AN EQUATION FOR THE DEGREES OF THE ABSOLUTELY IRREDUCIBLE REPRESENTATIONS OF A GROUP OF FINITE ORDER

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IF there is a nonsingular symmetric bilinear form[1] $f(\sum a^i C_i, \sum b^k C_k) = \sum\sum c_{ik} a^i b^k$ defined on a distributive algebra A with basis elements $C_1, C_2, \ldots, C_r$ over a field F such that $c_{ik} = c_{ki}$ $(i, k = 1, 2, \ldots r)$ and $(c_{ik})^{-1} = (c^{ik})$, then the so called *Casimir operator* [1,2]

$$C = \sum C_l C^l = \sum C^l C_l$$

is independent of the choice of the basis of A over F. Here C^i is defined as usually in tensor calculus by the formula $C^i = \sum c^{ik} C_k$. What can we say concerning C, if F is the rational number field, A the class ring of a group G of order N which is embedded in the group-ring S of G over F, r the number of classes of conjugate elements of G, C_i the sum of the elements of the ith class, C_1 the unity class and finally, with $f(X, Y)$ equal to $\mathrm{tr}(R(XY))/N^2$ where R denotes the regular representation of S?

In the multiplication table $C_l C_k = \sum c_{lk}{}^i C_i$ the non-negative integer $c_{lk}{}^i$ is equal to the number of representations of an arbitrary chosen element of the ith class as the product of an element of the lth class and an element of the kth class; hence it can be easily derived from the multiplication table of G. In particular, $c_{lk}{}^1 = \delta_{lk'} h_l$ where the k'th class is the inverse of the kth class and h_l is the number of elements in the lth class. Taking the elements of G as a basis of S in order to compute $R(C_l)$ we obtain the integral matrices $R(C_l) = (x_{l,A}{}^B)$ where the row index A and the column index B runs over G and

$$x_{l,A}{}^B = \sum_{X \epsilon C_l} \delta_{A,XB} = \begin{cases} 1, \text{ if } AB^{-1} \in C_l; \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

Denote the absolutely irreducible characters of G by $\chi^1, \chi^2, \ldots, \chi^r$ such that $\chi^1 = 1$, and denote the value of the kth character of any element of the ith class by $\chi_i{}^k$ so that $\chi_1{}^k = f^k$ is the degree of the kth irreducible representation of G. Since C_i is represented by a similarity transformation in any irreducible representation of G, we derive from these representations the r representations $D^k(\sum a^i C_i) = (\sum a^i h_i \chi_i{}^k / f^k)$ of degree 1 of A leading to the complete reduction $R_A \sim \sum_{m=1}^{r} (f^m)^2 D^m$ of the representation R_A of A induced by R, as is well known in the theory of representations. By application of the simple rule $\mathrm{tr}R(C_l) = N\delta_{1,l}$ which follows from (1) we obtain

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[1]The symbol of summation without index of summation means summation over each suffix occurring as upper index as well as lower index.

$$c_{ik} = \mathrm{tr}R(C_iC_k)/N^2 = \sum c_{ik}{}^j \mathrm{tr}C_j/N^2 = c_{ik}{}^1 N/N^2 = h_i\delta_{ik'}/N.$$

Since h_i is a divisor of N it follows that

$$c^{ik} = \delta_{ik'}N/h_i, \tag{2}$$

$$x^l{}_A{}^B = \sum c^{jl} x_{l,A}{}^B \tag{3}$$

are integers. Also, the coefficients of the matrix

$$R(C) = \sum R(C_l)R(C^l) = \sum (x_{l,A}{}^B)(x^l{}_A{}^B) = \sum\sum (x_{l,A}{}^T x^l{}_A{}^B), \tag{4}$$

and the coefficients of the characteristic polynomial of the matrix $R(C)$, i.e. det $(tE_N - R(C))$, are all integers. On the other hand it follows from

$$R_A \sim \sum_{m=1}^{r} (f^m)^2 D^m$$

and

$$D^m(C) = D^m(\sum C_lC^l) = D^m(\sum C_lc^{lj}C_j) = \sum_{l=1}^{r} N/h_l \,.\, D^m(C_l)D^m(C_{l'})$$

$$= (\sum_{l=1}^{r} N/h_l \,.\, h_l\chi_l{}^m/f^m \,.\, h_{l'}\chi_{l'}{}^m/f^m)$$

$$= N/(f^m)^2 \,.\, \sum_{l=1}^{r} h_l\chi_{l'}{}^m\chi_{l'}{}^m = (N/f^m)^2$$

that the matrix $R(C)$ is equivalent to the diagonal matrix with the numbers $(N/f^m)^2$ each $(f^m)^2$ times in the diagonal. Hence we have the formula

$$\det\,(tE_N - R(C)) = \prod_{m=1}^{r} (t - (N/f^m)^2)^{(f^m)^2}, \tag{5}$$

which means that the rational numbers N/f^m can be computed by solving an equation explicitly known from (1-4) and the multiplication table of G. Since the coefficients of that equation are rational integers and the highest coefficient is 1, it follows that f^m is a divisor of N.

References

[1] H. Casimir and B.L. van der Waerden, "Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen," *Math. Ann.*, vol. 111 (1935), 1-12.

[2] J. C. H. Whitehead, "Certain equations in the algebra of a semi-simple infinitesimal group," *Quart. J. Math.*, vol. 8 (1937), 220-237.

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