + (4-u)G \text{ if } 1 \leqslant u \leqslant 4, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\[2mm] \dfrac{u-1}{3}\widehat{A} \text{ if } 1 \leqslant u \leqslant 4, \end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the Zariski decomposition, respectively. Using this, we compute

$$S_X(G) = \frac{3}{10}\int_0^1 u^3\, du + \frac{3}{10}\int_1^4 u\, du = \frac{93}{40} < 3 = A_X(G).$$

As in the proof of Lemma 2.10, let us use results from [16, §4] to get a contradiction. Namely, applying [16, Corollary 4.18], we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{Q \in G} \delta_Q(G, V_{\bullet,\bullet}^G)\right\} = \min\left\{\frac{40}{31}, \inf_{Q \in G} \delta_Q(G, V_{\bullet,\bullet}^G)\right\},$$

where $\delta_Q(G, V_{\bullet,\bullet}^G)$ is defined in [16, §4]. So, there is $Q \in G$ such that $\delta_Q(G, V_{\bullet,\bullet}^G) < \frac{40}{31}$.

Let ℓ is a general line in $G \cong \mathbb{P}^2$ that contains Q. Then [16, Corollary 4.18] gives

$$\delta_Q(G, V_{\bullet,\bullet}^G) \geqslant \min\left\{\frac{1}{S(V_{\bullet,\bullet}^F; \ell)}, \frac{1}{S(W_{\bullet,\bullet,\bullet}^{G,\ell}; Q)}\right\}.$$

Let us compute $S(V_{\bullet,\bullet}^G; \ell)$ and $S(W_{\bullet,\bullet,\bullet}^{G,\ell}; Q)$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G - v\ell \sim_{\mathbb{R}} \begin{cases} (u-v)\ell \text{ if } 0 \leqslant u \leqslant 1, \\ (1-v)\ell \text{ if } 1 \leqslant u \leqslant 4. \end{cases}$$

Let $\widehat{\mathscr{C}} = \widehat{A}|_G$. Then $\widehat{\mathscr{C}}$ is a smooth cubic curve in $G \cong \mathbb{P}^2$. Let

$$N'(u) = N(u)\big|_G = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\[2mm] \dfrac{u-1}{3}\widehat{\mathscr{C}} \text{ if } 1 \leqslant u \leqslant 4. \end{cases}$$

Now, using [16, Theorem 4.8], we get

$$S(W^G_{\bullet,\bullet};\ell) = \frac{3}{10}\int_0^4\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu =$$

$$= \frac{3}{10}\int_0^1\int_0^u (u-v)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 (1-v)^2 dvdu = \frac{13}{40}.$$

Similarly, it follows from [16, Theorem 4.17] that $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q)$ can be computes as follows:

$$\frac{3}{10}\int_0^1\int_0^u \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + F_Q =$$

$$= \frac{3}{12}\int_0^1\int_0^u (u-v)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 (1-v)^2 dvdu + F_Q = \frac{13}{40} + F_Q,$$

where $F_Q = 0$ if $Q \notin \widehat{A}|_G$, and

$$F_Q = \frac{6}{10}\int_1^4\int_0^1 \frac{(1-v)(u-1)}{3}dvdu = \frac{9}{20}$$

otherwise. This gives $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) \leqslant \frac{31}{40}$. Combining the estimates, we get $\delta_Q(G,V^G_{\bullet,\bullet}) \geqslant \frac{40}{31}$, which is a contradiction. This completes the proof of the lemma. □

Hence, we see that the surface A has Du Val singularities. Let S be a general surface in the linear system $|H|$ that contains P. Then S is smooth, and $-K_X - uS \sim_{\mathbb{R}} (4-u)H - E$. Hence, the divisor $-K_X - uS$ is pseudoeffective $\iff$ it is nef $\iff$ $u \leqslant 1$. Then

$$S_X(S) = \frac{3}{10}\int_0^1 (-K_X - uS)^3 du = \frac{3}{10}\int_0^1 u(1-u)(7-u)du = \frac{13}{40} < 1.$$

Let $C = A|_S$. Then C is a reduced curve in $|-K_S|$ that is singular at P, and $C \cong \pi(C)$. Moreover, the curve $\pi(C)$ is a general hyperplane section of the cubic surface $\pi(A) \subset \mathbb{P}^3$ that passes through the point $\pi(P)$. Therefore, since $\pi(A)$ is not a cone by Lemma 4.3, we conclude that the curve C is irreducible. Hence, one of the following two cases holds:

(1) the curve C has an ordinary node at P,
(2) the curve C has an ordinary cusp at P.

Let $\Pi = \pi(S)$. Then Π is a plane in $\mathbb{P}^3$ such that $\pi(P) \in \Pi$ and $\Pi \cap \pi(A) = \pi(C)$, and the morphism $\pi|_S\colon S \to \Pi$ is a composition of blow ups of 9 intersection points $\Pi \cap \mathscr{C}$, which we denote by $O_1,\ldots,O_9$. Note that $\pi(C)$ is a reduced plane cubic curve that passes through these nine points, and $\pi(C)$ is smooth away from $\pi(P)$.

Lemma 4.4. *The curve C cannot have an ordinary double point at the point P.*

Proof. For each $i \in \{1,\ldots,9\}$, let L_i be the proper transform on S of the line in Π that passes through P and O_i. Then $L_i \neq L_j$ for $i \neq j$ since $\pi(C)$ is irreducible. We have

$$(-K_X - uS)\big|_S \sim_{\mathbb{R}} \frac{1-u}{6}\sum_{i=1}^9 L_i.$$

Let $\sigma\colon \widehat{S} \to S$ be the blow up of S at the point P, let $\mathbf{f}$ be the σ-exceptional curve and let $\widehat{C}, \widehat{L}_1, \ldots, \widehat{L}_9$ be the proper transforms on $\widehat{S}$ of the curves $C, L_1, \ldots, L_9$, respectively. Then $\widehat{L}_1, \ldots, \widehat{L}_9$ are disjoint. On the surface $\widehat{S}$, we have

$$\widehat{C}^2 = -4, \mathbf{f}^2 = \widehat{L}_1^2 = \cdots = \widehat{L}_9^2 = -1, \widehat{L}_1 \cdot \mathbf{f} = \cdots = \widehat{L}_9 \cdot \mathbf{f} = 1, \widehat{C} \cdot \mathbf{f} = 2,$$

and $\widehat{C} \cdot \widehat{L}_1 = \cdots = \widehat{C} \cdot \widehat{L}_9 = 0$.

Fix $u \in [0, 1]$. Let v be a nonnegative real number. Then

$$\sigma^*\left(\left(-K_X - uS\right)\big|_S\right) - v\mathbf{f} \sim_{\mathbb{R}} \frac{5+u}{6}\widehat{C} + \frac{1-u}{6}\sum_i^9 \widehat{L}_i + \frac{19 - 7u - 6v}{6}\mathbf{f}. \tag{4.1}$$

So, the divisor $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant \frac{19-7u}{6}$. Let $P(u,v)$ and $N(u,v)$ be the positive and the negative parts of its Zariski decomposition. Then, using Equation (4.1), we compute

$$P(u,v) = \begin{cases} \dfrac{5+u}{6}\widehat{C} + \dfrac{1-u}{6}\sum_i^9 \widehat{L}_i + \dfrac{19-7u-6v}{6}\mathbf{f} \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] \dfrac{19-7u-6v}{12}\widehat{C} + \dfrac{1-u}{6}\sum_i^9 \widehat{L}_i + \dfrac{19-7u-6v}{6}\mathbf{f} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 3-u, \\[3mm] \dfrac{19-7u-6v}{12}\left(\widehat{C} + 2\sum_i^9 \widehat{L}_i + 2\mathbf{f}\right) \text{ if } 3-u \leqslant v \leqslant \dfrac{19-7u}{6}, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] \dfrac{-3+3u+2v}{4}\widehat{C} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 3-u, \\[3mm] \dfrac{2v+3u-3}{4}\widehat{C} + (v+u-3)\sum_i^9 \widehat{L}_i \text{ if } 3-u \leqslant v \leqslant \dfrac{19-7u}{6}, \end{cases}$$

$$\mathrm{vol}\left(\sigma^*((-K_X-uS)|_S) - v\mathbf{f}\right) = \begin{cases} (1-u)(7-u) - v^2 \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] \dfrac{(1-u)(37-13u-12v)}{4} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 3-u, \\[3mm] \dfrac{(19-7u-6v)^2}{4} \text{ if } 3-u \leqslant v \leqslant \dfrac{19-7u}{6}, \end{cases}$$

$$P(u,v) \cdot \mathbf{f} = \begin{cases} v \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] \dfrac{3(1-u)}{2} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 3-u, \\[3mm] \dfrac{3(19-7u-6v)}{2} \text{ if } 3-u \leqslant v \leqslant \dfrac{19-7u}{6}. \end{cases}$$

Indeed, the divisor $P(u,v)$ above is effective, and it has a nonnegative intersection with every irreducible component of $\mathrm{Supp}(P(u,v))$, which implies that it is nef in every case. Now, using [3, Corollary 1.109], we get

$$S(W_{\bullet,\bullet}^{\widehat{S}}; \mathbf{f}) = \frac{3}{10} \int_0^1 \int_0^{\frac{19-7u}{6}} \mathrm{vol}\Big(\sigma^*\big((-K_X - uS)\big|_S\big) - v\mathbf{f}\Big) du\, dv =$$

$$= \frac{3}{10} \int_0^1 \int_0^{\frac{3-3u}{2}} \big((1-u)(7-u) - v^2\big) dv + \frac{3}{10} \int_0^1 \int_{\frac{3-3u}{2}}^{3-u} \frac{(1-u)(37 - 13u - 12v)}{4} dv +$$

$$+ \frac{3}{10} \int_0^1 \int_{3-u}^{\frac{19-7u}{6}} \frac{(19 - 7u - 6v)^2}{4} dv\, du = \frac{767}{480} < 2 = A_S(\mathbf{f}).$$

Moreover, if Q is a point in $\mathbf{f}$, then [3, Remark 1.113] gives

$$S\big(W_{\bullet,\bullet,\bullet}^{\widehat{S},\mathbf{f}}; Q\big) = F_Q + \frac{3}{10} \int_0^1 \int_0^{\frac{19-7u}{6}} \big(P(u,v) \cdot \mathbf{f}\big)^2 dv\, du = F_Q + \frac{3}{10} \int_0^1 \int_0^{\frac{3-3u}{2}} v^2 dv +$$

$$+ \frac{3}{10} \int_0^1 \int_{\frac{3-3u}{2}}^{3-u} \Big(\frac{3(1-u)}{2}\Big)^2 dv + \frac{3}{10} \int_0^1 \int_{3-u}^{\frac{19-7u}{6}} \Big(\frac{3(19 - 7u - 6v)}{2}\Big)^2 dv\, du = F_Q + \frac{147}{320},$$

where

$$F_Q = \frac{6}{10} \int_0^1 \int_0^{\frac{19-7u}{6}} \big(P(u,v) \cdot \mathbf{f}\big) \mathrm{ord}_Q\Big(N(u,v)\big|_{\mathbf{f}}\Big) dv\, du,$$

which implies the following assertions:

○ if $Q \notin \widehat{C} \cup \widehat{L}_1 \cup \cdots \cup \widehat{L}_9$, then $F_Q = 0$;

○ if $Q \in \widehat{L}_1 \cup \cdots \cup \widehat{L}_9$, then

$$F_Q = \frac{6}{10} \int_0^1 \int_{3-u}^{\frac{19-7u}{6}} \frac{3(19 - 7u - 6v)(v + u - 3)}{2} dv\, du = \frac{1}{960};$$

○ if $Q \in \widehat{C}$ and $\widehat{C}$ intersects $\mathbf{f}$ transversally at P, then

$$F_Q = \frac{6}{10} \int_0^1 \int_{\frac{3-3u}{2}}^{3-u} \frac{3(1-u)(2v + 3u - 3)}{8} dv\, du +$$

$$\frac{6}{10} \int_0^1 \int_{3-u}^{\frac{19-7u}{6}} \frac{3(19 - 7u - 6v)(2v + 3u - 3)}{8} dv\, du = \frac{643}{1920};$$

○ if $Q \in \widehat{C}$ and $\widehat{C}$ is tangent to $\mathbf{f}$ at the point P, then $F_Q = \frac{643}{960}$.

Thus, if C has a node at P, then $S\big(W_{\bullet,\bullet,\bullet}^{\widehat{S},\mathbf{f}}; Q\big) \leqslant \frac{305}{384}$, so [3, Remark 1.113] gives

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\Big\{\inf_{Q \in \mathbf{f}} \frac{1}{S\big(W_{\bullet,\bullet,\bullet}^{\widehat{S},\mathbf{f}}; Q\big)}, \frac{2}{S(V_{\bullet,\bullet}^{S}; \mathbf{f})}, \frac{1}{S_X(S)}\Big\} \geqslant \min\Big\{\frac{384}{305}, \frac{960}{767}, \frac{40}{13}\Big\} = \frac{960}{767} > 1,$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.5. *The curve C cannot have an ordinary cusp at the point P.*

Proof. Suppose C has a cusp. Let L be an irreducible curve in S such that $\pi(L)$ is a line and $\pi(L) \cap \pi(C) = \pi(P)$. Then $(-K_X - uS)|_S \sim_{\mathbb{R}} (1-u)L + C$.

Now, we consider the following commutative diagram:

$$
\begin{array}{ccccc}
S_1 & \xleftarrow{\ \sigma_2\ } & S_2 & \xleftarrow{\ \sigma_3\ } & S_3 \\
{\scriptstyle\sigma_1}\big\downarrow & & & & \big\downarrow{\scriptstyle v} \\
S & \xleftarrow{\qquad\sigma\qquad} & & & \widehat{S},
\end{array}
$$

where σ_1 is the blow up of P, σ_2 is the blow up of the point in the σ_1-exceptional curve contained in the proper transform of C, σ_3 is the blow up of the point in the σ_2-exceptional curve contained in the proper transform of C, v is the birational contraction of the proper transforms of $\sigma_1 \circ \sigma_2$-exceptional curves and σ is the birational contraction of the proper transform of the σ_3-exceptional curve. Then $\widehat{S}$ has two singular points:

(1) a cyclic quotient singularity of type $\frac{1}{2}(1,1)$, which we denote by Q_2;
(2) a cyclic quotient singularity of type $\frac{1}{3}(1,1)$, which we denote by Q_3.

Let $\mathbf{f}$ be the σ-exceptional curve, let $\widehat{C}$ be the proper transform on $\widehat{S}$ of the curve C and let $\widehat{L}$ be the proper transform of the curve L. Then the curves $\mathbf{f}, \widehat{C}, \widehat{L}$ are smooth. Moreover, it is not very difficult to check that $Q_2 \in \mathbf{f} \ni Q_3$, $Q_2 \notin \widehat{C} \not\ni Q_3$, $Q_2 \in \widehat{L} \not\ni Q_3$. Further, we have $A_S(\mathbf{f}) = 5$, $\sigma^*(C) \sim \widehat{C} + 6\mathbf{f}$, $\sigma^*(L) \sim \widehat{L} + 3\mathbf{f}$. On the surface $\widehat{S}$, we have

$$
\widehat{L}^2 = -\frac{1}{2},\ \widehat{L} \cdot \widehat{C} = 0,\ \widehat{C} \cdot \mathbf{f} = \frac{1}{2},\ \widehat{C}^2 = -6,\ \widehat{C} \cdot \mathbf{f} = 1,\ \mathbf{f}^2 = -\frac{1}{6}.
$$

Note that $Q_2 = \mathbf{f} \cap \widehat{L}$, and $\widehat{C}$ intersects $\mathbf{f}$ transversally by one point.

Fix $u \in [0,1]$. Let v be a nonnegative real number. Then

$$
\sigma^*\big((-K_X - uS)\big|_S\big) - v\mathbf{f} \sim_{\mathbb{R}} (1-u)\widehat{L} + \widehat{C} + (9 - 3u - v)\mathbf{f}. \tag{4.2}
$$

Thus, the divisor $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant 9 - 3u$. Let $P(u,v)$ be the positive part of the Zariski decomposition of $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$, and let $N(u,v)$ be the negative part. Then, using Equation (4.2), we compute

$$
P(u,v) =
\begin{cases}
(1-u)\widehat{L} + \widehat{C} + (9 - 3u - v)\mathbf{f} \text{ if } 0 \leqslant v \leqslant 3 - 3u, \\[2mm]
(1-u)\widehat{L} + \dfrac{9 - 3u - v}{6}\widehat{C} + (9 - 3u - v)\mathbf{f} \text{ if } 3 - 3u \leqslant v \leqslant 8 - 2u, \\[2mm]
\dfrac{9 - 3u - v}{6}\big(6\widehat{L} + \widehat{C} + 6\mathbf{f}\big) \text{ if } 8 - 2u \leqslant v \leqslant 9 - 3u,
\end{cases}
$$

and

$$
N(u,v) =
\begin{cases}
0 \text{ if } 0 \leqslant v \leqslant 3 - 3u, \\[2mm]
\dfrac{v + 3u - 3}{6}\widehat{C} \text{ if } 3 - 3u \leqslant v \leqslant 8 - 2u, \\[2mm]
\dfrac{v + 3u - 3}{6}\widehat{C} + (v + 2u - 8)\widehat{L} \text{ if } 8 - 2u \leqslant v \leqslant 9 - 3u.
\end{cases}
$$

This gives

$$\text{vol}\Big(\sigma^*\big((-K_X - uS)\big|_S\big) - v\mathbf{f}\Big) = \begin{cases} (1-u)(7-u) - \dfrac{v^2}{6} & \text{if } 0 \leqslant v \leqslant 3 - 3u, \\[2mm] \dfrac{(1-u)(17-5u-2v)}{2} & \text{if } 3 - 3u \leqslant v \leqslant 8 - 2u, \\[2mm] \dfrac{(9-3u-v)^2}{2} & \text{if } 8 - 2u \leqslant v \leqslant 9 - 3u, \end{cases}$$

and

$$P(u,v) \cdot \mathbf{f} = \begin{cases} \dfrac{v}{6} & \text{if } 0 \leqslant v \leqslant 3 - 3u, \\[2mm] \dfrac{1-u}{2} & \text{if } 3 - 3u \leqslant v \leqslant 8 - 2u, \\[2mm] \dfrac{9-3u-v}{2} & \text{if } 8 - 2u \leqslant v \leqslant 9 - 3u. \end{cases}$$

Now, we use [3, Corollary 1.109] to get

$$S(W_{\bullet,\bullet}^{\widehat{S}};\mathbf{f}) = \frac{3}{10}\int_0^1 \int_0^{3-3u} \left((1-u)(7-u) - \frac{v^2}{6}\right)dv +$$
$$+ \frac{3}{10}\int_0^1 \int_{3-3u}^{8-2u} \frac{(1-u)(17-5u-2v)}{2}dv + \frac{3}{10}\int_0^1 \int_{8-2u}^{9-3u} \frac{(9-3u-v)^2}{2}dv\,du = \frac{173}{40}.$$

Similarly, if Q is a point in $\mathbf{f}$, then [3, Remark 1.113] gives

$$S(W_{\bullet,\bullet,\bullet}^{\widehat{S},\mathbf{f}};Q) = F_Q + \frac{3}{10}\int_0^1 \int_0^{9-3u} \big(P(u,v)\cdot\mathbf{f}\big)^2 dv\,du =$$
$$= F_Q + \frac{3}{10}\int_0^1 \int_0^{3-3u} \left(\frac{v}{6}\right)^2 dv\,du + \frac{3}{10}\int_0^1 \int_{3-3u}^{8-2u} \left(\frac{1-u}{2}\right)^2 dv\,du +$$
$$+ \frac{3}{10}\int_0^1 \int_{8-2u}^{9-3u} \left(\frac{9-3u-v}{2}\right)^2 dv\,du = F_Q + \frac{5}{32},$$

where F_Q can be computes as follows:

○ if $Q \neq \widehat{C} \cap \mathbf{f}$ and $Q \neq \widehat{L} \cap \mathbf{f}$, then $F_Q = 0$;
○ if $Q = \widehat{L} \cap \mathbf{f}$, then

$$F_Q = \frac{6}{10}\int_0^1 \int_{8-2u}^{9-3u} \frac{(9-3u-v)(v+2u-8)}{2}dv\,du = \frac{1}{80};$$

○ if $Q = \widehat{C} \cap \mathbf{f}$, then

$$F_Q = \frac{6}{10}\int_0^1 \int_{3-3u}^{8-2u} \frac{(1-u)(v+3u-3)}{12}dv +$$
$$+ \frac{6}{10}\int_0^1 \int_{8-2u}^{9-3u} \frac{(9-3u-v)(v+3u-3)}{12}du\,dv = \frac{193}{480}.$$

Therefore, we conclude that

$$S\big(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q\big) = \begin{cases} \dfrac{5}{32} & \text{if } Q \notin \widehat{C} \cup \widehat{L}, \\[2mm] \dfrac{27}{80} & \text{if } Q = \widehat{L} \cap \mathbf{f}, \\[2mm] \dfrac{67}{120} & \text{if } Q = \widehat{C} \cap \mathbf{f}. \end{cases}$$

Let $\Delta_{\mathbf{f}} = \frac{1}{2}Q_2 + \frac{2}{3}Q_3$. Then, using [3, Remark 1.113] and our computations, we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\inf_{Q \in \mathbf{f}} \frac{1 - \operatorname{ord}_Q(\Delta_{\mathbf{f}})}{S\big(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q\big)}, \frac{5}{S\big(V^{S}_{\bullet,\bullet};\mathbf{f}\big)}, \frac{1}{S_X(S)}\right\} \geqslant \min\left\{\frac{40}{13}, \frac{200}{173}, \frac{120}{67}\right\} = \frac{200}{173} > 1,$$

which is a contradiction. $\qquad\square$

Combining Lemmas 4.4 and 4.5, we obtain a contradiction.

Corollary 4.6. *All smooth Fano threefolds in the family №2.4 are K-stable.*

5. Family №2.6 (Verra threefolds)

Smooth Fano threefolds in the family №2.6 can be described as follows:

(a) smooth divisors of degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, which are known as Verra threefolds,
(b) double covers of the (unique) smooth divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1,1)$ branched over smooth anticanonical K3 surfaces.

Note that every double cover of a smooth divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1,1)$ branched over a smooth anticanonical surface is K-stable [14, Example 4.4]. In fact, this also implies that general Verra threefold is K-stable [3, Example 3.14].

The goal of this section is to prove that all smooth Verra threefolds are K-stable.

Let X be a smooth divisor of degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, let $\pi_1 \colon X \to \mathbb{P}^2$ be the projection to the first factor of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $\pi_2 \colon X \to \mathbb{P}^2$ be the projection to the second factor. Then π_1 and π_2 are conic bundles [21]. Set $H_1 = \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $H_2 = \pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then

$$-K_X \sim H_1 + H_2,$$

and the group $\operatorname{Pic}(X)$ is generated by H_1 and H_2. Note that $\operatorname{Aut}(X)$ is finite [10]. Thus, the threefold X is K-stable $\iff$ it is K-polystable. See [22] for details.

Lemma 5.1. *Fix a point $P \in X$. Let C_1 be the fiber of the conic bundle π_1 that contains P, and let C_2 be the fiber of π_2 that contains P. Then C_1 or C_2 is smooth at P.*

Proof. Local computations. $\qquad\square$

Let Δ_1 and Δ_2 be the discriminant curves of the conic bundles π_1 and π_2, respectively. Then Δ_1 and Δ_2 are reduced curves of degree 6 that have at most ordinary double points as singularities. For basic properties of the discriminant curves Δ_1 and Δ_2, see [21, §3.8]. In particular, we know that no line in $\mathbb{P}^2$ can be an irreducible component of these curves.

Lemma 5.2. *Fix a point $P \in X$. Let C_2 be the fiber of the conic bundle π_2 that contains P, let S be a general surface in $|H_2|$ that contains C_2. Then one of the following cases holds:*

(1) *C_2 is smooth, S is smooth, the divisor $-K_S$ is ample;*
(2) *C_2 is smooth, S is smooth, the divisor $-K_S$ is nef, the surface S has exactly four irreducible curves that have trivial intersection with the divisor $-K_S$, these curves are disjoint and none of them passes through P, and $C_2 \subset \mathrm{Sing}(\pi_1^{-1}(\Delta_1))$;*
(3) *C_2 is singular and reduced, S is smooth, the divisor $-K_S$ is ample;*
(4) *C_2 is not reduced, $P \notin \mathrm{Sing}(S) \subset \mathrm{Supp}(C_2)$, and $\mathrm{Sing}(S)$ consists of two points, which are ordinary double points of the surface S, and $-K_S$ is ample.*

Proof. The assertion about the singularities of the surface S are local and well known.

We have $-K_S \sim H_1|_S$ and $K_S^2 = 2$, so S is a weak del Pezzo surface of degree 2, and the restriction $\pi_1|_S \colon S \to \mathbb{P}^2$ is the anticanonical morphism. Let ℓ be an irreducible curve in the surface S such that

$$-K_S \cdot \ell = 0.$$

Then ℓ is an irreducible component of the fiber $\pi_1^{-1}(\pi_1(\ell))$. But $\pi_2(\ell)$ is the line $\pi_2(S)$, which implies that the scheme fiber $\pi_1^{-1}(\pi_1(\ell))$ is singular, which implies that $\pi_1(\ell) \in \Delta_1$. Since $\ell \cap C_2 \neq \varnothing$ and $\pi_2(S)$ is a general line in $\mathbb{P}^2$ that passes through the point $\pi_2(P)$, we see that $\pi_1(C_2) \subset \Delta_1$. This implies that C_2 is irreducible and reduced.

Let $R = \pi_1^{-1}(\pi_1(C_2))$. Then the surface R is singular along a curve that is isomorphic to the conic $\pi_1(C_2) \cong C_2$. Let $f \colon \widetilde{R} \to R$ be the blow up of this curve. Then $\widetilde{R}$ is smooth, and the composition morphism $\pi_1 \circ f$ induces a $\mathbb{P}^1$-bundle $\widetilde{R} \to \widetilde{C}_2$, where $\widetilde{C}_2$ is double cover of the conic $\pi_1(C_2)$ that is branched over the eight points $\pi_1(C_2) \cap (\Delta_1 - \pi_1(C_2))$. In particular, we see that $\widetilde{C}_2$ is an irrational curve, which implies that $C_2 = \mathrm{Sing}(R)$.

Vice versa, if the fiber C_2 is a smooth conic, and the conic $\pi_1(C_2)$ is an irreducible component of the discriminant curve Δ_1, then it follows from the Bertini theorem that

$$S \cdot \pi_1^{-1}(\pi_1(C_2)) = 2C_2 + \ell_1 + \ell_2 + \cdots + \ell_k,$$

where $\ell_1, \ell_2, \ldots, \ell_k$ are k distinct irreducible reduced curves in X that are irreducible components of the fibers of the natural projection $\pi_1^{-1}(\pi_1(C_2)) \to \pi_1(C_2)$. Since

$$4 = 2H_2^2 \cdot H_1 = H_2 \cdot S \cdot \pi_1^{-1}(\pi_1(C_2)) = H_2 \cdot \left(2C_2 + \sum_{i=1}^{k} \ell_i\right) = k,$$

we see that S contains exactly 4 irreducible curves that intersects trivially with $-K_S$. Now, the generality in the choice of S implies that none of these curves contains P. $\qquad\square$

Example 5.3. Actually, the case (2) in Lemma 5.2 can happen. Indeed, in the assumption and notations of Lemma 5.2, let $P = ([0:0:1], [0:0:1])$, and suppose that X is given by

$$(u^2 + 2uw + v^2 + 2w^2)x^2 + (uv - w^2)xy + (uw - 2uv + 3v^2 + w^2)y^2 + (uw + v^2)z^2 = 0,$$

where $([u:v:w], [x:y:z])$ are coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$. Then the threefold X is smooth. For instance, this can be checked using the following Magma script:

```
Q:=RationalField();
PxP<x,y,z,u,v,w>:=ProductProjectiveSpace(Q,[2,2]);
X:=Scheme(PxP,[(u^2+2*u*w+v^2+2*w^2)*x^2+(u*v-w^2)*x*y+
              ((-2*v+w)*u+3*v^2+w^2)*y^2+(u*w+v^2)*z^2]);
IsNonsingular(X);
```

Observe that the fiber C_1 is a singular reduced curve given by $u = v = 2x^2 - xy + y^2 = 0$, the fiber C_2 is a smooth curve that is given by $x = y = uw + v^2 = 0$, and the discriminant curve Δ_1 is a union of the conic $\pi_1(C_2)$ and the irreducible quartic plane curve given by

$$8u^3v - 4u^3w - 11u^2v^2 + 16u^2vw - 12u^2w^2 + 8uv^3 -$$
$$-28uv^2w + 14uvw^2 - 16uw^3 - 12v^4 - 28v^2w^2 - 7w^4 = 0.$$

As in case (2) in Lemma 5.2, we have $C_2 = \mathrm{Sing}(\pi_1^{-1}(\pi_1(C_2)))$.

Let us prove that X is K-stable. Suppose it is not. Using the valuative criterion [15, 18], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface by [3, Theorem 3.17].

Let P be any point in Z, let C_1 be the fiber of the conic bundle π_1 that contains P and let C_2 be the fiber of the conic bundle π_2 that contains P. By Lemma 5.1, at least one curve among C_1 or C_2 is smooth at P. We may assume that C_2 is smooth at P.

Let S be a general surface in $|H_2|$ that contains C_2. Then S is smooth by Lemma 5.2. Moreover, one of the following three cases holds:

(1) C_2 is smooth, the divisor $-K_S$ is ample;
(2) C_2 is smooth, $\pi_1(C_2) \subset \Delta_1$, the divisor $-K_S$ is nef, the surface S has exactly four irreducible curves that have trivial intersection with the divisor $-K_S$, these curves are disjoint and none of them passes through P;
(3) C_2 is singular and reduced, the divisor $-K_S$ is ample.

Let C be the curve in X that is defined as follows:

○ if C_2 is smooth and irreducible, we let $C = C_2$.
○ if C_2 is reducible, we let C be its irreducible component that contains P.

Note that $-K_S \sim H_1|_S$ and $K_S^2 = 2$. Let $\eta: S \to \mathbb{P}^2$ be the restriction morphism $\pi_1|_S$. Then η is an anticanonical morphism of the surface S, which is generically two-to-one. Hence, the morphism η induces an involution $\tau \in \mathrm{Aut}(S)$. We let $C' = \tau(C)$.

Now, let u be a nonnegative real number. Then we have $-K_X - uS \sim_{\mathbb{R}} H_1 + (1 - u)H_2$, so $-K_X - uS$ is nef $\iff -K_X - uS$ is pseudoeffective $\iff u \leqslant 1$. Then $S_X(S) = \frac{5}{12}$. Now, let us use notations introduced in [3, §1.7]. Using [3, Theorem 1.112], we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W_{\bullet,\bullet}^S; C)}, \frac{1}{S(W_{\bullet,\bullet,\bullet}^{S,C}; P)}\right\},$$

where $S(W_{\bullet,\bullet}^S; C)$ and $S(W_{\bullet,\bullet,\bullet}^{S,C}; P)$ are defined in [3, §1.7]. Hence, since $S_X(S) < 1$, we get

$$\max\left\{S(W_{\bullet,\bullet}^S; C), S(W_{\bullet,\bullet,\bullet}^{S,C}; P)\right\} \geqslant 1. \tag{5.1}$$

Moreover, if $Z = P$, then it also follows from [3, Theorem 1.112] that

$$\max\left\{S(W_{\bullet,\bullet}^S; C), S(W_{\bullet,\bullet,\bullet}^{S,C}; P)\right\} > 1. \tag{5.2}$$

Let us estimate $S(W_{\bullet,\bullet}^S; C)$ and $S(W_{\bullet,\bullet,\bullet}^{S,C}; P)$ using results obtained in [3, §1.7].

Let $P(u) = -K_X - uS$. Then [3, Corollary 1.109] gives

$$S(W_{\bullet,\bullet}^S; C) = \frac{1}{4} \int_0^1 \int_0^\infty \mathrm{vol}(P(u)|_S - vC)\,dv\,du.$$

Let $P(u,v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_S - vC$, and let $N(u,v)$ be its negative part, where $u \in [0,1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = F_P + \frac{1}{4} \int_0^1 \int_0^\infty (P(u,v) \cdot C)^2\,dv\,du$$

by [3, Theorem 1.112], where

$$F_P = \frac{1}{2} \int_0^1 \int_0^\infty (P(u,v) \cdot C)\,\mathrm{ord}_P\Big(N(u,v)\big|_C\Big)\,dv\,du.$$

Lemma 5.4. *Suppose that C_2 is smooth, and $-K_S$ is ample. Then*

$$S(W_{\bullet,\bullet}^S; C) = \frac{13}{24},$$
$$S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = 1.$$

Proof. We have $C \cdot C' = 4$ and $(C')^2 = C^2 = 0$. Fix $u \in [0,1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)|_S - vC \sim_{\mathbb{Q}} \Big(\frac{3}{2} - u - v\Big)C + \frac{1}{2}C',$$

which implies that $P(u)|_S - vC$ is pseudoeffective $\iff$ $P(u)|_S - vC$ is nef $\iff$ $v \leqslant \frac{3}{2} - u$. If $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant \frac{3}{2} - u$, then $P(u,v) = (\frac{3}{2} - u - v)C + \frac{1}{2}C'$ and $N(u,v) = 0$, so

$$S(W_{\bullet,\bullet}^S; C) = \frac{1}{4} \int_0^1 \int_0^{\frac{3}{2}-u} \Big(\Big(\frac{3}{2} - u - v\Big)C + \frac{1}{2}C'\Big)^2\,dv\,du = \frac{1}{4} \int_0^1 \int_0^{\frac{3}{2}-u} 6 - 4u - 4v\,dv\,du = \frac{13}{24}.$$

Similarly, we see that $F_P = 0$ and $P(u,v) \cdot C = 2$, which gives $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = 1$. $\square$

Lemma 5.5. *Suppose that C_2 is smooth, $-K_S$ is not ample. Then*

$$S(W_{\bullet,\bullet}^S; C) = \frac{7}{12},$$
$$S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = \frac{5}{6}.$$

Proof. In this case, we have the following commutative diagram:

$$
\begin{array}{ccc}
 & S & \\
{\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\eta} \\
\overline{S} & \xrightarrow{\;\;\pi\;\;} & \mathbb{P}^2,
\end{array}
$$
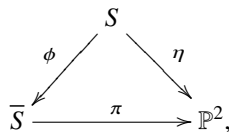
where ϕ is a birational map that contracts four disjoint (-2)-curves, and $\overline{S}$ is a del Pezzo surface of degree 2 that has 4 isolated ordinary double points, and π is a double cover that is ramified in a reducible quartic curve that is a union of two irreducible conics such that $\eta(C)$ is one of these two conics. In particular, we have $C = \tau(C)$.

Let E_1, E_2, E_3, E_4 be the ϕ-exceptional curves. Fix $u \in [0, 1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)|_S - vC \sim_{\mathbb{Q}} (2 - u - v)C + \frac{1}{2}(E_1 + E_2 + E_3 + E_4)$$

so the divisor $P(u)|_S - vC$ is pseudoeffective $\iff v \leqslant 2 - u$. Moreover, we have

$$P(u, v) = \begin{cases} (2 - u - v)C + \dfrac{1}{2}(E_1 + E_2 + E_3 + E_4) & \text{if } 0 \leqslant v \leqslant 1 - u, \\[2mm] (2 - u - v)\left(C + \dfrac{1}{2}(E_1 + E_2 + E_3 + E_4)\right) & \text{if } 1 - u \leqslant v \leqslant 2 - u, \end{cases}$$

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1 - u, \\[2mm] \dfrac{u + v - 1}{2}(E_1 + E_2 + E_3 + E_4) & \text{if } 1 - u \leqslant v \leqslant 2 - u, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_S - vC\big) = \begin{cases} (6 - 4u) - 4 & \text{if } 0 \leqslant v \leqslant 1 - u, \\[1mm] 2(2 - u - v)^2 & \text{if } 1 - u \leqslant v \leqslant 2 - u. \end{cases}$$

Now, integrating $\mathrm{vol}(P(u)|_S - vC)$, we obtain $S(W^S_{\bullet,\bullet}; C) = \frac{7}{12}$.

To compute $S(W^{S;C}_{\bullet,\bullet,\bullet}; P)$, we first observe that $F_P = 0$ because $P \notin E_1 \cup E_2 \cup E_3 \cup E_4$ since S is a general surface in $|H_2|$ that contains C_2. On the other hand, we have

$$P(u, v) \cdot C = \begin{cases} 2 & \text{if } 0 \leqslant v \leqslant 1 - u, \\ 4 - 2u - 2v & \text{if } 1 - u \leqslant v \leqslant 2 - u. \end{cases}$$

Hence, integrating $(P(u, v) \cdot C)^2$, we get $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) = \frac{5}{6}$ as required. $\qquad\square$

Lemma 5.6. *Suppose that C_2 is singular. Then*

$$S(W^S_{\bullet,\bullet}; C) = \frac{3}{4},$$

$$S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \leqslant \frac{11}{12}.$$

Proof. The curve C_2 consists of two irreducible components: the curve C and another curve, which we denote by L. The curves C and L are smooth and intersects transversally at one point. Note that $P \neq C \cap L$ since C_2 is smooth at the point P by assumption.

The intersections of the curves C, L and $C' = \tau(C)$ on S are given in the table below.

$\bullet$	C	L	C'
C	-1	1	2
L	1	-1	0
C'	2	0	-1

Fix $u \in [0, 1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Since $C + C' \sim -K_S$, we have

$$P(u)|_S - vC \sim_{\mathbb{Q}} (2 - u - v)C + (1 - u)L + C',$$

so $P(u)|_S - vC$ is pseudoeffective $\iff v \leqslant 2 - u$. Moreover, if $0 \leqslant u \leqslant \frac{1}{2}$, then

$$P(u, v) = \begin{cases} (2 - u - v)C + (1 - u)L + C' \text{ if } 0 \leqslant v \leqslant 1, \\ (2 - u - v)(C + L) + C' \text{ if } 1 \leqslant v \leqslant \dfrac{3}{2} - u, \\ (2 - u - v)(C + L + C') \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u, \end{cases}$$

$$N(u, v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\ (v - 1)L \text{ if } 1 \leqslant v \leqslant \dfrac{3}{2} - u, \\ (v - 1)L + (2v + 2u - 3)C' \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_S - vC\big) = \begin{cases} 6 - v^2 - 4u - 2v \text{ if } 0 \leqslant v \leqslant 1, \\ 7 - 4u - 4v \text{ if } 1 \leqslant v \leqslant \dfrac{3}{2} - u, \\ 4(u + v - 2)^2 \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u. \end{cases}$$

Similarly, if $\frac{1}{2} \leqslant u \leqslant 1$, then

$$P(u, v) = \begin{cases} (2 - u - v)C + (1 - u)L + C' \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\ (2 - u - v)(C + 2C') + (1 - u)L \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 1, \\ (2 - u - v)(C + L + C') \text{ if } 1 \leqslant v \leqslant 2 - u, \end{cases}$$

$$N(u, v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\ (2v + 2u - 3)C' \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 1, \\ (v - 1)L + (2v + 2u - 3)C' \text{ if } 1 \leqslant v \leqslant 2 - u, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_S - vC\big) = \begin{cases} 6 - v^2 - 4u - 2v \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\ (5 - 2u - 3v)(3 - 2u - v) \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 1, \\ 4(u + v - 2)^2 \text{ if } 1 \leqslant v \leqslant 2 - u. \end{cases}$$

Hence, integrating $\mathrm{vol}(P(u)|_S - vC)$, we get $S(W^S_{\bullet,\bullet}; C) = \frac{3}{4}$.

Now, let us compute $S(W^{S,C}_{\bullet,\bullet,\bullet}; P)$. If $0 \leqslant u \leqslant \frac{1}{2}$, then

$$P(u, v) \cdot C = \begin{cases} 1 + v \text{ if } 0 \leqslant v \leqslant 1, \\ 2 \text{ if } 1 \leqslant v \leqslant \dfrac{3}{2} - u, \\ 8 - 4u - 4v \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u. \end{cases}$$

Similarly, if $\frac{1}{2} \leqslant u \leqslant 1$, then

$$P(u, v) \cdot C = \begin{cases} 1 + v \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\[2mm] 7 - 4u - 3v \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 1, \\[2mm] 8 - 4u - 4v \text{ if } 1 \leqslant v \leqslant 2 - u. \end{cases}$$

Then integrating, we get $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = \frac{145}{192} + F_P$. To compute F_P, let us recall that $P \notin L$. Hence, if $P \notin C'$, then $F_P = 0$, which implies that $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = \frac{145}{192} < \frac{11}{12}$ as required.

We may assume that $P \in C \cap C'$. If C' intersects C transversally at P, then

$$\mathrm{ord}_P\left(N(u, v)\big|_C\right) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\[2mm] 2u + 2v - 3 \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u, \end{cases}$$

which gives

$$F_P = \frac{1}{2} \int_0^{\frac{1}{2}} \int_{\frac{3}{2}-u}^{2-u} (8 - 4u - 4v)(2u + 2v - 3) \, dv \, du +$$

$$+ \frac{1}{2} \int_{\frac{1}{2}}^1 \int_1^{\frac{3}{2}-u} (7 - 4u - 3v)(2u + 2v - 3) \, dv \, du$$

$$+ \frac{1}{2} \int_{\frac{1}{2}}^1 \int_{\frac{3}{2}-u}^{2-u} (8 - 4u - 4v)(2u + 2v - 3) \, dv \, du = \frac{31}{384},$$

so $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = \frac{145}{192} + \frac{31}{384} = \frac{107}{128} < \frac{11}{12}$. If C' is tangent to C at the point P, then

$$\mathrm{ord}_P\left(N(u, v)\big|_C\right) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{3}{2} - u, \\[2mm] 2(2u + 2v - 3) \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2 - u, \end{cases}$$

which gives $F_P = \frac{31}{192}$, so $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) = \frac{145}{192} + \frac{31}{192} = \frac{11}{12}$. □

Now, using Equation (5.2) and Lemmas 5.4, 5.5, 5.6, we see that Z is a curve.

Lemma 5.7. *One has $H_1 \cdot Z \geqslant 1$ and $H_2 \cdot Z \geqslant 1$.*

Proof. If $H_2 \cdot Z = 0$, then $Z = C$, which is impossible by Equation (5.1) and Lemmas 5.4, 5.5, 5.6. Hence, we see that $H_2 \cdot Z \geqslant 1$ and $\pi_2(Z)$ is a curve. Let us show that $H_1 \cdot Z \geqslant 1$.

Suppose that $H_1 \cdot Z = 0$. Then Z must be an irreducible component of the curve C_1. If C_1 is reduced, then arguing exactly as in the proofs of Lemmas 5.4, 5.5, 5.6, we obtain a contradiction with [3, Corollary 1.109]. Thus, we see that C_1 is not reduced.

So far, the point P was a point in Z. Let us choose $P \in Z$ such that $\pi_2(P) \in \pi_2(Z) \cap \Delta_2$. Then C_2 is singular. But it is smooth at P by Lemma 5.1, which fits our assumption above. Then $S(W_{\bullet,\bullet}^S; C) = \frac{3}{4}$ and $S(W_{\bullet,\bullet,\bullet}^{S,C}; P) \leqslant \frac{11}{12}$ by Lemma 5.6, which contradicts Equation (5.1). □

Both $\pi_1(Z)$ and $\pi_2(Z)$ are curves. Similar to what we did in the proof of Lemma 5.7, let us choose the point $P \in Z$ such that $\pi_1(P) \in \Delta_1$. Then C_1 is singular at P, which implies that C_2 is smooth at P by Lemma 5.1. Now, using Equation (5.1) and Lemmas 5.5 and 5.6, we see that $C = C_2$, the curve C_2 is smooth, the divisor $-K_S$ is ample.

We see that S is a del Pezzo surface, and $\eta\colon S \to \mathbb{P}^2$ is a double cover ramified in a smooth quartic curve, so we are almost in the same position as in the proof of Lemma 5.4. But now, we have one small advantage: the point $\eta(P)$ is contained in the ramification divisor of the double cover η because C_1 is singular at P. Then $|-K_S|$ contains a unique curve that is singular at P. Denote this curve by R. We have the following possibilities:

(a) R is an irreducible curve that has a nodal singularity at P;
(b) R is an irreducible curve that has a cuspidal singularity at P;
(c) $R = R_1 + R_2$ for two (-1)-curves in S that intersect transversally at P;
(d) $R = R_1 + R_2$ for two (-1)-curves in S that are tangent at P.

Let $f\colon \widetilde{S} \to S$ be the blow up of the point P. Denote by $\widetilde{R}$ and $\widetilde{C}$ the proper transforms on the surface $\widetilde{S}$ of the curves R and C, respectively. Fix $u \in [0,1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$f^*\big(P(u)|_S\big) - vE \sim_{\mathbb{Q}} \widetilde{R} + (1-u)\widetilde{C} + (3-u-v)E.$$

Let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of $\widetilde{R} + (1-u)\widetilde{C} + (3-u-v)E$, and let $\widetilde{N}(u,v)$ its negative part. Then it follows from [3, Remark 1.113] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{2}{S(W^{\widetilde{S}}_{\bullet,\bullet};E)}, \inf_{O \in E} \frac{1}{S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)}\right\}, \tag{5.3}$$

where $S(W^{\widetilde{S}}_{\bullet,\bullet};E)$ and $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)$ are defined in [3] similar to $S(W^{S}_{\bullet,\bullet};C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$. These two numbers can be computed using [3, Remark 1.113]. Namely, we have

$$S(W^{\widetilde{S}}_{\bullet,\bullet};E) = \frac{1}{4}\int_0^1\int_0^\infty \big(\widetilde{P}(u,v)\big)^2 dv\, du$$

and

$$S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O) = \frac{1}{4}\int_0^1\int_0^\infty \Big(\big(\widetilde{P}(u,v)\cdot E\big)\Big)^2 dv\, du + F_O,$$

where O is a point in E and

$$F_O = \frac{1}{2}\int_0^1\int_0^\infty \big(\widetilde{P}(u,v)\cdot E\big)\mathrm{ord}_O\Big(\widetilde{N}(u,v)\big|_E\Big) dv\, du.$$

Let us use these formulas to estimate $S(W^{\widetilde{S}}_{\bullet,\bullet};E)$ and $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)$.

If the curve R is irreducible, the intersections of the curves $\widetilde{C}, \widetilde{R}, E$ can be computed as follows: $\widetilde{C}^2 = -1$, $\widetilde{C}\cdot\widetilde{R} = 0$, $\widetilde{C}\cdot E = 1$, $\widetilde{R}^2 = -2$, $\widetilde{R}\cdot E = 2$, $E^2 = -1$. If R is reducible, then $\widetilde{R} = \widetilde{R}_1 + \widetilde{R}_2$ for two smooth irreducible curves $\widetilde{R}_1 + \widetilde{R}_2$ such that the intersection form of the curves $\widetilde{C}, \widetilde{R}_1, \widetilde{R}_1$ and E is given in the following table:

$\bullet$	$\widetilde{C}$	$\widetilde{R}_1$	$\widetilde{R}_1$	E
$\widetilde{C}$	-1	0	0	1
$\widetilde{R}_1$	0	-2	1	1
$\widetilde{R}_1$	0	1	-2	1
E	1	1	1	-1

Then $\widetilde{R} + (1-u)\widetilde{C} + (3-u-v)E$ is pseudoeffective $\iff v \leqslant 3-u$. Moreover, we have

$$\widetilde{P}(u,v) = \begin{cases} \widetilde{R} + (1-u)\widetilde{C} + (3-u-v)E \text{ if } 0 \leqslant v \leqslant 2-u, \\ (1-u)\widetilde{C} + (3-u-v)(E+\widetilde{R}) \text{ if } 2-u \leqslant v \leqslant 2, \\ (3-u-v)(E+\widetilde{R}+\widetilde{C}) \text{ if } 2 \leqslant v \leqslant 3-u, \end{cases}$$

$$\widetilde{N}(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 2-u, \\ (v+u-2)\widetilde{R} \text{ if } 2-u \leqslant v \leqslant 2, \\ (v+u-2)\widetilde{R} + (v-2)\widetilde{C} \text{ if } 2 \leqslant v \leqslant 3-u, \end{cases}$$

$$\left(\widetilde{P}(u,v)\right)^2 = \begin{cases} 6 - v^2 - 4u \text{ if } 0 \leqslant v \leqslant 2-u, \\ 14 + 2u^2 + 4uv + v^2 - 12u - 8v \text{ if } 2-u \leqslant v \leqslant 2, \\ 2(3-u-v)^2 \text{ if } 2 \leqslant v \leqslant 3-u, \end{cases}$$

Now, integrating, we obtain $S(W^{\widetilde{S}}_{\bullet,\bullet}; E) = \frac{17}{12}$.

Fix a point $O \in E$. To estimate $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet}; O)$, first we observe that

$$\widetilde{P}(u,v) \cdot E = \begin{cases} v \text{ if } 0 \leqslant v \leqslant 2-u, \\ 4 - 2u - v \text{ if } 2-u \leqslant v \leqslant 2, \\ 6 - 2u - 2v \text{ if } 2 \leqslant v \leqslant 3-u. \end{cases}$$

Therefore, integrating $(\widetilde{P}(u,v) \cdot E)^2$, we obtain $S(W^{Y,\widetilde{S}}_{\bullet,\bullet,\bullet}; O) = \frac{13}{24} + F_O$.

If $O \notin \widetilde{C} \cup \widetilde{R}$, then $F_O = 0$. Similarly, if $O \in \widetilde{C}$, then $O \notin \widetilde{R}$, which gives

$$F_O = \frac{1}{2} \int_0^1 \int_2^{3-u} (6-2u-2v)(v-2)\,dv\,du = \frac{1}{24}.$$

Finally, if $O \in \widetilde{R}$, then $O \notin \widetilde{C}$, which gives

$$F_O \leqslant \frac{1}{2} \int_0^1 \int_{2-u}^2 2(4-2u-v)(v+u-2)\,dv\,du + \frac{1}{2} \int_0^1 \int_2^{3-u} 2(6-2u-2v)(v+u-2)\,dv\,du = \frac{7}{24}.$$

Summarizing, we get $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet}; O) \leqslant \frac{5}{6}$ for every point $O \in E$, which contradicts Equation (5.3) because $S_X(S) = \frac{5}{12}$ and $S(W^{\widetilde{S}}_{\bullet,\bullet}; E) = \frac{17}{12} < 2$. This shows that X is K-stable.

Corollary 5.8. *All smooth Fano threefolds in the family №2.6 are K-stable.*

6. Family №2.7

Now, let us fix three smooth quadric hypersurfaces $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$ in $\mathbb{P}^4$ such that their intersection is a smooth curve of degree 8 and genus 5. We set $\mathscr{C} = \mathcal{Q} \cap \mathcal{Q}' \cap \mathcal{Q}''$. Let $\pi \colon X \to \mathcal{Q}$ be the blow up of the smooth curve $\mathscr{C}$. Then X is a smooth Fano threefold, which is contained in the family №2.7. Moreover, all smooth Fano threefolds in this family can be obtained in this way. Note that $-K_X^3 = 14$ and $\mathrm{Aut}(X)$ is finite [10].

The pencil generated by the surfaces $\mathcal{Q}'|_\mathcal{Q}$ and $\mathcal{Q}''|_\mathcal{Q}$ gives a rational map $\mathcal{Q} \dashrightarrow \mathbb{P}^1$, which fits the following commutative diagram:

$$
\begin{array}{ccc}
 & X & \\
\pi \swarrow & & \searrow \phi \\
\mathcal{Q} \dashrightarrow & & \mathbb{P}^1,
\end{array}
$$
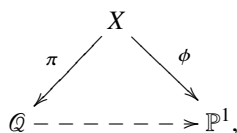
where ϕ is a fibration into quartic del Pezzo surfaces. Let E be the π-exceptional surface, and let $H = \pi^*(\mathcal{O}_{\mathbb{P}^4}(1)|_\mathcal{Q})$. Then $-K_X \sim 3H - E$, the morphism ϕ is given by the linear system $|2H - E|$, and $E \cong \mathscr{C} \times \mathbb{P}^1$.

Lemma 6.1. *Let S be a fiber of the morphism ϕ. Then S has at most Du Val singularities.*

Proof. Since $E|_S$ is a smooth curve, the surface S is smooth along $E|_S$, which implies that it has at most isolated singularities. Hence, we conclude that S is normal and irreducible.

Note that $S \cong \pi(S)$. Suppose that the singularities of the surface $\pi(S)$ are not Du Val. Then it follows from [5, Theorem 1] that $\pi(S)$ is a cone in $\mathbb{P}^4$ over a quartic elliptic curve. Let P be the vertex of the cone $\pi(S)$, and let T_P be the hyperplane section of the quadric hypersurface $\mathcal{Q}$ that is singular at P. Then T_P contains all lines in $\mathcal{Q}$ that pass through P, which implies that T_P contains $\pi(S)$. This is impossible since T_P is a quadric cone. $\square$

The goal of this section is to prove that X is K-stable. To do this, we fix a point $P \in X$. By [15, 18], to prove that X is K-stable, it is enough to show that $\delta_P(X) > 1$.

Let S be the fiber of the morphism ϕ containing P. Then S is a quartic del Pezzo surface, and S has at most Du Val singularity at P by Lemma 6.1. Moreover, if S is singular at P, then P is a singular point of the surface S of type $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{D}_4$ or $\mathbb{D}_5$, see [12].

Lemma 6.2. *If $\delta_P(S) > \frac{54}{55}$ or $P \in \mathrm{Sing}(S)$ and $\delta_P(S) > \frac{27}{28}$, then $\delta_P(X) > 1$.*

Proof. Take $u \in \mathbb{R}_{\geq 0}$. Since $S \sim 2H - E$, we have

$$
-K_X - uS \sim_{\mathbb{R}} (3 - 2u)H - (u-1)E \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right)S + \frac{1}{2}E.
$$

Using this, we conclude that the divisor $-K_X - uS$ is pseudoeffective if and only if $u \leqslant \frac{3}{2}$. For $u \leqslant \frac{3}{2}$, let $P(u)$ be the positive part of Zariski decomposition of the divisor $-K_X - uS$, and let $N(u)$ be the negative part of Zariski decomposition of the divisor $-K_X - uS$. Then

$$
P(u) = \begin{cases} -K_X - uS \text{ if } 0 \leqslant u \leqslant 1, \\ (3 - 2u)H \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}
$$

and

$$
N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)E \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}
$$

This gives

$$
S_X(S) = \frac{1}{14} \int_0^{\frac{3}{2}} (P(u))^3 \, du = \frac{1}{14} \int_0^1 (14 - 12u) du + \frac{1}{14} \int_1^{\frac{3}{2}} 2(3 - 2u)^3 \, du = \frac{33}{56} < 1.
$$

Now, using [1, Theorem 3.3] and [3, Corollary 1.102], we get

$$\delta_P(X) \geqslant \min\left\{ \frac{1}{S_X(S)}, \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S(W_{\bullet,\bullet}^S; F)} \right\}, \tag{6.1}$$

where the infimum is taken by all prime divisors F over the surface S such that $P \in C_S(F)$, and $S(W_{\bullet,\bullet}^S; F)$ can be computed using [3, Corollary 1.108] as follows:

$$S(W_{\bullet,\bullet}^S; F) = \frac{3}{14} \int_0^{\frac{3}{2}} \left(P(u)\big|_S\right)^2 \mathrm{ord}_F\left(N(u)\big|_S\right) du + \frac{3}{14} \int_0^{\frac{3}{2}} \int_0^{\infty} \mathrm{vol}\left(P(u)\big|_S - vF\right) dv du.$$

Let $E_S = E|_S$. Then $E_S \in |-2K_S|$ and $E_S \cong \mathscr{C}$. Moreover, the surface S is smooth in a neighborhood of the curve E_S. Furthermore, we have

$$P(u)\big|_S = \begin{cases} -K_S \text{ if } 0 \leqslant u \leqslant 1, \\ (3 - 2u)(-K_S) \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(u)\big|_S = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u - 1)E_S \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Let F be any prime divisor over S such that $P \in C_S(F)$. Then

$$\begin{aligned}
S(W_{\bullet,\bullet}^S; F) &= \frac{3}{14} \int_1^{\frac{3}{2}} 4(u-1)(3-2u)^2 \mathrm{ord}_F(E_S) du + \\
&\quad + \frac{3}{14} \int_0^1 \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv du + \\
&\quad + \frac{3}{14} \int_1^{\frac{3}{2}} \int_0^{\infty} \mathrm{vol}\left((3-2u)(-K_S) - vF\right) dv du = \\
&= \frac{\mathrm{ord}_F(E_S)}{56} + \frac{3}{14} \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv + \frac{3}{14} \int_1^{\frac{3}{2}} (3-2u)^3 \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv du = \\
&= \frac{\mathrm{ord}_F(E_S)}{56} + \frac{3}{14} \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv + \frac{3}{112} \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv = \\
&= \frac{\mathrm{ord}_F(E_S)}{56} + \frac{27}{112} \int_0^{\infty} \mathrm{vol}\left(-K_S - vF\right) dv = \frac{\mathrm{ord}_F(E_S)}{56} + \frac{27}{28} S_S(F) \leqslant \frac{A_S(F)}{56} + \frac{27 A_S(F)}{28 \delta_P(S)}
\end{aligned}$$

because log pair (S, E_S) is log canonical. Therefore, if $\delta_P(S) > \frac{54}{55}$, Equation (6.1) gives $\delta_P(X) > 1$. Similarly, if $P \in \mathrm{Sing}(S)$, $P \notin E_S$, so $\mathrm{ord}_F(E_S) = 0$, which implies that

$$S(W_{\bullet,\bullet}^S; F) = \frac{27}{28} S_S(F) \leqslant \frac{27 A_S(F)}{28 \delta_P(S)}.$$

Hence, in this case, it follows from Equation (6.1) that $\delta_P(X) > 1$ provided that $\delta_P(S) > \frac{27}{28}$. □

Corollary 6.3. *If S is smooth, then $\delta_P(X) > 1$.*

*Proof.* If S is smooth, then $\delta(S) = \frac{4}{3}$ by [3, Lemma 2.12]. Now, apply Lemma 6.2. □

Corollary 6.4. *If P is not contained in a line in S, then $\delta_P(X) > 1$.*

Proof. If P is not contained in a line in S, then P is a smooth point of the surface S, and the blow up of the surface S at this point is a (possibly singular) cubic surface in $\mathbb{P}^3$. Thus, arguing exactly as in the end of the proof of [3, Lemma 2.12], we obtain $\delta_P(S) \geqslant \frac{3}{2}$, which implies that $\delta_P(X) > 1$ by Lemma 6.2. $\qquad\square$

Now, let T be a surface in the linear system $|H|$ such that $P \in T$, and let $\mathcal{Q} = \pi(T)$. Then $\mathcal{Q}$ is a hyperplane section of the hypersurface $\mathcal{Q}$, so both $\mathcal{Q}$ and T are irreducible. In the following, we will choose T such that the surface $\mathcal{Q}$ is smooth so that $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 6.5. *Suppose that T is a general surface in the linear system $|H|$ such that $P \in T$. Then the (scheme) intersection $S \cap T$ is an irreducible reduced curve.*

Proof. Let $\rho\colon \widetilde{S} \to S$ be a blow up of the quartic del Pezzo surface S at the point P, and let Z be the proper transform of the curve $T|_S$ on the surface $\widetilde{S}$. Then $|Z|$ has no base points and gives the morphism $\eta\colon \widetilde{S} \to \mathbb{P}^3$ that fits the following commutative diagram:

$$
\begin{array}{ccc}
 & \widetilde{S} & \\
\rho \swarrow & & \searrow \eta \\
S \dashrightarrow & \cdots \cdots \rightarrow & \mathbb{P}^3,
\end{array}
$$
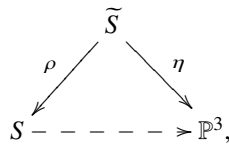
where $S \dashrightarrow \mathbb{P}^3$ is a projection from P. Moreover, if P is a smooth point of the surface S, then $Z^2 = 3$, and the image of the morphism η is an irreducible cubic surface in $\mathbb{P}^3$. Similarly, if P is a singular point of the surface S, then we have $Z^2 = 4 - \mathrm{mult}_P(S) = 2$, and the image of the morphism η is an irreducible quadric surface. Therefore, we conclude that the curve Z must be irreducible and reduced (by Bertini theorem), which implies that the intersection $S \cap T$ is also irreducible and reduced. $\qquad\square$

Remark 6.6. Suppose that S is singular at P, and T is a general surface in $|H|$ that passes through the point P. Then, choosing appropriate coordinates $[x : y : z : t : w]$ on $\mathbb{P}^4$, we may assume that $\pi(P) = [0 : 0 : 0 : 0 : 1]$, and the surface $\pi(S)$ is given in $\mathbb{P}^4$ by

$$
\begin{cases}
at^2 + btx + f_2(x, y, z) = 0, \\
wt = g_2(x, y, z),
\end{cases}
$$

where a and b are complex numbers, $f_2(x, y, z)$ and $g_2(x, y, z)$ are nonzero quadratic homogeneous polynomials. In the chart $w \neq 0$, the surface $\pi(S)$ is given by

$$
\begin{cases}
at^2 + btx + f_2(x, y, z) = 0, \\
t = g_2(x, y, z),
\end{cases}
$$

where now we consider x, y, z, t as affine coordinates on $\mathbb{C}^4$. Then $\pi(S) \cap \mathcal{Q}$ is cut out on the surface $\pi(S)$ by $c_1 x + c_2 y + c_3 z + c_4 t = 0$, where c_1, c_2, c_3, c_4 are general numbers. The affine part of the surface $\pi(S)$ is isomorphic to the hypersurface in $\mathbb{C}^3$ given by

$$
ag_2^2(x, y, z) + bxg_2(x, y, z) + f_2(x, y, z) = 0,
$$

and the affine part of the curve $\pi(S) \cap \mathcal{Q}$ is cut out by $c_1 x + c_2 y + c_3 z + c_4 g_2(x, y, z) = 0$. If P is a singular point of the surface S of type $\mathbb{D}_4$ or $\mathbb{D}_5$, then $S \cap T$ has an ordinary cusp at the point P, which easily implies that the intersection $S \cap T$ is reduced and irreducible. Similarly, if P is a Du Val singular point of the surface S of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$, then the intersection $S \cap T$ has an isolated ordinary double singularity at P.

Observe that the morphism $\pi\colon X \to \mathcal{Q}$ induces a birational morphism $\varpi\colon T \to \mathcal{Q}$, and the morphism $\phi\colon X \to \mathbb{P}^1$ induces a fibration $\varphi\colon T \to \mathbb{P}^1$ that both fit the following commutative diagram:

$$
\begin{array}{ccc}
 & T & \\
\varpi \swarrow & & \searrow \varphi \\
\mathcal{Q} \dashrightarrow & & \mathbb{P}^1,
\end{array}
$$
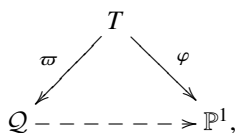
where $\mathcal{Q} \dashrightarrow \mathbb{P}^1$ is a map given by the pencil generated by the curves $\mathcal{Q}'|_\mathcal{Q}$ and $\mathcal{Q}''|_\mathcal{Q}$. In the following, we will always choose $T \in |H|$ such that the quadric surface $\mathcal{Q}$ is smooth, and T is either smooth or has one isolated ordinary singularity, which would imply that a general fiber of the induced fibration $\varphi\colon T \to \mathbb{P}^1$ is a smooth elliptic curve. Let $\mathcal{C} = S|_T$. Then $\mathcal{C}$ is the fiber of the (elliptic) fibration φ that contains the point P.

Let u be a nonnegative real number. Then $-K_X - uT \sim_\mathbb{R} (3 - u)H - E \sim_\mathbb{R} (1 - u)H + S$, which implies that $-K_X - uT$ is nef $\iff -K_X - uT$ is pseudoeffective $\iff u \in [0, 1]$. Integrating, we get $S_X(T) = \frac{9}{28} < 1$. For simplicity, we let $P(u) = -K_X - uT$.

Lemma 6.7. *Suppose that S is singular at P. Then $\delta_P(X) > 1$.*

Proof. Now, let us choose $T \in |H|$ such that T is a general surface in $|H|$ that contains P. Then T and $\mathcal{Q}$ are smooth, and ϖ is a blow up of the eight intersection points $\mathscr{C} \cap \mathcal{Q}$. Moreover, by Lemma 6.5, the curve $\mathcal{C}$ is an irreducible singular curve of arithmetic genus 1. Thus, we have $P = \mathrm{Sing}(\mathcal{C})$. Furthermore, using Remark 6.6, we see that

○ either $\mathcal{C}$ has an isolated ordinary double singularity at P,
○ or the curve $\mathcal{C}$ has an ordinary cusp at the point P.

Recall that $\mathcal{Q}$ is a smooth quadric surface, so that it contains exactly two lines that pass though $\pi(P)$. Since T is chosen to be general, these lines are disjoint from $\mathscr{C} \cap \mathcal{Q}$. Denote by L_1 and L_2 the proper transforms of these lines on T. Then

$$
P(u)\big|_T \sim_\mathbb{R} \big((1 - u)H + S\big)\big|_T \sim_\mathbb{R} (1 - u)(L_1 + L_2) + \mathcal{C}.
$$

Now, we let $\sigma\colon \widetilde{T} \to T$ be the blow up of the point P, we let $\mathbf{e}$ be the σ-exceptional curve, and we denote by $\widetilde{\mathcal{C}}, \widetilde{L}_1, \widetilde{L}_2$ the proper transforms on $\widetilde{T}$ of the curves $\mathcal{C}, L_1, L_2$, respectively. Take a nonnegative real number v. Then

$$
\sigma^*\big(P(u)\big|_T\big) - v\mathbf{e} \sim_\mathbb{R} \widetilde{\mathcal{C}} + (1 - u)(\widetilde{L}_1 + \widetilde{L}_2) + (4 - 2u - v)\mathbf{e}. \tag{6.2}
$$

Note that the curves $\widetilde{\mathcal{C}}, \widetilde{L}_1, \widetilde{L}_2$ are disjoint. Moreover, we have $\widetilde{L}_1^2 = \widetilde{L}_2^2 = -1$ and $\widetilde{\mathcal{C}}^2 = -4$. Thus, using Equation (6.2), we see that $\sigma^*(P(u)|_T) - v\mathbf{e}$ is pseudoeffective $\iff v \leqslant 4 - 2u$.

Let $\widetilde{P}(u, v)$ be the positive part of Zariski decomposition of the divisor $\sigma^*(P(u)|_T) - v\mathbf{e}$, and let $\widetilde{N}(u, v)$ be its negative part. Then it follows from [3, Remark 1.113] that

$$
\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(T)}, \frac{2}{S(W^{\widetilde{T}}_{\bullet,\bullet}; \mathbf{e})}, \inf_{O \in \mathbf{e}} \frac{1}{S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet}; O)}\right\}, \tag{6.3}
$$

where $S(W^{\widetilde{T}}_{\bullet,\bullet}; \mathbf{e})$ and $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet}; O)$ are defined in [3, §1.7], and these two numbers can be computed using formulas described in [3, Remark 1.113]. Namely, we have

$$
S(W^{\widetilde{T}}_{\bullet,\bullet}; \mathbf{e}) = \frac{3}{14} \int_0^1 \int_0^{4 - 2u} \big(\widetilde{P}(u, v)\big)^2 \, dv\, du
$$

and

$$S(W_{\bullet,\bullet,\bullet}^{\widetilde{T};\mathbf{e}};O) = \frac{3}{14}\int_0^1\int_0^{4-2u}\left((\widetilde{P}(u,v)\cdot\mathbf{e})\right)^2 dvdu + \frac{3}{7}\int_0^1\int_0^{4-2u}(\widetilde{P}(u,v)\cdot\mathbf{e})\mathrm{ord}_O\left(\widetilde{N}(u,v)\big|_{\mathbf{e}}\right)dvdu,$$

where O is a point in $\mathbf{e}$. Moreover, using Equation (6.2), we compute $\widetilde{P}(u,v)$ and $\widetilde{N}(u,v)$ as follows:

$$\widetilde{P}(u,v) = \begin{cases} \widetilde{C} + (1-u)(\widetilde{L}_1+\widetilde{L}_2) + (4-2u-v)\mathbf{e} \text{ if } 0 \le v \le 2-2u, \\ \dfrac{4-2u-v}{2}\widetilde{C} + (1-u)(\widetilde{L}_1+\widetilde{L}_2) + (4-2u-v)\mathbf{e} \text{ if } 2-2u \le v \le 3-u, \\ \dfrac{4-2u-v}{2}(\widetilde{C} + 2(\widetilde{L}_1+\widetilde{L}_2) + 2\mathbf{e}) \text{ if } 3-u \le v \le 4-2u, \end{cases}$$

and

$$\widetilde{N}(u,v) = \begin{cases} 0 \text{ if } 0 \le v \le 2-2u, \\ \dfrac{v-2+2u}{2}\widetilde{C} \text{ if } 2-2u \le v \le 3-u, \\ \dfrac{v-2+2u}{2}\widetilde{C} + (v-3+u)(\widetilde{L}_1+\widetilde{L}_2) \text{ if } 3-u \le v \le 4-2u. \end{cases}$$

Thus, we have

$$(\widetilde{P}(u,v))^2 = \begin{cases} 2(1-u)(5-u) - v^2 \text{ if } 0 \le v \le 2-2u, \\ 2(1-u)(7-3u-2v) \text{ if } 2-2u \le v \le 3-u, \\ 2(4-2u-v)^2 \text{ if } 3-u \le v \le 4-2u, \end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e} = \begin{cases} v \text{ if } 0 \le v \le 2-2u, \\ 2(1-u) \text{ if } 2-2u \le v \le 3-u, \\ 2(4-2u-v) \text{ if } 3-u \le v \le 4-2u. \end{cases}$$

Now, integrating, we get $S(W_{\bullet,\bullet}^{\widetilde{T}};\mathbf{e}) = \frac{51}{28} < 2$.

Let O be an arbitrary point in $\mathbf{e}$. If $O \notin \widetilde{L}_1\cup\widetilde{L}_2\cup\widetilde{C}$, then we compute $S(W_{\bullet,\bullet,\bullet}^{\widetilde{T};\mathbf{e}};O) = \frac{4}{7}$. Similarly, if $O \in \widetilde{L}_1\cup\widetilde{L}_2$, then $S(W_{\bullet,\bullet,\bullet}^{\widetilde{T};\mathbf{e}};O) = \frac{17}{28}$. Finally, if $O \in \widetilde{C}$, then

$$S(W_{\bullet,\bullet,\bullet}^{\widetilde{T};\mathbf{e}};O) = \frac{4}{7} + \frac{3}{7}\int_0^1\int_{2-2u}^{3-u} 2(1-u)\frac{v-2+2u}{2}\mathrm{ord}_O(\widetilde{C}|_{\mathbf{e}})dvdu +$$
$$+ \frac{3}{7}\int_0^1\int_{3-u}^{4-2u} 2(4-2u-v)\frac{v-2+2u}{2}\mathrm{ord}_O(\widetilde{C}|_{\mathbf{e}})dvdu = \frac{4}{7} + \frac{17}{56}\mathrm{ord}_O(\widetilde{C}|_{\mathbf{e}}).$$

Hence, if $\widetilde{C}$ intersects $\mathbf{e}$ transversally, then $S(W_{\bullet,\bullet,\bullet}^{\widetilde{T};\mathbf{e}};O) < 1$ so that $\delta_P(X) > 1$ by Equation (6.3).

Therefore, to complete the proof of the lemma, we may assume that $\widetilde{C}$ is tangent to $\mathbf{e}$. This means that C has a cusp at P, and the intersection $\widetilde{C}\cap\mathbf{e}$ consists of a single point.

Now, as in the proof of Lemma 4.5, we consider the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{T} & \xleftarrow{\;\;\gamma\;\;} & \overline{T} \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\upsilon} \\
T & \xleftarrow{\;\;\varsigma\;\;} & \widehat{T},
\end{array}
$$

where γ is a composition of the blow up of the point $\widetilde{\mathcal{C}} \cap \mathbf{e}$ with the blow up of the unique intersection point of the proper transforms of the curves $\widetilde{\mathcal{C}}$ and $\mathbf{e}$, υ is the birational contraction of all $(\sigma \circ \gamma)$-exceptional curves that are not (-1)-curve, and ς is the birational contraction of the proper transform of the unique γ-exceptional curve that is (-1)-curve. Then $\widehat{T}$ has two singular points:

(1) a cyclic quotient singularity of type $\frac{1}{2}(1,1)$, which we denote by O_2;
(2) a cyclic quotient singularity of type $\frac{1}{3}(1,1)$, which we denote by O_3.

Let $\mathbf{f}$ be the ς-exceptional curve, let $\widehat{\mathcal{C}}$ be the proper transform on $\widehat{T}$ of the curve $\mathcal{C}$ and let $\widehat{L}_1$ and $\widehat{L}_2$ be the proper transforms on $\widehat{T}$ of the curves L_1 and L_2, respectively. Then the curves $\mathbf{f}$, $\widehat{\mathcal{C}}$, $\widehat{L}_1$, $\widehat{L}_2$ are smooth, and the curve $\mathbf{f}$ contains both points O_2 and O_3, which are not contained in $\widehat{\mathcal{C}}$. Moreover, we have

$$
\widehat{L}_1 \cap \widehat{L}_2 = \widehat{L}_1 \cap \mathbf{f} = \widehat{L}_2 \cap \mathbf{f} = O_3.
$$

Furthermore, we have $A_T(\mathbf{f}) = 5$, $\varsigma^*(\mathcal{C}) \sim \widehat{\mathcal{C}} + 6\mathbf{f}$, $\varsigma^*(L_1) \sim \widehat{L}_1 + 2\mathbf{f}$, $\varsigma^*(L_2) \sim \widehat{L}_2 + 2\mathbf{f}$, and the intersection form of the curves $\mathbf{f}$, $\widehat{\mathcal{C}}$, $\widehat{L}_1$ and $\widehat{L}_2$ is given in the following table:

	$\mathbf{f}$	$\widehat{\mathcal{C}}$	$\widehat{L}_1$	$\widehat{L}_2$
$\mathbf{f}$	$-\frac{1}{6}$	1	$\frac{1}{3}$	$\frac{1}{3}$
$\widehat{\mathcal{C}}$	1	-6	0	0
$\widehat{L}_1$	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
$\widehat{L}_2$	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$

For a nonnegative real number v, we have

$$
\varsigma^*\big(P(u)\big|_T\big) - v\mathbf{f} \sim_{\mathbb{R}} \widehat{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10 - 4u - v)\mathbf{f},
$$

which implies that the divisor $\varsigma^*(P(u)|_T) - v\mathbf{f}$ is pseudoeffective if and only if $v \leqslant 10 - 4u$ because the intersection form of the curves $\widehat{\mathcal{C}}$, $\widehat{L}_1$, $\widehat{L}_2$ is negative definite.

Let $\widehat{P}(u,v)$ be the positive part of Zariski decomposition of the divisor $\varsigma^*(P(u)|_T) - v\mathbf{f}$, and let $\widehat{N}(u,v)$ be its negative part. Set $\Delta_{\mathbf{f}} = \frac{1}{2}O_2 + \frac{2}{3}O_3$. By [3, Remark 1.113], we get

$$
\delta_P(X) \geqslant \min\left\{ \frac{1}{S_X(T)},\ \frac{5}{S(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f})},\ \inf_{O \in \mathbf{f}} \frac{1 - \operatorname{ord}_O(\Delta_{\mathbf{f}})}{S(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)} \right\}, \tag{6.4}
$$

where $S(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f})$ and $S(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)$ are defined as $S(W^{\widetilde{T}}_{\bullet,\bullet}; \mathbf{e})$ and $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet}; O)$ used earlier. Moreover, it follows from [3, Remark 1.113] that

$$
S(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f}) = \frac{3}{14} \int_0^1 \int_0^{10-4u} \big(\widehat{P}(u,v)\big)^2 dv\,du
$$

and

$$S(W_{\bullet,\bullet,\bullet}^{\widehat{T},\mathbf{f}}; O) = \frac{3}{14} \int_0^1 \int_0^{10-4u} \left(\left(\widehat{P}(u,v) \cdot \mathbf{f} \right) \right)^2 dv\,du + \frac{3}{7} \int_0^1 \int_0^{10-4u} \left(\widehat{P}(u,v) \cdot \mathbf{f} \right) \mathrm{ord}_O \left(\widehat{N}(u,v)\big|_{\mathbf{f}} \right) dv\,du.$$

Moreover, we compute $\widehat{P}(u,v)$ and $\widehat{N}(u,v)$ as follows:

$$\widehat{P}(u,v) = \begin{cases} \widetilde{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10 - 4u - v)\mathbf{f} & \text{if } 0 \leqslant v \leqslant 4 - 4u, \\[2mm] \dfrac{10 - 4u - v}{6}\widetilde{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10 - 4u - v)\mathbf{f} & \text{if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\[2mm] \dfrac{10 - 4u - v}{6}\left(\widehat{\mathcal{C}} + 6(\widehat{L}_1 + \widehat{L}_2) + 6\mathbf{f} \right) & \text{if } 9 - 3u \leqslant v \leqslant 10 - 4u, \end{cases}$$

and

$$\widehat{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 4 - 4u, \\[2mm] \dfrac{v - 4 + 4u}{6}\widehat{\mathcal{C}} & \text{if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\[2mm] \dfrac{v - 4 + 4u}{6}\widehat{\mathcal{C}} + (v - 9 + 3u)(\widetilde{L}_1 + \widetilde{L}_2) & \text{if } 9 - 3u \leqslant v \leqslant 10 - 4u. \end{cases}$$

This gives

$$\left(\widehat{P}(u,v) \right)^2 = \begin{cases} 2(1-u)(5-u) - \dfrac{v^2}{6} & \text{if } 0 \leqslant v \leqslant 4 - 4u, \\[2mm] \dfrac{2(1-u)(19 - 7u - 2v)}{3} & \text{if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\[2mm] \dfrac{2(10 - 4u - v)^2}{3} & \text{if } 9 - 3u \leqslant v \leqslant 10 - 4u, \end{cases}$$

and

$$\widehat{P}(u,v) \cdot \mathbf{f} = \begin{cases} \dfrac{v}{6} & \text{if } 0 \leqslant v \leqslant 4 - 4u, \\[2mm] \dfrac{2(1-u)}{3} & \text{if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\[2mm] \dfrac{2(10 - 4u - v)}{3} & \text{if } 9 - 3u \leqslant v \leqslant 10 - 4u. \end{cases}$$

Now, integrating, we get $S(W_{\bullet,\bullet}^{\widehat{T}}; \mathbf{f}) = \frac{135}{28} < A_T(\mathbf{f}) = 5$.

Let O be a point in $\mathbf{f}$. If $O \notin \widehat{L}_1 \cup \widehat{L}_2 \cup \widehat{\mathcal{C}}$, then

$$S(W_{\bullet,\bullet,\bullet}^{\widehat{T},\mathbf{f}}; O) = \frac{3}{14} \int_0^1 \int_0^{4-4u} \left(\frac{v}{6} \right)^2 dv\,du + \frac{3}{14} \int_0^1 \int_{4-4u}^{9-3u} \left(\frac{2(1-u)}{3} \right)^2 dv\,du + $$

$$+ \frac{3}{14} \int_0^1 \int_{9-3u}^{10-4u} \left(\frac{2(10 - 4u - v)}{3} \right)^2 dv\,du = \frac{13}{63}.$$

Similarly, if $O = \mathbf{f} \cap \widetilde{\mathcal{C}}$, then

$$S(W_{\bullet,\bullet,\bullet}^{\widehat{T},\mathbf{f}}; O) = \frac{13}{63} + \frac{3}{7} \int_0^1 \int_{4-4u}^{9-3u} \frac{2(1-u)}{3} \times \frac{v-4+4u}{6} dv\, du +$$

$$+ \frac{3}{7} \int_0^1 \int_{9-3u}^{10-4u} \frac{2(10-4u-v)}{3} \times \frac{v-4+4u}{6} dv\, du = \frac{13}{63} + \frac{193}{504} = \frac{33}{56}.$$

Likewise, if $O = O_3$, we compute $S(W_{\bullet,\bullet,\bullet}^{\widehat{T},\mathbf{f}}; O) = \frac{3}{14}$. So, using Equation (6.4), we get $\delta_P(X) > 1$.   □

Thus, to prove that $\delta_P(X) > 1$, we may assume that S is singular, but $P \notin \mathrm{Sing}(S)$.

Lemma 6.8. *Suppose that $P \notin E$. Then $\delta_P(X) > 1$.*

Proof. Recall that E is the π-exceptional surface. Using Corollary 6.4, we may assume that S contains a line L that passes through P. Then $\pi(L)$ is a line in $\mathcal{Q}$, $\pi(L) \cap \mathcal{C} \neq \varnothing$, and one of the following cases holds:

Case 1: the line $\pi(L)$ intersects the curve $\mathcal{C}$ transversally at two points,
Case 2: the line $\pi(L)$ is tangent to the curve $\mathcal{C}$ at their single intersection point.

Now, let us choose T to be a sufficiently general surface in $|H|$ that passes through L. If the intersection $\pi(L) \cap \mathcal{C}$ consists of two points, then $\varpi \colon T \to \mathcal{Q}$ is a blow up of eight distinct points of the transversal intersection $\mathcal{Q} \cap \mathcal{C}$, which implies that T is smooth. On the other hand, if $L \cap \mathcal{C}$ consists of one point, then T has one ordinary double point, which is not contained in the curve L. We have $\mathcal{C} = S|_T = L + Z$, where Z is a smooth rational irreducible curve such that $\pi(Z)$ is a smooth twisted cubic in $\mathcal{Q}$. The twisted cubic curve $\pi(Z)$ in $\mathcal{Q}$ intersects the curve $\mathcal{C}$ transversally by six distinct points, which we denote by $Q_3, Q_4, Q_5, Q_6, Q_7, Q_8$. Moreover, if $\pi(L) \cap \mathcal{C}$ consists of two distinct points, we denote these points by Q_1 and Q_2. Likewise, if $\pi(L) \cap \mathcal{C}$ consists of one point, we let $Q_1 = Q_2 = \pi(Z) \cap \mathcal{C}$. Then

Case 1: the morphism $\varpi \colon T \to \mathcal{Q}$ is the blow up of the points $Q_1, Q_2, \ldots, Q_8$,
Case 2: the morphism $\varpi \colon T \to \mathcal{Q}$ is a composition of the blow up of the points $Q_3, \ldots, Q_8$ with a weighted blow up with weights $(1, 2)$ of the point $Q_1 = Q_2$.

Since T is a general surface in $|H|$ that contains the line L, we may assume that $P \notin Z$. Likewise, we may assume further that Z is contained in the smooth locus of the surface T. Moreover, we may also assume that the quadric surface $\mathcal{Q}$ does not contain lines that pass through one point in the set $\{Q_1, Q_2, \pi(P)\}$ and one point in $\{Q_3, Q_4, Q_5, Q_6, Q_7, Q_8\}$. Indeed, let $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{Q}'''$ be the hyperplane sections of the quadric $\mathcal{Q}$ that are singular at the points Q_1, Q_2, $\pi(P)$, respectively. Then $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{Q}'''$ are cones, $\pi(L) \subset \mathcal{Q}' \cap \mathcal{Q}'' \cap \mathcal{Q}'''$, and every line in $\mathcal{Q}$ containing a point in $\{Q_1, Q_2, \pi(P)\}$ is a ruling of one of these cones. On the other hand, we have $\mathcal{C} \not\subset \mathcal{Q}' \cup \mathcal{Q}'' \cup \mathcal{Q}'''$ because $\mathcal{C}$ is not contained in a hyperplane. This implies that the quadric threefold $\mathcal{Q}$ contains at most finitely many lines that pass through a point in $\{Q_1, Q_2, \pi(P)\}$ and a point in $\mathcal{C} \setminus \{Q_1, Q_2\}$. Therefore, we can choose the surface $T \in |H|$ such that $L \subset T$, but $\mathcal{Q} = \pi(T)$ does not contain any of these lines.

Let us identify $\mathcal{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ such that the line $\pi(L)$ is a curve in $\mathcal{Q}$ of degree $(0, 1)$. Denote by $\mathbf{e}_1, \ldots, \mathbf{e}_8$ the ϖ-exceptional curves such that $\varpi(\mathbf{e}_1) = Q_1, \ldots, \varpi(\mathbf{e}_8) = Q_8$. Let $\mathbf{g}_3, \ldots, \mathbf{g}_8$ be the strict transforms on T of the curves in $\mathcal{Q}$ of degree $(1, 0)$ that pass through the points $Q_3, \ldots, Q_8$, respectively. Then $L, Z, \mathbf{e}_1, \ldots, \mathbf{e}_8, \mathbf{g}_3, \ldots, \mathbf{g}_8$ are smooth irreducible rational curves. In Case 1, their intersections are given in the following table:

	L	Z	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_8$	$\mathbf{g}_3$	$\mathbf{g}_4$	$\mathbf{g}_5$	$\mathbf{g}_6$	$\mathbf{g}_7$	$\mathbf{g}_8$
L	-2	2	1	1	0	0	0	0	0	0	1	1	1	1	1	1
Z	2	-2	0	0	1	1	1	1	1	1	0	0	0	0	0	0
$\mathbf{e}_1$	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_2$	1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_3$	0	1	0	0	-1	0	0	0	0	0	1	0	0	0	0	0
$\mathbf{e}_4$	0	1	0	0	0	-1	0	0	0	0	0	1	0	0	0	0
$\mathbf{e}_5$	0	1	0	0	0	0	-1	0	0	0	0	0	1	0	0	0
$\mathbf{e}_6$	0	1	0	0	0	0	0	-1	0	0	0	0	0	1	0	0
$\mathbf{e}_7$	0	1	0	0	0	0	0	0	-1	0	0	0	0	0	1	0
$\mathbf{e}_8$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	1
$\mathbf{g}_3$	1	0	0	0	1	0	0	0	0	0	-1	0	0	0	0	0
$\mathbf{g}_4$	1	0	0	0	0	1	0	0	0	0	0	-1	0	0	0	0
$\mathbf{g}_5$	1	0	0	0	0	0	1	0	0	0	0	0	-1	0	0	0
$\mathbf{g}_6$	1	0	0	0	0	0	0	1	0	0	0	0	0	-1	0	0
$\mathbf{g}_7$	1	0	0	0	0	0	0	0	1	0	0	0	0	0	-1	0
$\mathbf{g}_8$	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	-1

In Case 2, we have $\mathbf{e}_1 = \mathbf{e}_2$, and $\mathbf{e}_1$ contains the singular point of T so that $\mathbf{e}_1^2 = -\frac{1}{2}$. The remaining intersection numbers are exactly the same as in Case 1.

Observe that $P \notin Z \cup \mathbf{g}_3 \cup \mathbf{g}_4 \cup \mathbf{g}_5 \cup \mathbf{g}_6 \cup \mathbf{g}_7 \cup \mathbf{g}_8 \cup \mathbf{e}_1 \cup \mathbf{e}_2$ since $P \notin E$ by assumption.

Recall that $P(u) = -K_X - uT$ is nef $\iff$ $P(u)$ is pseudoeffective $\iff$ $u \in [0,1]$. Let v be a nonnegative real number. Then, in both Cases 1 and 2, we have

$$P(u)\big|_T - vL \sim_{\mathbb{R}} \frac{9 - 5u - 4v}{4}L + \frac{3 + u}{4}Z + \frac{5 - 5u}{4}(\mathbf{e}_1 + \mathbf{e}_2) + \frac{1 - u}{4}\sum_{i=3}^{8}\mathbf{g}_i, \qquad (6.5)$$

which implies that the divisor $P(u)\big|_T - vL$ is pseudoeffective $\iff$ $v \leqslant \frac{9 - 5u}{4}$.

Let $P(u, v)$ be the positive part of Zariski decomposition of the divisor $P(u)\big|_T - vL$, and let $N(u, v)$ be its negative part. Then it follows from [3, Theorem 1.112] that

$$\delta_P(X) \geqslant \min\left\{ \frac{1}{S_X(T)}, \frac{1}{S(W^T_{\bullet,\bullet}; L)}, \frac{1}{S(W^{T,L}_{\bullet,\bullet,\bullet}; P)} \right\}, \qquad (6.6)$$

where

$$S\big(W^T_{\bullet,\bullet}; L\big) = \frac{3}{14}\int_0^1 \int_0^{\frac{9-5u}{4}} \big(P(u, v)\big)^2 \, dv \, du$$

and

$$S\big(W^{T,L}_{\bullet,\bullet,\bullet}; P\big) = \frac{3}{14}\int_0^1 \int_0^{\frac{9-5u}{4}} \big((P(u,v) \cdot L)\big)^2 \, dv \, du + \frac{3}{7}\int_0^1 \int_0^{\frac{9-5u}{4}} \big(P(u,v) \cdot L\big) \operatorname{ord}_P\big(N(u,v)\big|_L\big) \, dv \, du.$$

Let us compute $S(W^T_{\bullet,\bullet};L)$ and $S(W^{T,L}_{\bullet,\bullet,\bullet};P)$. If $0 \leqslant u \leqslant \frac{1}{3}$, then, using Equation (6.5), we get

$$P(u,v) = \begin{cases} \dfrac{9-5u-4v}{4}L + \dfrac{3+u}{4}Z + \dfrac{5-5u}{4}(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 0 \leqslant v \leqslant 1, \\[4mm] \dfrac{9-5u-4v}{4}(L+\mathbf{e}_1+\mathbf{e}_2) + \dfrac{3+u}{4}Z + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 1 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[4mm] \dfrac{9-5u-4v}{4}(L+Z+\mathbf{e}_1+\mathbf{e}_2) + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 2-u, \\[4mm] \dfrac{9-5u-4v}{4}\left(L+Z+\mathbf{e}_1+\mathbf{e}_2+\displaystyle\sum_{i=3}^{8}\mathbf{g}_i\right) \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\[2mm] (v-1)(\mathbf{e}_1+\mathbf{e}_2) \text{ if } 1 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] (v-1)(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{2v+3u-3}{2}Z \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 2-u, \\[3mm] (v-1)(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{2v+3u-3}{2}Z + (v-2+u)\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

$$\left(P(u,v)\right)^2 = \begin{cases} 2u^2 + (2v-12)u - 2v^2 - 2v + 10 \text{ if } 0 \leqslant v \leqslant 1, \\[2mm] 2u^2 + (2v-12)u - 6v + 12 \text{ if } 1 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] \dfrac{13u^2 + 16uv + 4v^2 - 42u - 24v + 33}{2} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 2-u, \\[3mm] \dfrac{(9-5u-4v)^2}{2} \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

and

$$P(u,v) \cdot L = \begin{cases} 1-u+2v \text{ if } 0 \leqslant v \leqslant 1, \\[2mm] 3-u \text{ if } 1 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[3mm] 6-4u-2v \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 2-u, \\[3mm] 2(9-5u-4v) \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}. \end{cases}$$

Similarly, if $\frac{1}{3} \leqslant u \leqslant 1$, then, using Equation (6.5), we get

$$P(u,v) = \begin{cases} \dfrac{9-5u-4v}{4}L + \dfrac{3+u}{4}Z + \dfrac{5-5u}{4}(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[4mm] \dfrac{9-5u-4v}{4}(L+Z) + \dfrac{5-5u}{4}(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 1, \\[4mm] \dfrac{9-5u-4v}{4}(L+Z+\mathbf{e}_1+\mathbf{e}_2) + \dfrac{1-u}{4}\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 1 \leqslant v \leqslant 2-u, \\[4mm] \dfrac{9-5u-4v}{4}\left(L+Z+\mathbf{e}_1+\mathbf{e}_2+\displaystyle\sum_{i=3}^{8}\mathbf{g}_i\right) \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[2mm] \dfrac{2v+3u-3}{2}Z \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 1, \\[2mm] (v-1)(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{2v+3u-3}{2}Z \text{ if } 1 \leqslant v \leqslant 2-u, \\[2mm] (v-1)(\mathbf{e}_1+\mathbf{e}_2) + \dfrac{2v+3u-3}{2}Z + (v-2+u)\displaystyle\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

$$\big(P(u,v)\big)^2 = \begin{cases} 2u^2 + (2v-12)u - 2v^2 - 2v + 10 \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[2mm] \dfrac{(1-u)(29-13u-16v)}{2} \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 1, \\[2mm] \dfrac{13u^2 + 16uv + 4v^2 - 42u - 24v + 33}{2} \text{ if } 1 \leqslant v \leqslant 2-u, \\[2mm] \dfrac{(9-5u-4v)^2}{2} \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}, \end{cases}$$

and

$$P(u,v) \cdot L = \begin{cases} 1-u+2v \text{ if } 0 \leqslant v \leqslant \dfrac{3-3u}{2}, \\[2mm] 4-4u \text{ if } \dfrac{3-3u}{2} \leqslant v \leqslant 1, \\[2mm] 6-4u-2v \text{ if } 1 \leqslant v \leqslant 2-u, \\[2mm] 2(9-5u-4v) \text{ if } 2-u \leqslant v \leqslant \dfrac{9-5u}{4}. \end{cases}$$

Therefore, we have $P \notin \mathrm{Supp}(N(u,v))$ because $P \notin Z \cup \mathbf{g}_3 \cup \mathbf{g}_4 \cup \mathbf{g}_5 \cup \mathbf{g}_6 \cup \mathbf{g}_7 \cup \mathbf{g}_8 \cup \mathbf{e}_1 \cup \mathbf{e}_2$. So, integrating $(P(u,v))^2$ and $(P(u,v) \cdot L)^2$, we get $S(W_{\bullet,\bullet}^T; L) = \frac{423}{448}$ and $S(W_{\bullet,\bullet,\bullet}^{T,L}; P) = \frac{79}{84}$, which implies that $\delta_P(X) > 1$ by Equation (6.6). $\qquad\square$

By Lemma 6.8, to show that $\delta_P(X) > 1$, we may assume that $P \in E$. Then $\pi(P) \in \mathscr{C}$. Now, let us choose T to be a sufficiently general surface in $|H|$ that contains the point P so that $\mathcal{Q}$ is a sufficiently general hyperplane section of the quadric $\mathcal{Q}$ that contains $\pi(P)$. Then T is smooth, and $\varpi \colon T \to \mathcal{Q}$ is a blow up of eight points of the intersection $\mathcal{Q} \cap \mathscr{C}$.

Let $Q_1 = \pi(P)$, let $Q_2, \ldots, Q_8$ be the remaining seven points of the intersection $\mathcal{Q} \cap \mathscr{C}$ and let $\mathbf{e}_1, \ldots, \mathbf{e}_8$ be the ϖ-exceptional curves such that $\varpi(\mathbf{e}_1) = Q_1, \ldots, \varpi(\mathbf{e}_8) = Q_8$. For every $u \in [0,1]$, set

$$t(u) = \inf\left\{v \in \mathbb{R}_{\geqslant 0} \,\middle|\, \text{the divisor } P(u)\big|_T - v\mathbf{e}_1 \text{ is pseudoeffective}\right\},$$

and fix a real number $v \in [0, t(u)]$. Let $P(u,v)$ and $N(u,v)$ be the positive and the negative parts of the Zariski decomposition of the $\mathbb{R}$-divisor $P(u)\big|_T - v\mathbf{e}_1$, respectively. Then

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(T)}, \frac{1}{S(W_{\bullet,\bullet}^T; \mathbf{e}_1)}, \frac{1}{S(W_{\bullet,\bullet,\bullet}^{T;\mathbf{e}_1}; P)}\right\} \qquad (6.7)$$

by [3, Theorem 1.112], where

$$S\big(W_{\bullet,\bullet}^T; \mathbf{e}_1\big) = \frac{3}{14}\int_0^1 \int_0^{t(u)} \big(P(u,v)\big)^2 \, dv\, du$$

and

$$S\bigl(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet}; P\bigr) = \frac{3}{14} \int_0^1 \int_0^{t(u)} \bigl((P(u,v) \cdot \mathbf{e}_1)\bigr)^2 dv\, du +$$

$$+ \frac{3}{7} \int_0^1 \int_0^{t(u)} (P(u,v) \cdot \mathbf{e}_1)\, \mathrm{ord}_P\bigl(N(u,v)\big|_{\mathbf{e}_1}\bigr) dv\, du.$$

Let us compute $S(W^T_{\bullet,\bullet}; \mathbf{e}_1)$ and $S(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet}; P)$.

Recall that $\varphi\colon T \to \mathbb{P}^1$ is an elliptic fibration, which is given by the linear system $|-K_T|$. As in the proof of Lemma 6.8, let us identify $\mathcal{Q} = \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 6.9. *Let F be a curve in $|-K_T|$. Then F is irreducible and reduced.*

Proof. Suppose that F is reducible or nonreduced. Then the curve $\pi(F)$ is also reducible or nonreduced, and every irreducible component of the curve F is a smooth (-2)-curve. But $\pi(F)$ is a curve in $\mathcal{Q}$ of degree $(2,2)$ that passes through the points $Q_1, Q_2, \ldots, Q_8$. Therefore, we have one of the following possibilities:

(1) $\mathcal{Q}$ contains a line that passes through Q_1 and one point among $Q_2, \ldots, Q_8$,
(2) $\mathcal{Q}$ contains a line that passes through two points among $Q_2, \ldots, Q_8$,
(3) $\mathcal{Q}$ contains a conic that passes through Q_1 and three points among $Q_2, \ldots, Q_8$.

Recall that $\mathcal{Q}$ is a general hyperplane section of the quadric $\mathcal{\widehat{Q}}$ that contains $Q_1 = \pi(P)$. As we already mentioned in the proof of Lemma 6.8, the quadric $\mathcal{\widehat{Q}}$ contains finitely many lines that pass through Q_1 and a point in $\mathscr{C} \setminus Q_1$. Thus, since $\mathcal{Q}$ is assumed to be general, the quadric $\mathcal{Q}$ does not contain any of these lines so that $\mathcal{Q}$ does not contain a line that passes through Q_1 and a point among $Q_2, \ldots, Q_8$.

Similarly, a parameter count implies that $\mathcal{Q}$ does not contain secant lines of the curve $\mathscr{C}$ so that $\mathcal{Q}$ does not contain a line that passes through two points among $Q_2, \ldots, Q_8$,

Finally, we suppose that $\mathcal{Q}$ contains an irreducible conic C that passes through Q_1 and three points among $Q_2, \ldots, Q_8$. Let $\rho\colon \mathcal{\widehat{Q}} \dashrightarrow \mathbb{P}^3$ be a linear projection from the point Q_1. Then $\rho(\mathscr{C})$ is a curve of degree 7, and the induced map $\mathscr{C} \dashrightarrow \rho(\mathscr{C})$ is an isomorphism because $\mathscr{C}$ is an intersection of quadrics. Similarly, all points $\rho(Q_2), \ldots, \rho(Q_8)$ are distinct. Then $\rho(C)$ is a three-secant line of the curve $\rho(\mathscr{C})$. Note that $\rho(\mathscr{C})$ contains one-parameter family of three-secants [11, Appendix A]. But $\rho(\mathcal{Q})$ is a general plane in $\mathbb{P}^3$, which implies that $\rho(\mathcal{Q})$ does not contain three-secant lines of the curve $\rho(\mathscr{C})$ – a contradiction. □

Corollary 6.10. *Let γ be a class in the group $\mathrm{Pic}(T)$ such that $-K_T \cdot \gamma = 1$ and $\gamma^2 = -1$. Then the linear system $|\gamma|$ consists of a unique (-1)-curve.*

Proof. Apply the Riemann–Roch theorem, Serre duality and Lemma 6.9. □

Let us use Corollary 6.10 to describe infinitely many (-1)-curves in the surface T. Namely, let ℓ_1 and ℓ_2 be any curves in $\mathcal{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ of degrees $(1,0)$ and $(0,1)$, respectively. For $n \in \mathbb{Z}_{\geqslant 0}$ and $i \in \{2, 3, 4, 5, 6, 7, 8\}$, let $B_{n,1,1}, B_{n,1,2}, B_{n,2,i}, B_{n,3}, B_{n,4,i}$ be the classes

$$\varpi^*\bigl(a_1\ell_1 + a_2\ell_2\bigr) - \sum_{i=1}^{8} b_i \mathbf{e}_i \in \mathrm{Pic}(T),$$

where $a_1, a_2, b_1, \ldots, b_8$ are nonnegative integers given in the following table:

	a_1	a_2	b_1	$b_2, b_3, b_4, b_5, b_6, b_7, b_8$
$B_{n,1,1}$	$14n^2 + 7n + 1$	$14n^2 + 7n$	$7n^2 + 7n + 1$	$\forall j \; b_j = 7n^2 + 3n$
$B_{n,1,2}$	$14n^2 + 7n$	$14n^2 + 7n + 1$	$7n^2 + 7n + 1$	$\forall j \; b_j = 7n^2 + 3n$
$B_{n,2,i}$	$14n^2 + 13n + 3$	$14n^2 + 13n + 3$	$7n^2 + 10n + 3$	$b_i = 7n^2 + 6n + 2$ $\forall j \neq i \; b_j = 7n^2 + 6n + 1$
$B_{n,3}$	$14n^2 + 21n + 7$	$14n^2 + 21n + 7$	$7n^2 + 14n + 6$	$\forall j \; b_j = 7n^2 + 10n + 3$
$B_{n,4,i}$	$14n^2 + 29n + 15$	$14n^2 + 29n + 15$	$7n^2 + 18n + 11$	$b_i = 7n^2 + 14n + 6$ $\forall j \neq i \; b_j = 7n^2 + 14n + 7$

By Corollary 6.10, each linear system $|B_{n,1,1}|, |B_{n,1,2}|, |B_{n,2,i}|, |B_{n,3}|, |B_{n,4,i}|$ contains a unique (-1)-curve. Hence, we can identify the classes $B_{n,1,1}, B_{n,1,2}, B_{n,2,i}, B_{n,3}, B_{n,4,i}$ with (-1)-curves in $|B_{n,1,1}|$, $|B_{n,1,2}|, |B_{n,2,i}|, |B_{n,3}|, |B_{n,4,i}|$, respectively. Set

$$B_{n,1} = B_{n,1,1} + B_{n,1,2},$$
$$B_{n,2} = B_{n,2,2} + B_{n,2,3} + B_{n,2,4} + B_{n,2,5} + B_{n,2,6} + B_{n,2,7} + B_{n,2,8},$$
$$B_{n,4} = B_{n,4,2} + B_{n,4,3} + B_{n,4,4} + B_{n,4,5} + B_{n,4,6} + B_{n,4,7} + B_{n,4,8}.$$

Note that irreducible components of each curve $B_{n,1}, B_{n,2}, B_{n,4}$ are disjoint (-1)-curves, and $B_{n,1} \cap B_{n,2} = \varnothing$, $B_{n,2} \cap B_{n,3} = \varnothing$, $B_{n,3} \cap B_{n,4} = \varnothing$, $B_{n,4} \cap B_{n+1,1} = \varnothing$ for each $n \geqslant 0$.

Now, we let $I'_{0,1} = [0, \frac{1}{3}]$ and $I''_{0,1} = [\frac{1}{3}, \frac{3}{8}]$. For every $n \in \mathbb{Z}_{>0}$, we also let

$$I'_{n,1} = \left[\frac{-1 + 4n + 14n^2}{6n + 14n^2}, \frac{1 + 13n + 21n^2}{3 + 16n + 21n^2} \right],$$
$$I''_{n,1} = \left[\frac{1 + 13n + 21n^2}{3 + 16n + 21n^2}, \frac{3 + 35n + 49n^2}{8 + 42n + 49n^2} \right].$$

For every $n \in \mathbb{Z}_{\geqslant 0}$, we let

$$I'_{n,2} = \left[\frac{3 + 35n + 49n^2}{8 + 42n + 49n^2}, \frac{3 + 22n + 28n^2}{6 + 26n + 28n^2} \right],$$
$$I''_{n,2} = \left[\frac{3 + 22n + 28n^2}{6 + 26n + 28n^2}, \frac{2 + 7n}{3 + 7n} \right],$$
$$I'_{n,3} = \left[\frac{2 + 7n}{3 + 7n}, \frac{21 + 50n + 28n^2}{26 + 54n + 28n^2} \right],$$
$$I''_{n,3} = \left[\frac{21 + 50n + 28n^2}{26 + 54n + 28n^2}, \frac{39 + 91n + 49n^2}{48 + 98n + 49n^2} \right],$$
$$I'_{n,4} = \left[\frac{39 + 91n + 49n^2}{48 + 98n + 49n^2}, \frac{19 + 41n + 21n^2}{23 + 44n + 21n^2} \right],$$
$$I''_{n,4} = \left[\frac{19 + 41n + 21n^2}{23 + 44n + 21n^2}, \frac{17 + 32n + 14n^2}{20 + 34n + 14n^2} \right].$$

Set $I_{n,1} = I'_{n,1} \cup I''_{n,1}$, $I_{n,2} = I'_{n,2} \cup I''_{n,2}$, $I_{n,3} = I'_{n,3} \cup I''_{n,3}$, $I_{n,4} = I'_{n,4} \cup I''_{n,4}$. Then

$$[0, 1) = \bigcup_{n \in \mathbb{Z}_{\geqslant 0}} \left(I_{n,1} \cup I_{n,2} \cup I_{n,3} \cup I_{n,4} \right),$$

the intervals $I_{n,1}$, $I_{n,2}$, $I_{n,3}$, $I_{n,4}$ have positive volumes, and all their interiors are disjoint. Let us analyze $P(u,v)$ and $N(u,v)$ when u is contained in one of these intervals.

First, we deal with $u \in I_{n,1}$. If $u \in I'_{n,1}$ and $v \in \left[0, \frac{2+14n+28n^2-u(1+14n+28n^2)}{1+7n+7n^2}\right]$, then

$$
P(u,v) = \frac{19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)}{8+28n+21n^2}\mathbf{e}_1+
$$
$$
+\frac{3+35n+49n^2-u(8+42n+49n^2)}{8+28n+21n^2}B_{n,1} + \frac{1-4n-14n^2+u(6n+14n^2)}{8+28n+21n^2}B_{n,2}
$$

and $N(u,v) = 0$. The same holds if $u \in I''_{n,1}$ and $v \in \left[0, \frac{7+26n+28n^2-u(6+26n+28n^2)}{3+10n+7n^2}\right]$. Then

$$
(P(u,v))^2 = 10 - 12u + 2u^2 - 2v - v^2,
$$
$$
P(u,v) \cdot \mathbf{e}_1 = 1 + v.
$$

Similarly, if $u \in I'_{n,1}$ and $v \in \left[\frac{2+14n+28n^2-u(1+14n+28n^2)}{1+7n+7n^2}, \frac{7+26n+28n^2-u(6+26n+28n^2)}{3+10n+7n^2}\right]$, then

$$
P(u,v) = \frac{19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)}{8+28n+21n^2}\mathbf{e}_1+
$$
$$
+\frac{\left(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)\right)(1+7n+7n^2)}{8+28n+21n^2}B_{n,1}+
$$
$$
+\frac{1-4n-14n^2+u(6n+14n^2)}{8+28n+21n^2}B_{n,2}
$$

and

$$
N(u,v) = \left(u\left(1+14n+28n^2\right)+v\left(1+7n+7n^2\right)-2-14n-28n^2\right)B_{n,1}.
$$

Then

$$
(P(u,v))^2 = 10 - 12u + 2u^2 - 2v - v^2+
$$
$$
+2\left(u\left(1+14n+28n^2\right)+v\left(1+7n+7n^2\right)-2-14n-28n^2\right)^2
$$

and

$$
P(u,v) \cdot \mathbf{e}_1 = 5 + 56n + 280n^2 + 588n^3 + 392n^4-
$$
$$
-2u(1+7n+7n^2)(1+14n+28n^2) - v(1+28n+126n^2+196n^3+98n^4).
$$

Likewise, if $u \in I''_{n,1}$ and $v \in \left[\frac{7+26n+28n^2-u(6+26n+28n^2)}{3+10n+7n^2}, \frac{2+14n+28n^2-u(1+14n+28n^2)}{1+7n+7n^2}\right]$, then

$$
P(u,v) = \frac{\left(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)\right)}{8+28n+21n^2}\mathbf{e}_1+
$$
$$
+\frac{\left(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)\right)(1+n)(3+7n)}{8+28n+21n^2}B_{n,2}+
$$
$$
+\frac{3+35n+49n^2-u(8+42n+49n^2)}{8+28n+21n^2}B_{n,1}
$$

and

$$N(u, v) = \left(u\left(6 + 26n + 28n^2 \right) + v\left(3 + 10n + 7n^2 \right) - 7 - 26n - 28n^2 \right) B_{n,2}.$$

Then

$$\left(P(u, v) \right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$

$$+ 7\left(u\left(6 + 26n + 28n^2 \right) + v\left(3 + 10n + 7n^2 \right) - 7 - 26n - 28n^2 \right)^2$$

and

$$P(u, v) \cdot \mathbf{e}_1 = 148 + 1036n + 2751n^2 + 3234n^3 + 1372n^4 -$$

$$- 14u(1 + n)(1 + 2n)(3 + 7n)^2 - v\left(62 + 420n + 994n^2 + 980n^3 + 343n^4 \right).$$

If $u \in I'_{n,1}$ and $v \in \left[\frac{7 + 26n + 28n^2 - u\left(6 + 26n + 28n^2 \right)}{3 + 10n + 7n^2}, \frac{19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right)}{8 + 28n + 21n^2} \right]$, then

$$P(u, v) = \frac{19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right) - v\left(8 + 28n + 21n^2 \right)}{8 + 28n + 21n^2} \mathbf{e}_1 +$$

$$+ \frac{\left(19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right) - v\left(8 + 28n + 21n^2 \right) \right)\left(1 + 7n + 7n^2 \right)}{8 + 28n + 21n^2} B_{n,1} +$$

$$+ \frac{\left(19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right) - v\left(8 + 28n + 21n^2 \right) \right)(1 + n)(3 + 7n)}{8 + 28n + 21n^2} B_{n,2}$$

and

$$N(u, v) = \left(u\left(1 + 14n + 28n^2 \right) + v\left(1 + 7n + 7n^2 \right) - 2 - 14n - 28n^2 \right) B_{n,1} +$$

$$+ \left(u\left(6 + 26n + 28n^2 \right) + v\left(3 + 10n + 7n^2 \right) - 7 - 26n - 28n^2 \right) B_{n,2}.$$

The same holds if $u \in I''_{n,1}$ and $v \in \left[\frac{2 + 14n + 28n^2 - u\left(1 + 14n + 28n^2 \right)}{1 + 7n + 7n^2}, \frac{19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right)}{8 + 28n + 21n^2} \right]$. In both cases, we have

$$\left(P(u, v) \right)^2 = \left(19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right) - v\left(8 + 28n + 21n^2 \right) \right)^2,$$

$$P(u, v) \cdot \mathbf{e}_1 = \left(8 + 28n + 21n^2 \right)\left(19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right) - v\left(8 + 28n + 21n^2 \right) \right).$$

Hence, if $u \in I_{n,1}$, then

$$t(u) = \frac{19 + 70n + 84n^2 - u\left(16 + 70n + 84n^2 \right)}{8 + 28n + 21n^2}.$$

Now, we deal with $u \in I_{n,2}$. If $u \in I'_{n,2}$ and $v \in \left[0, \frac{7 + 26n + 28n^2 - u\left(6 + 26n + 28n^2 \right)}{3 + 10n + 7n^2} \right]$, then

$$P(u, v) = \frac{17 + 56n + 56n^2 - u\left(15 + 56n + 56n^2 \right) - 7v(1 + n)(1 + 2n)}{7(1 + n)(1 + 2n)} \mathbf{e}_1 +$$

$$+ \frac{(1 + n)(2 + 7n) - u(1 + n)(3 + 7n)}{7(1 + n)(1 + 2n)} B_{n,2} + \frac{u\left(8 + 42n + 49n^2 \right) - 3 - 35n - 49n^2}{7(1 + n)(1 + 2n)} B_{n,3}$$

and $N(u, v) = 0$. The same holds if $u \in I''_{n,2}$ and $v \in \left[0, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$. Then

$$\left(P(u, v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2,$$
$$P(u, v) \cdot \mathbf{e}_1 = v + 1.$$

If $u \in I'_{n,2}$ and $v \in \left[\frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$, then

$$P(u, v) = \frac{17 + 56n + 56n^2 - u(15 + 56n + 56n^2) - 7v(1 + n)(1 + 2n)}{7(1 + n)(1 + 2n)}\mathbf{e}_1 +$$
$$+ \frac{(1 + n)(3 + 7n)\left(17 + 56n + 56n^2 - u(15 + 56n + 56n^2) - 7v(1 + n)(1 + 2n)\right)}{7(1 + n)(1 + 2n)}B_{n,2} +$$
$$+ \frac{u\left(8 + 42n + 49n^2\right) - 3 - 35n - 49n^2}{7(1 + n)(1 + 2n)}B_{n,3},$$

$$N(u, v) = \left(u\left(6 + 26n + 28n^2\right) + v\left(3 + 10n + 7n^2\right) - 7 - 26n - 28n^2\right)B_{n,2},$$

$$\left(P(u, v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$
$$+ 7\left(u\left(6 + 26n + 28n^2\right) + v\left(3 + 10n + 7n^2\right) - 7 - 26n - 28n^2\right)^2,$$

$$P(u, v) \cdot \mathbf{e}_1 = 148 + 1036n + 2751n^2 + 3234n^3 + 1372n^4 -$$
$$- 14u(1 + n)(1 + 2n)(3 + 7n)^2 -$$
$$- v\left(62 + 420n + 994n^2 + 980n^3 + 343n^4\right).$$

Similarly, if $u \in I''_{n,2}$ and $v \in \left[\frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{7+26n+28n^3-u\left(6+26n+28n^2\right)}{3+10n+7n^2}\right]$, then

$$P(u, v) = \frac{17 + 56 + 56n^2 - u\left(15 + 56n + 56n^2\right) - 7v(1 + n)(1 + 2n)}{7(1 + n)(1 + 2n)}\mathbf{e}_1 +$$
$$+ \frac{(6 + 14n + 7n^2)\left(17 + 56 + 56n^2 - u\left(15 + 56n + 56n^2\right) - 7v(1 + n)(1 + 2n)\right)}{7(1 + n)(1 + 2n)}B_{n,3} +$$
$$+ \frac{(1 + n)(2 + 7n) - u(1 + n)(3 + 7n)}{7(1 + n)(1 + 2n)}B_{n,2},$$

$$N(u, v) = \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right)B_{n,3},$$

$$\left(P(u, v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$
$$+ \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right)^2,$$

$$P(u, v) \cdot \mathbf{e}_1 = 7(1 + n)(13 + 53n + 70n^2 + 28n^3) -$$
$$- 14u(1 + n)(1 + 2n)\left(6 + 14n + 7n^2\right) - 7v(1 + n)^2\left(5 + 14n + 7n^2\right).$$

Likewise, if $u \in I'_{n,2}$ and $v \in \left[\frac{15+42n+28n^2-u(14+42n+28n^2)}{6+14n+7n^2}, \frac{17+56n+56n^2-u(15+56n+56n^2)}{7(1+n)(1+2n)} \right]$, then

$$
\begin{aligned}
P(u,v) = &\frac{17 + 56n + 56n^2 - u(15 + 56n + 56n^2) - v(7(1+n)(1+2n))}{7(1+n)(1+2n)} \mathbf{e}_1 + \\
&+ \frac{(1+n)(3+7n)(17 + 56n + 56n^2 - u(15 + 56n + 56n^2) - 7v(1+n)(1+2n))}{7(1+n)(1+2n)} B_{n,2} + \\
&+ \frac{(6 + 14n + 7n^2)(17 + 56n + 56n^2 - u(15 + 56n + 56n^2) - 7v(1+n)(1+2n))}{7(1+n)(1+2n)} B_{n,3}
\end{aligned}
$$

and

$$
\begin{aligned}
N(u,v) = &\left(u(6 + 26n + 28n^2) + v(3 + 10n + 7n^2) - 7 - 26n - 28n^2\right) B_{n,2} + \\
&+ \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right) B_{n,3}.
\end{aligned}
$$

The same holds if $u \in I''_{n,2}$ and $v \in \left[\frac{7+26n+28n^2-u(6+26n+28n^2)}{3+10n+7n^2}, \frac{17+56n+56n^2-u(15+56n+56n^2)}{7(1+n)(1+2n)} \right]$. Moreover, in both cases, we have

$$
P(u,v) \cdot \mathbf{e}_1 = 14(1+n)(1+2n)\left(17 + 56n + 56n^2 - u\left(15 + 56n + 56n^2\right) - 7v(1+n)(1+2n)\right)
$$

and

$$
\left(P(u,v)\right)^2 = 2\left(17 + 56n + 56n^2 - u\left(15 + 56n + 56n^2\right) - 7v(1+n)(1+2n)\right)^2.
$$

Thus, if $u \in I_{n,2}$, then

$$
t(u) = \frac{17 + 56n + 56n^2 - u\left(15 + 56n + 56n^2\right)}{7(1+n)(1+2n)}.
$$

Now, we deal with $u \in I_{n,3}$. If $u \in I'_{n,3}$ and $v \in \left[0, \frac{15+42n+28n^2-u(14+42n+28n^2)}{6+14n+7n^2} \right]$, then

$$
\begin{aligned}
P(u,v) = &\frac{59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n)}{7(1+n)(3+2n)} \mathbf{e}_1 + \\
&+ \frac{39 + 91n + 49n^2 - u(48 + 98n + 49n^2)}{7(1+n)(3+2n)} B_{n,3} + \frac{u(1+n)(3+7n) - (1+n)(2+7n)}{7(1+n)(3+2n)} B_{n,4}
\end{aligned}
$$

and $N(u,v) = 0$. The same holds if $u \in I''_{n,3}$ and $v \in \left[0, \frac{31+58n+28n^2-u(30+58n+28n^2)}{11+18n+7n^2} \right]$. Then

$$
\left(P(u,v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2,
$$
$$
P(u,v) \cdot \mathbf{e}_1 = 1 + v.
$$

If $u \in I'_{n,3}$ and $v \in \left[\frac{15+42n+28n^2 - u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{31+58n+28n^2 - u\left(30+58n+28n^2\right)}{11+18n+7n^2} \right]$, then

$$P(u,v) = \frac{59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n)}{7(1+n)(3+2n)} \mathbf{e}_1 +$$

$$+ \frac{(6 + 14n + 7n^2)(59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n))}{7(1+n)(3+2n)} B_{n,3} +$$

$$+ \frac{u(1+n)(3+7n) - (1+n)(2+7n)}{7(1+n)(3+2n)} B_{n,4}$$

$$N(u,v) = \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right) B_{n,3},$$

$$\left(P(u,v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$

$$+ \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right)^2,$$

$$P(u,v) \cdot \mathbf{e}_1 = 7(1+n)\left(13 + 53n + 70n^2 + 28n^3\right) -$$

$$- 14u(1+n)(1+2n)\left(6 + 14n + 7n^2\right) - 7v(1+n)^2\left(5 + 14n + 7n^2\right).$$

Similarly, if $u \in I''_{n,3}$ and $v \in \left[\frac{31+58n+28n^2 - u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{15+42n+28n^2 - u\left(14+42n+28n^2\right)}{6+14n+7n^2} \right]$, then

$$P(u,v) = \frac{59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n)}{7(1+n)(3+2n)} \mathbf{e}_1 +$$

$$+ \frac{(1+n)(11+7n)(59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n))}{7(1+n)(3+2n)} B_{n,4} +$$

$$+ \frac{39 + 91n + 49n^2 - u\left(48 + 98n + 49n^2\right)}{7(1+n)(3+2n)} B_{n,3},$$

$$N(u,v) = \left(u\left(30 + 58n + 28n^2\right) + v\left(11 + 18n + 7n^2\right) - 31 - 58n - 28n^2\right) B_{n,4},$$

$$\left(P(u,v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$

$$+ 7\left(u\left(30 + 58n + 28n^2\right) + v\left(11 + 18n + 7n^2\right) - 31 - 58n - 28n^2\right)^2,$$

$$P(u,v) \cdot \mathbf{e}_1 = 2388 + 8372n + 10983n^2 + 6370n^3 + 1372n^4 -$$

$$- 14u(1+n)^2(11+7n)(15+14n) - v\left(846 + 2772n + 3346n^2 + 1764n^3 + 343n^4\right).$$

If $u \in I'_{n,3}$ and $v \in \left[\frac{31+58n+28n^2 - u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{59+112n+56n^2 - u\left(57+112n+56n^2\right)}{7(1+n)(3+2n)} \right]$, then

$$P(u,v) = \frac{59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n)}{7(1+n)(3+2n)} \mathbf{e}_1 +$$

$$+ \frac{(6 + 14n + 7n^2)(59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n))}{7(1+n)(3+2n)} B_{n,3} +$$

$$+ \frac{(1+n)(11+7n)(59 + 112n + 56n^2 - u(57 + 112n + 56n^2) - 7v(1+n)(3+2n))}{7(1+n)(3+2n)} B_{n,4}$$

and

$$N(u,v) = \left(u\left(14 + 42n + 28n^2\right) + v\left(6 + 14n + 7n^2\right) - 15 - 42n - 28n^2\right)B_{n,3} +$$
$$+ \left(u\left(30 + 58n + 28n^2\right) + v\left(11 + 18n + 7n^2\right) - 31 - 58n - 28n^2\right)B_{n,4}.$$

The same holds if $u \in I''_{u,3}$ and $v \in \left[\frac{15 + 42n + 28n^2 - u\left(14 + 42n + 28n^2\right)}{6 + 14n + 7n^2}, \frac{59 + 112n + 56n^2 - u\left(57 + 112n + 56n^2\right)}{7(1+n)(3+2n)}\right]$. In both cases, we have

$$P(u,v) \cdot \mathbf{e}_1 = 14(1+n)(3+2n)\left(59 + 112n + 56n^2 - u\left(57 + 112n + 56n^2\right) - 7v(1+n)(3+2n)\right)$$

and

$$\left(P(u,v)\right)^2 = 2\left(59 + 112n + 56n^2 - u\left(57 + 112n + 56n^2\right) - 7v(1+n)(3+2n)\right)^2.$$

Therefore, if $u \in I_{n,3}$, then

$$t(u) = \frac{59 + 112n + 56n^2 - u\left(57 + 112n + 56n^2\right)}{7(1+n)(3+2n)}.$$

Finally, we deal with $u \in I_{n,4}$. If $u \in I'_{n,4}$ and $v \in \left[0, \frac{31 + 58n + 28n^2 - u\left(30 + 58n + 28n^2\right)}{11 + 18n + 7n^2}\right]$, then

$$P(u,v) = \frac{103 + 182n + 84n^2 - u\left(100 + 182n + 84n^2\right) - v\left(36 + 56n + 21n^2\right)}{36 + 56n + 21n^2}\mathbf{e}_1 +$$
$$+ \frac{17 + 32n + 14n^2 - u\left(20 + 34n + 14n^2\right)}{36 + 56n + 21n^2}B_{n,4} + \frac{u\left(48 + 98n + 49n^2\right) - 39 - 91n - 49n^2}{36 + 56n + 21n^2}B_{n+1,1}$$

and $N(u,v) = 0$. The same holds when $u \in I''_{n,4}$ and $v \in \left[0, \frac{44 + 70n + 28n^2 - u\left(43 + 70n + 28n^2\right)}{15 + 21n + 7n^2}\right]$. In both cases, we compute

$$\left(P(u,v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2,$$
$$P(u,v) \cdot \mathbf{e}_1 = 1 + v.$$

If $u \in I'_{n,4}$ and $v \in \left[\frac{31 + 58n + 28n^2 - u\left(30 + 58n + 28n^2\right)}{11 + 18n + 7n^2}, \frac{44 + 70n + 28n^2 - u\left(43 + 70n + 28n^2\right)}{15 + 21n + 7n^2}\right]$, then

$$P(u,v) = \frac{103 + 182n + 84n^2 - u\left(100 + 182n + 84n^2\right) - v\left(36 + 56n + 21n^2\right)}{36 + 56n + 21n^2}\mathbf{e}_1 +$$
$$+ \frac{(1+n)(11+7n)\left(103 + 182n + 84n^2 - u\left(100 + 182n + 84n^2\right) - v\left(36 + 56n + 21n^2\right)\right)}{36 + 56n + 21n^2}B_{n,4} +$$
$$+ \frac{u\left(48 + 98n + 49n^2\right) - 39 - 91n - 49n^2}{36 + 56n + 21n^2}B_{n+1,1},$$

$$N(u,v) = \left(u\left(30 + 58n + 28n^2\right) + v\left(11 + 18n + 7n^2\right) - 31 - 58n - 28n^2\right)B_{n,4},$$

$$\left(P(u,v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$
$$+ 7\left(u\left(30 + 58n + 28n^2\right) + v\left(11 + 18n + 7n^2\right) - 31 - 58n - 28n^2\right)^2,$$

$$P(u, v) \cdot \mathbf{e}_1 = 2388 + 8372n + 10983n^2 + 6370n^2 + 1372n^4 -$$
$$- 14u(1+n)^2(11+7n)(15+14n) -$$
$$- v\left(846 + 2772n + 3346n^2 + 1764n^3 + 343n^4\right).$$

If $u \in I''_{n,4}$ and $v \in \left[\frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2}, \frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}\right]$, then

$$P(u, v) = \frac{103 + 182n + 84n^2 - u(100+182n+84n^2) - v(36+56n+21n^2)}{36+56n+21n^2}\mathbf{e}_1 +$$
$$+ \frac{(15+21n+7n^2)\left(103+182n+84n^2 - u(100+182n+84n^2) - v(36+56n+21n^2)\right)}{36+56n+21n^2}B_{n+1,1} +$$
$$+ \frac{17+32n+14n^2 - u\left(20+34n+14n^2\right)}{36+56n+21n^2}B_{n,4}$$

and

$$N(u, v) = \left(u\left(43+70n+28n^2\right) + v\left(15+21n+7n^2\right) - 44 - 70n - 28n^2\right)B_{n+1,1}.$$

Moreover, in this case, we have

$$\left(P(u, v)\right)^2 = 10 - 12u + 2u^2 - 2v - v^2 +$$
$$+ 2\left(u\left(43+70n+28n^2\right) + v\left(15+21n+7n^2\right) - 44 - 70n - 28n^2\right)^2$$

and

$$P(u, v) \cdot \mathbf{e}_1 = 1321 + 3948n + 4396n^2 + 2156n^3 + 392n^4 -$$
$$- 2u\left(15+21n+7n^2\right)\left(43+70n+28n^2\right) -$$
$$- v\left(449 + 1260n + 1302n^2 + 588n^3 + 98n^4\right).$$

If $u \in I'_{n,4}$ and $v \in \left[\frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2}, \frac{103+182n+84n^2-u\left(100+182n+84n^2\right)}{36+56n+21n^2}\right]$, then

$$P(u, v) = \frac{103 + 182n + 84n^2 - u(100+182n+84n^2) - v(36+56n+21n^2)}{36+56n+21n^2}\mathbf{e}_1 +$$
$$+ \frac{(1+n)(11+7n)\left(103+182n+84n^2 - u(100+182n+84n^2) - v(36+56n+21n^2)\right)}{36+56n+21n^2}B_{n,4} +$$
$$+ \frac{(15+21n+7n^2)\left(103+182n+84n^2 - u(100+182n+84n^2) - v(36+56n+21n^2)\right)}{36+56n+21n^2}B_{n+1,1}$$

and

$$N(u, v) = \left(u(30+58n+28n^2) + v(11+18n+7n^2) - 31 - 58n - 28n^2\right)B_{n,4} +$$
$$+ \left(u(43+70n+28n^2) + v(15+21n+7n^2) - 44 - 70n - 28n^2\right)B_{n+1,1}.$$

The same holds when $u \in I''_{n,4}$ and $v \in \left[\frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{103+182n+84n^2-u\left(100+182n+84n^2\right)}{36+56n+21n^2}\right]$.
Moreover, in both cases, we have

$$P(u, v) \cdot \mathbf{e}_1 = (36 + 56n + 21n^2)(103 + 182n + 84n^2 - u(100 + 182n + 84n^2) - v(36 + 56n + 21n^2))$$

and

$$(P(u, v))^2 = (103 + 182n + 84n^2 - u(100 + 182n + 84n^2) - v(36 + 56n + 21n^2))^2$$

Thus, if $u \in I_{n,4}$, then

$$t(u) = \frac{103 + 182n + 84n^2 - u(100 + 182n + 84n^2)}{36 + 56n + 21n^2}.$$

Now, we are ready to compute $S(W_{\bullet,\bullet}^T; \mathbf{e}_1)$. Namely, for every $i \in \{1, 2, 3, 4\}$, we set

$$S_{n,i} = \frac{3}{14} \int_{I_{n,i}} \int_0^{t(u)} (P(u, v))^2 \, dv \, du.$$

Then

$$S(W_{\bullet,\bullet}^T; \mathbf{e}_1) = \sum_{n=0}^{\infty} \left(S_{n,1} + S_{n,2} + S_{n,3} + S_{n,4} \right).$$

On the other hand, integrating, we get

$$S_{n,1} = \begin{cases} \dfrac{84365}{114688} & \text{if } n = 0, \\[2ex] \dfrac{(8 + 28n + 21n^2)A_{n,1}}{448n^4(1 + n)(2 + 7n)^4(3 + 7n)^4(4 + 7n)^4(1 + 7n + 7n^2)} & \text{if } n \geqslant 1, \end{cases}$$

where

$$\begin{aligned} A_{n,1} = {} & 1536 + 109312n + 2935552n^2 + 42681728n^3 + 386407488n^4 + 2335296292n^5 + \\ & + 9789648099n^6 + 29038364761n^7 + 61312905318n^8 + 91454579804n^9 + \\ & + 94035837280n^{10} + 63317750608n^{11} + 25088413952n^{12} + 4427367168n^{13}. \end{aligned}$$

Similarly, we get

$$S_{n,2} = \frac{(1 + 2n)A_{n,2}}{4(1 + n)(2 + 7n)^4(3 + 7n)^4(4 + 7n)^4(6 + 14n + 7n^2)},$$

where

$$\begin{aligned} A_{n,2} = {} & 1618654 + 31459234n + 271069253n^2 + 1362423916n^3 + 4419070194n^4 + 9654348284n^5 + \\ & + 14368501182n^6 + 14362052096n^7 + 9209328422n^8 + 3412762192n^9 + 553420896n^{10}. \end{aligned}$$

Likewise, we get

$$S_{n,3} = \frac{(3 + 2n)A_{n,3}}{4(1 + n)(3 + 7n)^4(6 + 7n)^4(8 + 7n)^4(11 + 7n)(6 + 14n + 7n^2)},$$

where

$$A_{n,3} = 1167997914 + 15454923336n + 91492878645n^2 + 319934133575n^3 +$$
$$+ 734395997090n^4 + 1162203105378n^5 + 1294197714054n^6 + 1014406754242n^7 +$$
$$+ 548632346402n^8 + 195059453722n^9 + 41045383120n^{10} + 3873946272n^{11}.$$

Finally, we get

$$S_{n,4} = \frac{(36 + 56n + 21n^2)A_{n,4}}{448(1 + n)^4(6 + 7n)^4(8 + 7n)^4(10 + 7n)^4(11 + 7n)(15 + 21n + 7n^2)},$$

where

$$A_{n,4} = 365613573312 + 4021500121920n + 20341847967024n^2 +$$
$$+ 62650071283024n^3 + 131072047236004n^4 + 196698030664492n^5 + 217823761840153n^6 +$$
$$+ 180219167765455n^7 + 111395400841326n^8 + 50802960251820n^9 + 16615457209344n^{10} +$$
$$+ 3690223711216n^{11} + 498816700928n^{12} + 30991570176n^{13}.$$

Then, adding, we get

$$S\big(W_{\bullet,\bullet}^T; \mathbf{e}_1\big) = \sum_{n=0}^{\infty} \Big(S_{n,1} + S_{n,2} + S_{n,3} + S_{n,4}\Big) \approx 0.976712233\ldots < 1.$$

Finally, let us compute $S(W_{\bullet,\bullet,\bullet}^{T;\mathbf{e}_1}; P)$. For every $i \in \{1, 2, 3, 4\}$, we set

$$M'_{n,i} = \frac{3}{14} \int_{I'_{n,i}} \int_0^{t(u)} \Big((P(u, v) \cdot \mathbf{e}_1)\Big)^2 dv du,$$
$$M''_{n,i} = \frac{3}{14} \int_{I''_{n,i}} \int_0^{t(u)} \Big((P(u, v) \cdot \mathbf{e}_1)\Big)^2 dv du.$$

Then

$$S\big(W_{\bullet,\bullet,\bullet}^{T;\mathbf{e}_1}; P\big) = \sum_{n=0}^{\infty} \sum_{i=1}^{4} \Big(M'_{n,i} + M''_{n,i}\Big) + \frac{3}{7} \int_0^1 \int_0^{t(u)} \big(P(u, v) \cdot \mathbf{e}_1\big) \mathrm{ord}_P\Big(N(u, v)\big|_{\mathbf{e}_1}\Big) dv du.$$

On the other hand, integrating, we get

$$M'_{n,1} = \begin{cases} \dfrac{1403}{22268} & \text{if } n = 0, \\[3mm] \dfrac{(1 + n)A'_{n,1}}{448n^4(1 + 3n)^4(3 + 7n)^4(1 + 7n + 7n^2)} & \text{if } n \geqslant 1, \end{cases}$$

where

$$A'_{n,1} = 1 + 81n + 2535n^2 + 37209n^3 + 301046n^4 + 1459736n^5 +$$
$$+ 4420190n^6 + 8425410n^7 + 9821448n^8 + 6392736n^9 + 1778112n^{10}.$$

Similarly, we get

$$M''_{n,1} = \frac{(1 + 7n + 7n^2)A''_{n,1}}{28(1 + n)(1 + 3n)^4(2 + 7n)^4(3 + 7n)^4(4 + 7n)^4},$$

where

$$\begin{aligned}
A''_{n,1} = {} & 480574 + 12906866n + 157271760n^2 + 1149521334n^3 + 5612285145n^4 + \\
& + 19278934535n^5 + 47770884833n^6 + 86016481159n^7 + 111679016743n^8 + \\
& + 101939513907n^9 + 62077730148n^{10} + 22635902898n^{11} + 3735591048n^{12}.
\end{aligned}$$

Likewise, we have

$$M'_{n,2} = \frac{(6 + 14n + 7n^2)A'_{n,2}}{224(1 + n)(1 + 2n)^3(2 + 7n)^4(3 + 7n)^4(4 + 7n)^4},$$

and

$$M''_{n,2} = \frac{11780 + 111142n + 430951n^2 + 875637n^3 + 978656n^4 + 566832n^5 + 131712n^6}{224(1 + 2n)^3(3 + 7n)^4(6 + 14n + 7n^2)},$$

where

$$\begin{aligned}
A'_{n,2} = {} & 1561176 + 35176776n + 356105548n^2 + 2137950448n^3 + \\
& + 8458603286n^4 + 23158717414n^5 + 44778314889n^6 + 61151030584n^7 + \\
& + 57807289939n^8 + 36026947376n^9 + 13321631568n^{10} + 2213683584n^{11}.
\end{aligned}$$

Similarly, we have

$$M'_{n,3} = \frac{(11 + 7n)A'_{n,3}}{224(1 + n)^3(3 + 7n)^4(13 + 14n)^4(6 + 14n + 7n^2)},$$

where

$$\begin{aligned}
A'_{n,3} = {} & 13726028 + 164541190n + 859036123n^2 + 2564002455n^3 + 4823323519n^4 + \\
& + 5933644367n^5 + 4776917782n^6 + 2428774768n^7 + 708314208n^8 + 90354432n^9.
\end{aligned}$$

Likewise, we have $M''_{n,3} = \frac{(6+14n+7n^2)A''_{n,3}}{224(1+n)^3(6+7n)^4(8+7n)^4(11+7n)(13+14n)^4}$, where

$$\begin{aligned}
A''_{n,3} = {} & 67760261208 + 703706084640n + 3313300067388n^2 + 9335574166156n^3 + \\
& + 17489294547578n^4 + 22873117200584n^5 + 21308562209725n^6 + 14139587568253n^7 + \\
& + 6548997703738n^8 + 2016283621072n^9 + 371345421216n^{10} + 30991570176n^{11}.
\end{aligned}$$

Similarly, we see that

$$M'_{n,4} = \frac{(15 + 21n + 7n^2)A'_{n,4}}{28(1 + n)^4(6 + 7n)^4(8 + 7n)^4(11 + 7n)(23 + 21n)^4},$$

where

$$A'_{n,4} = 88135013250 + 967134809574n + 4853884596732n^2 +$$
$$+ 14732868828434n^3 + 30120687035243n^4 + 43697011451345n^5 + 46124583653603n^6 +$$
$$+ 35692827118809n^7 + 20096052100397n^8 + 8028312817917n^9 +$$
$$+ 2160120347280n^{10} + 351456857766n^{11} + 26149137336n^{12}.$$

Finally, we have

$$M''_{n,4} = \frac{(11 + 7n)A''_{n,4}}{448(1 + n)^4(10 + 7n)^4(23 + 21n)^4(15 + 21n + 7n^2)},$$

where

$$A''_{n,4} = 7582266167 + 59702225967n + 210973884925n^2 + 440580768679n^3 +$$
$$+ 602090743422n^4 + 562572998512n^5 + 363945674554n^6 + 160955181870n^7 +$$
$$+ 46566357768n^8 + 7957643904n^9 + 609892416n^{10}.$$

Now, adding terms together, we see that

$$S\big(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet}; P\big) \leqslant 0.974 + \frac{3}{7} \int_0^1 \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big) dv\,du. \tag{6.8}$$

Now, for every $i \in \{1, 2, 3, 4\}$ and any irreducible component ℓ of the curve $B_{n,i}$, we let

$$F_{n,i} = \frac{3}{7} \int_0^1 \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell \cdot \mathbf{e}_1\big) dv\,du =$$
$$= \frac{3}{7} \int_{I_{n,i-1}} \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell \cdot \mathbf{e}_1\big) dv\,du +$$
$$+ \frac{3}{7} \int_{I_{n,i}} \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell \cdot \mathbf{e}_1\big) dv\,du +$$
$$+ \frac{3}{7} \int_{I_{n,i+1}} \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell \cdot \mathbf{e}_1\big) dv\,du,$$

where we set $I_{0,0} = \varnothing$, $I_{n,5} = I_{n+1,1}$ for $n \geqslant 0$ and $I_{n,0} = I_{n-1,4}$ for $n \geqslant 1$. Since irreducible components of the curve $B_{n,i}$ are disjoint, we get

$$\frac{3}{7} \int_0^1 \int_0^{t(u)} \big(P(u,v) \cdot \mathbf{e}_1\big) \mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big) dv\,du \leqslant \sum_{n=0}^{\infty} \sum_{i=1}^{4} F_{n,i}.$$

On the other hand, each $F_{n,i}$ is not difficult to compute. For instance, we have

$$F_{0,1} = \frac{3}{7} \int_0^{\frac{1}{3}} \int_{2-u}^{\frac{7-6u}{3}} (v + u - 2)(5 - 2u - v)dvdu+$$

$$+ \frac{3}{7} \int_0^{\frac{1}{3}} \int_{\frac{7-6u}{3}}^{\frac{19-16u}{8}} 8(u + v - 2)(19 - 16u - 8v)dvdu+$$

$$+ \frac{3}{7} \int_{\frac{1}{3}}^{\frac{3}{8}} \int_{2-u}^{\frac{19-16u}{8}} 8(u + v - 2)(19 - 16u - 8v)dvdu = \frac{281}{32256}.$$

Similarly, we see that

$$F_{n,1} = \frac{3(1 + 7n + 7n^2)^2}{2n^2(1 + 3n)(-1 + 7n)(1 + 7n)(2 + 7n)(3 + 7n)^2(4 + 7n)(2 + 21n)}$$

for $n \geqslant 1$. Likewise, we get

$$F_{n,2} = \begin{cases} \dfrac{5}{3584} & \text{if } n = 0, \\ \dfrac{1+n}{112n(1 + 2n)(1 + 3n)(2 + 7n)(3 + 7n)^2(4 + 7n)} & \text{if } n \geqslant 1. \end{cases}$$

Likewise, for every $n \geqslant 0$, we have

$$F_{n,3} = \frac{15(6 + 14n + 7n^2)^2}{4(1 + n)^2(1 + 2n)(2 + 7n)(3 + 7n)^2(4 + 7n)(6 + 7n)(8 + 7n)(13 + 14n)}$$

and

$$F_{n,4} = \frac{(11 + 7n)^2}{112(1 + n)^2(3 + 7n)(6 + 7n)(8 + 7n)(10 + 7n)(13 + 14n)(23 + 21n)}.$$

Now, one can easily check that the total sum of all $F_{n,1}$, $F_{n,2}$, $F_{n,3}$, $F_{n,4}$ is at most 0.014. This and Equation (6.8) give $S(W_{\bullet,\bullet,\bullet}^{T;\mathbf{e}_1}; P) \leqslant 0.974 + 0.014 = 0.988$. Using Equation (6.7), we get $\delta_P(X) > 1$.

Corollary 6.11. *All smooth Fano threefolds in the family №2.7 are K-stable.*

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