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SOLUTIONS OF PERTURBED SCHRÖDINGER EQUATIONS WITH ELECTROMAGNETIC FIELDS AND CRITICAL NONLINEARITY

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Abstract We consider the existence and multiplicity of standing-wave solutions

$$\psi(x,t) = \exp\left(-\frac{\mathrm{i}Et}{\hbar}\right)u(x)$$

of nonlinear Schrödinger equations with electromagnetic fields and critical nonlinearity

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + \mathrm{i}A(x))^2\psi + W(x)\psi - K(x)|\psi|^{2^*-2}\psi - h(x,|\psi|^2)\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ N \geqslant 3.$$

Under suitable assumptions, we prove that it has at least one solution and that, for any $m \in \mathbb{N}$, it has at least m pairs of solutions.

Keywords: nonlinear Schrödinger equation; critical nonlinearity; magnetic fields; variational methods

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1. Introduction

The linear Schrödinger equation is a basic tool of quantum mechanics, and it provides a description of the dynamics of a particle in a non-relativistic setting. The nonlinear Schrödinger equation arises in different physical theories, e.g. the description of Bose–Einstein condensates and nonlinear optics (see [**6**] and the references therein). Both the linear and the nonlinear Schrödinger equations have been widely considered in the literature. The main purpose of this paper is to study the existence and multiplicity of semiclassical solutions of the perturbed Schrödinger equations with electromagnetic fields and critical nonlinearity of the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + \mathrm{i}A(x))^2\psi + W(x)\psi - K(x)|\psi|^{2^*-2}\psi - h(x,|\psi|^2)\psi \quad \text{for } x \in \mathbb{R}^N, \tag{1.1}$$

where i is the imaginary unit, $\hbar$ is Planck's constant,

$$A(x) = (A_1(x), A_2(x), \dots, A_N(x)) \colon \mathbb{R}^N \to \mathbb{R}^N$$

is a real vector (magnetic) potential with magnetic field $B = \operatorname{curl} A$ and $W(x) \colon \mathbb{R}^N \to \mathbb{R}^N$ is a scalar electric potential.

We are interested in standing-wave solutions, i.e. solutions to (1.1) of the type

$$\psi(x,t) = \exp\left(-\frac{\mathrm{i}Et}{\hbar}\right) u(x),$$

when $\hbar$ is sufficiently small, when E is a real number and $u(x)$ is a complex-valued function which satisfies

$$-(\nabla + \mathrm{i}A(x))^2 u(x) + \lambda(W(x) - E)u(x) = \lambda K(x)|u|^{2^*-2}u + \lambda h(x,|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\lambda^{-1} = \hbar^2/2m$. The transition from quantum mechanics to classical mechanics can be formally performed by letting $\hbar \to 0$. Thus, the existence of solutions for $\hbar$ small, semi-classical solutions has important physical interest.

In recent years, a lot of work has been devoted to investigating standing-wave solutions in the case $A(x) \equiv 0$. In this case one is led to look for positive solutions $u \colon \mathbb{R}^N \to \mathbb{R}$ of the semilinear equation with more general nonlinearity:

$$-\Delta u(x) + \lambda(W(x) - E)u(x) = \lambda g(x,u). \tag{1.3}$$

Different approaches have been taken to attack this problem under various hypotheses on the potential and the nonlinearity (see, for example, [**16**,**17**,**21**,**24**,**27**] and references therein). Observe that in all the aforementioned papers, the nonlinearities are assumed to be subcritical:

$$|g(x,u)| \leqslant c(1 + |u|^{p-1}) \quad \text{with } p \in (2, 2^*), \tag{1.4}$$

together with some other technical conditions, of course. Under the condition

$$\inf_{x\in\mathbb{R}^N} (W(x) - E) > 0,$$

there have been extensive investigations on problem (1.3). In [**21**], using a Lyapunov–Schmidt reduction, Floer and Weinstein established the existence of standing-wave solutions of (1.3); $W(x) - E$ is a bounded function having a non-degenerate critical point for sufficiently small $\hbar > 0$. Moreover, they showed that u concentrates near the given non-degenerate critical point of $W(x) - E$ when $\hbar \to 0$. Their method and results were later generalized by Oh [**24**] to the higher-dimensional case, and the existence of multi-bump solutions concentrating near several non-degenerate critical points of $W(x) - E$ as $\hbar \to 0$ was obtained. For more results, we refer the reader to [**1**,**2**,**11**,**12**,**16**,**17**]. Clapp and Ding [**15**] studied problem (1.3) in the case when the nonlinearities are assumed to be critical, with $g(x,u) = \mu u + u^{2^*-1}$; here $W(x) - E \geqslant 0$ has a potential well and is invariant under an orthogonal involution of $\mathbb{R}^N$. They established the existence and multiplicity of solutions which change sign exactly once and these solutions localize near the potential well for μ small and λ large. Ding and Lin [**18**] showed the existence and multiplicity of semiclassical solutions of perturbed nonlinear Schrödinger equations with critical nonlinearity. Ding and Wei [**19**] established the existence and multiplicity of semiclassical bound states of the nonlinear Schrödinger equations under the assumption that $W(x) - E$ changes sign and g is superlinear with critical or supercritical growth as $|u| \to \infty$.

There are also many works dealing with the magnetic case when $A(x) \not\equiv 0$ and $g(x, u)$ is subcritical growth. The first one would appear to be [**20**], in which the existence of standing waves was obtained for $\hbar > 0$ fixed and for special classes of magnetic fields. If A and W are periodic functions, the existence of various types of solutions for fixed $\hbar > 0$ has been proved in [**3**] by applying minimax arguments. Concerning semiclassical bound states, it is proved in [**23**] that $\hbar > 0$ small admits a least-energy solution which concentrates near the global minimum of W. A multiplicity result for solutions had been obtained in [**10**] by using a topological argument. It is also proved therein that the magnetic potential A only contributes to the phase factor of the solitary solutions for $\hbar > 0$ sufficiently small. In [**13**], single-bump bound states were obtained by using perturbation methods. These concentrate near a non-degenerate critical point of W as $\hbar \to 0$. If $g(x, u)$ is of critical growth, in this case, Wang [**27**] studied the electromagnetic Schrödinger equations

$$-(\nabla + \mathrm{i}A(x))^2 u(x) + \lambda V(x) u(x) = K(x)|u|^{2^*-2}u \quad \text{for } x \in \mathbb{R}^N. \tag{1.5}$$

By using the linking theorem twice with the corresponding functional, Wang established the existence results. Chabrowski and Szulkin [**9**] considered (1.5) under the assumption that $V(x)$ changes sign, by using a min–max type argument based on a topological linking. They obtained a solution in the Sobolev space which is defined in their paper. Assuming $K(x) \equiv 1$, Han [**22**] studied problem (1.5) and established the existence of non-trivial solutions in the critical case by means of the variational method. For other results, we refer the reader to [**4, 8, 14, 25, 26, 28**].

In the present paper, we consider the standing waves of problem (1.1) under the condition $\inf_{x\in\mathbb{R}^N}(W(x) - E) = 0$ and critical nonlinearity. It seems that Byeon and Wang [**6**] were the first to study the energy level and the asymptotic behaviour of positive solutions to Schrödinger equations under the condition $\inf_{x\in\mathbb{R}^N}(W(x) - E) = 0$. In [**7**], Cao and Noussair extended the results of [**6**]. However, it seems there is almost no work on the existence of semi-classical solutions to problems on $\mathbb{R}^N$ involving critical nonlinearities and electromagnetic fields. We mainly follow the idea of [**18, 19**]. Note that although the idea was used before for other problems, the adaptation of the procedure to our problem is not trivial at all: because of the appearance of electromagnetic potential $A(x)$, we must consider our problem for complex-valued functions and so we need more delicate estimates.

The paper is organized as follows: in § 2, we describe our main results (Theorem 2.3). Section 3 is devoted to behaviour of Palais–Smale (PS) sequences. Section 4 contains the proofs of the main results.

2. Main results

We set $V(x) = W(x) - E$ and rewrite (1.2) in the following form:

$$-(\nabla + \mathrm{i}A(x))^2 u(x) + \lambda V(x)u(x) = \lambda K(x)|u|^{2^*-2}u + \lambda h(x, |u|^2)u \quad \text{for } x \in \mathbb{R}^N. \quad (2.1)$$

We make the following assumptions on $A(x)$ and $V(x)$ throughout this paper:

(V) $V(x) \in C(\mathbb{R}^N, R)$, $V(x_0) = \min V = 0$, and there exists $b > 0$ such that the set $V^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

(A) $A_j(x) \in C(\mathbb{R}^N, R)$, $j = 1, 2, \dots, N$, and $A(x_0) = 0$;

(K) $K \in C(\mathbb{R}^N \times R, R)$, $0 < m := \inf K \leqslant M := \sup K < \infty$;

(h_1) $h \in C(\mathbb{R}^N \times [0, +\infty), R)$ and $h(x, t) = o(1)$ uniformly in x as $t \to 0$;

(h_2) there are $C_0 > 0$ and $q \in (2, 2^*)$ such that $|h(x, t)| \leqslant C_0(1 + t^{(q-2)/2})$;

(h_3) there are $a_0 > 0$, $p > 2$ and $2 < \mu < 2^*$ such that $H(x, t) \geqslant a_0 t^{p/2}$ and $\frac{1}{2}\mu H(x, t) \leqslant h(x, t)t$ for all (x, t), where

$$H(x, t) = \int_0^t h(x, s)\, \mathrm{d}s.$$

A typical nonlinearity considered in physical problems is $h(x, |u|^2)u = |u|^{p-2}u$. Here we allow h to be more general as specified in assumptions (h_1)–(h_3).

Let

$$\nabla_A u = (\nabla + \mathrm{i}A)u$$

and

$$H^1_A(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\}.$$

Hence, $H^1_A(\mathbb{R}^N)$ is the Hilbert space under the scalar product

$$(u, v) = \mathrm{Re} \int_{\mathbb{R}^N} ((\nabla u + \mathrm{i}A(x)u)\overline{(\nabla v + \mathrm{i}A(x)v)} + u\bar{v}),$$

the norm induced by the product $(\cdot\,,\cdot)$ is

$$\begin{aligned}\|u\|_{H^1_A(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2)\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^N} (|\nabla u + \mathrm{i}A(x)u|^2 + |u|^2)\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (|\mathrm{i}A(x)|^2 + 1)|u|^2) - 2\,\mathrm{Re}\int_{\mathbb{R}^N} \mathrm{i}A(x)\bar{u}\nabla u\right)^{1/2}.\end{aligned}$$

Let

$$E := \left\{u \in H^1_A(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V(x)|u|^2).$$

Remark 2.1. We have the following diamagnetic inequality (see, for example, [**20**]):

$$|\nabla_A u(x)| \geqslant |\nabla |u(x)|| \quad \text{for } u \in H_A^1(\mathbb{R}^N).$$

Indeed, since A is real-valued,

$$|\nabla |u|(x)| = \left|\mathrm{Re}\left(\nabla u \frac{\bar{u}}{|u|}\right)\right| = \left|\mathrm{Re}(\nabla u + \mathrm{i}Au)\frac{\bar{u}}{|u|}\right| \leqslant |\nabla u + \mathrm{i}Au|$$

(the bar denotes complex conjugation); this fact means that if $u \in H_A^1(\mathbb{R}^N)$, then $|u| \in H^1(\mathbb{R}^N)$ and therefore $u \in L^p(\mathbb{R}^N)$ for any $p \in [2, 2^*]$.

Remark 2.2. The spaces $H_A^1(\mathbb{R}^N)$ and the spaces $H^1(\mathbb{R}^N)$ are not comparable; more precisely, in general $H_A^1(\mathbb{R}^N) \not\subseteq H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \not\subseteq H_A^1(\mathbb{R}^N)$. However, it was proved by Arioli and Szulkin [**3**] that if K is a bounded domain with regular boundary, then $H_A^1(K)$ and $H^1(K)$ are equivalent, where $H_A^1(K) = \{u \in L^2(K)\colon \nabla u \in L^2(K)\}$ with the norm

$$\|u\|_{H_A^1(K)} = \left(\int_K (|\nabla_A u|^2 + |u|^2)\right)^{1/2}.$$

Let

$$E_\lambda := \left\{u \in H_A^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\right\}$$

with the norms

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + \lambda V(x)|u|^2).$$

Thus, it is easy to see that the norm $\|\cdot\|_E$ is equivalent to $\|\cdot\|_\lambda$ for each $\lambda > 0$. From Remark 2.1, for each $s \in [2, 2^*]$, there is $c_s > 0$ (independent of λ) such that, if $\lambda > 1$,

$$\left(\int_{\mathbb{R}^N} |u|^s\right)^{1/s} \leqslant c_s \left(\int_{\mathbb{R}^N} |\nabla |u||^2\right)^{1/2} \leqslant c_s \left(\int_{\mathbb{R}^N} |\nabla_A u|^2\right)^{1/2} \leqslant c_s \|u\|_\lambda. \tag{2.2}$$

Consider the functional

$$\begin{aligned} J_\lambda(u) &:= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + \lambda V(x)|u|^2) - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*} - \frac{\lambda}{2}\int_{\mathbb{R}^N} H(x, |u|^2) \\ &= \tfrac{1}{2}\|u\|_\lambda^2 - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*} - \frac{\lambda}{2}\int_{\mathbb{R}^N} H(x, |u|^2). \end{aligned}$$

Under the assumptions, standard arguments [**29**] show that $J_\lambda \in C^1(E, R)$ and its critical points are solutions of (2.1).

Our main result is as follows.

Theorem 2.3. *Let (V), (A), (K) and (h_1)–(h_3) be satisfied. Thus, we have the following.*

(i) *For any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that problem (2.1) has at least one solution u_λ for each $\lambda \geqslant \Lambda_\sigma$ satisfying $0 < J_\lambda(u_\lambda) \leqslant \sigma\lambda^{1-N/2}$.*

(ii) *Assume additionally that $h(x,t)$ is odd in t; for any $m \in N$ and $\sigma > 0$ there is $\Lambda_{m\sigma} > 0$ such that problem (2.1) has at least m pairs of solutions u_λ with $0 < J_\lambda(u_\lambda) \leqslant \sigma\lambda^{1-N/2}$ whenever $\lambda \geqslant \Lambda_{m\sigma}$.*

3. Behaviour of (PS) sequences

The main result of this section is the following compactness result.

Proposition 3.1. *Assume that (V), (A), (K) and (h_1)–(h_3) are satisfied. Then there is a constant $\alpha_0 > 0$ independent of λ such that, for any $(PS)_c$ sequence (u_n) for $J_\lambda(u)$,*

$$J_\lambda(u_n) \to c, \tag{3.1}$$

$$J'_\lambda(u_n) \to 0 \quad \text{strongly in } E^*, \tag{3.2}$$

and either $u_n \to u$ or $c - J_\lambda(u) \geqslant \alpha_0\lambda^{1-N/2}$.

As a consequence, we obtain the following result.

Corollary 3.2. *Under the assumptions of Proposition 3.1, $J_\lambda(u)$ satisfies the $(PS)_c$ condition for all $c < \alpha_0\lambda^{1-N/2}$.*

The proof of Proposition 3.1 consists of a series of lemmas which occupy this section.

Lemma 3.3. *Suppose that a sequence $\{u_n\} \subset E$ satisfies (3.1) and (3.2). Then there exists a constant $M(c)$ which is independent of $\lambda \geqslant 0$ such that $c \geqslant 0$ and*

$$\limsup_{n\to\infty} \|u\|_\lambda^2 \leqslant M(c).$$

Proof. On the one hand, by (3.1) and (3.2) one has

$$J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n = c + o(1) + \varepsilon_n\|u_n\|_\lambda, \tag{3.3}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

On the other hand,

$$\begin{aligned} J_\lambda(u_n) &- \frac{1}{\mu}J'_\lambda(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N}(|\nabla_A u_n|^2 + \lambda V(x)|u_n|^2) + \left(\frac{1}{\mu} - \frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N} K(x)|u_n|^{2^*} \\ &\quad + \frac{\lambda}{\mu}\int_{\mathbb{R}^N} h(x,|u_n|^2)|u_n|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} H(x,|u_n|^2). \end{aligned} \tag{3.4}$$

Assumption (h_3) implies that

$$\frac{1}{\mu}h(x,|u_n|^2)|u_n|^2 - \tfrac{1}{2}H(x,|u_n|^2) \geqslant 0.$$

Thus, it follows from (K), (3.3) and (3.4) that

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_\lambda^2 \leqslant c + o(1) + \varepsilon_n\|u_n\|_\lambda.$$

Hence, for n large enough, we have

$$\|u_n\|_\lambda^2 \leqslant \frac{2\mu}{\mu-2}c.$$

Thus, $\|u_n\|_\lambda$ is bounded as $n \to \infty$. Taking the limit in (3.4) shows that $c \geqslant 0$. Then $\{u_n\}$ is bounded and $c \geqslant 0$. □

From Lemma 3.3 we may assume without loss of generality $u_n \rightharpoonup u$ in $E(H_A^1(\mathbb{R}^N))$, $u_n \to u$ in $L_{\mathrm{loc}}^s(\mathbb{R}^N)$ for $1 \leqslant s < 2^*$ and $u_n(x) \to u(x)$ a.e., for $x \in \mathbb{R}^N$. Clearly, u is a critical point of J_λ.

Lemma 3.4. *If $t \in [2, 2^*)$, there is a subsequence $\{u_{m_i}\}$ of $\{u_n\}$ such that, for each $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that*

$$\limsup_{i\to\infty} \int_{B_i \setminus B_s} |u_{m_i}|^t \leqslant \varepsilon,$$

for $s \geqslant s_\varepsilon$, where $B_R = \{x \in \mathbb{R}^N : |u| \leqslant R\}$.

Proof. From $u_n \to u$ in $L_{\mathrm{loc}}^t(\mathbb{R}^N)$ we have

$$\int_{B_i} |u_m|^t \to \int_{B_i} |u|^t \quad \text{as } m \to \infty,$$

and for each $i \in \mathbb{N}$ there exists $\tilde{m}_i \in \mathbb{N}$ such that

$$\int_{B_i} (|u_{m_j}|^t - |u|^t) < \frac{1}{i} \quad \text{for all } m_j = \tilde{m}_i + j, \quad j = 1, 2, \dots .$$

Let $\tilde{m}_{i+1} \geqslant \tilde{m}_i$. In particular, for $m_i = \tilde{m}_{i+1} + i$, we have

$$\int_{B_i} (|u_{m_i}|^t - |u|^t) < \frac{1}{i} \quad \text{for all } m_i = \tilde{m}_i + i, \quad i = 1, 2, \dots .$$

Observe that there exists an s_ε such that

$$\int_{\mathbb{R}^N \setminus B_s} |u|^t < \tfrac{1}{3}\varepsilon \tag{3.5}$$

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for all $s \geqslant s_\varepsilon$ since

$$\begin{aligned}\int_{B_i \setminus B_s} |u_{m_i}|^t &= \int_{B_i} (|u_{m_i}|^t - |u|^t) + \int_{B_i \setminus B_s} |u|^t + \int_{B_s} (|u|^t - |u_{m_i}|^t) \\ &\leqslant \frac{1}{i} + \int_{\mathbb{R}^N \setminus B_s} |u|^t + \int_{B_s} (|u|^t - |u_{m_i}|^t) \\ &\leqslant \varepsilon \quad \text{as } n \to +\infty.\end{aligned}$$

□

Remark 3.5. From the proof of Lemma 3.4 we can find a subsequence (u_{n_j}) such that the result of the lemma holds for both $s = 2$ and $s = q$.

Let $\eta\colon [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta = 1$ if $t \leqslant \frac{1}{2}$, $\eta = 0$ if $t \geqslant 1$. Define

$$\hat{u}_j(x) = \eta\biggl(\frac{2|x|}{j}\biggr) u(x).$$

Obviously,

$$\|u - \hat{u}_j\|_E \to 0 \quad \text{as } j \to +\infty. \tag{3.6}$$

We have the following lemma.

Lemma 3.6. *Let $\{u_m\}$ and $\{\hat{u}_m\}$ be as defined above. Then*

$$\lim_{j\to\infty} \operatorname{Re} \int_{\mathbb{R}^N} (h(x, |u_{m_j}|^2) u_{m_j} - h(x, |u_{m_j} - \hat{u}_j|^2)(u_{m_j} - \hat{u}_j) - h(x, |\hat{u}_j|^2)\hat{u}_j)\bar{w} \to 0$$

uniformly in $w \in E$ with $\|w\| \leqslant 1$.

Proof. Remark 2.2, (3.6) and the local compactness of Sobolev embedding imply that, for any $r > 0$,

$$\lim_{j\to\infty} \operatorname{Re} \int_{B_r} (h(x, |u_{m_j}|^2) u_{m_j} - h(x, |u_{m_j} - \hat{u}_j|^2)(u_{m_j} - \hat{u}_j) - h(x, |\hat{u}_j|^2)\hat{u}_j)\bar{w} = 0$$

uniformly in $w \in E$ with $\|w\| \leqslant 1$. For any $\varepsilon > 0$ it follows from (3.5) that

$$\limsup_{j\to\infty} \int_{B_j \setminus B_s} |\hat{u}_j|^t \leqslant \int_{\mathbb{R}^N \setminus B_s} |u|^t \leqslant \varepsilon$$

for all $s \geqslant s_\varepsilon$. By Remark 3.5 and (h_1), (h_2) we have

$$\begin{aligned}&\limsup_{j\to\infty} \operatorname{Re} \int_{\mathbb{R}^N} (h(x, |u_{m_j}|^2) u_{m_j} - h(x, |u_{m_j} - \hat{u}_j|^2)(u_{m_j} - \hat{u}_j) - h(x, |\hat{u}_j|^2)\hat{u}_j)\bar{w} \\ &\quad = \limsup_{j\to\infty} \operatorname{Re} \int_{B_j \setminus B_s} (h(x, |u_{m_j}|^2) u_{m_j} - h(x, |u_{m_j} - \hat{u}_j|^2)(u_{m_j} - \hat{u}_j) - h(x, |\hat{u}_j|^2)\hat{u}_j)\bar{w} \\ &\quad \leqslant c \limsup_{j\to\infty} \int_{B_j \setminus B_s} (|u_{n_j}| + |\hat{u}_j|)|w| + C \limsup_{j\to\infty} \int_{B_j \setminus B_s} (|u_{n_j}|^{q-1} + |\hat{u}_j|^{q-1})|w|\end{aligned}$$

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$$\begin{aligned}
&\leqslant c \limsup_{j\to\infty}(|u_{n_j}|_{L^2(B_j\setminus B_s)} + |\hat{u}_j|_{L^2(B_j\setminus B_s)})|w|_2 \\
&\qquad\qquad + C \limsup_{j\to\infty}(|u_{n_j}|^{q-1}_{L^q(B_j\setminus B_s)} + |\hat{u}_j|^{q-1}_{L^q(B_j\setminus B_s)})|w|_q \\
&\leqslant c'\varepsilon^{1/2} + C'\varepsilon^{(q-1)/q},
\end{aligned}$$

where c, C, c' and C' are positive constants. This completes the proof of Lemma 3.6. □

Lemma 3.7. *Let $\{u_m\}$ and $\{\hat{u}_m\}$ be as defined above. Then the following conclusions hold:*

(i) $J_\lambda(u_n - \hat{u}_n) \to c - J_\lambda(u)$;

(ii) $J'_\lambda(u_n - \hat{u}_n) \to 0$.

Proof. Since

$$\begin{aligned}
J_\lambda(u_n - \hat{u}_n) = \frac{1}{2}\int_{\mathbb{R}^N} &(|\nabla_A(u_n - \hat{u}_n)|^2 + \lambda V(x)|u_n - \hat{u}_n|^2) \\
&- \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u_n - \hat{u}_n|^{2^*} - \frac{\lambda}{2}\int_{\mathbb{R}^N} H(x, |u_n - \hat{u}_n|^2) \\
= J_\lambda(u_n) &- J_\lambda(\hat{u}_n) \\
&+ \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*} - |u_n - \hat{u}_n|^{2^*} - |\hat{u}_n|^{2^*}) \\
&+ \frac{\lambda}{2}\int_{\mathbb{R}^N} (H(x, |u_n|^2) - H(x, |u_n - \hat{u}_n|^2) - H(x, |\hat{u}_n|^2)).
\end{aligned}$$

On the one hand, using (3.6) and along the lines of proving the Brézis–Lieb Lemma [**29**], we have

$$\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*} - |u_n - \hat{u}_n|^{2^*} - |\hat{u}_n|^{2^*}) \to 0$$

and

$$\int_{\mathbb{R}^N} (H(x, |u_n|^2) - H(x, |u_n - \hat{u}_n|^2) - H(x, |\hat{u}_n|^2)) \to 0.$$

On the other hand, together with $J_\lambda(u_n) \to c$, $J_\lambda(\hat{u}_n) \to J_\lambda(u)$ as $n \to \infty$. This completes the proof of Lemma 3.7 (i).

In order to prove (ii), observe that, for any $w \in E$,

$$\begin{aligned}
&J'_\lambda(u_n - \hat{u}_n)w \\
&\quad = \operatorname{Re}\bigg\{\int_{\mathbb{R}^N} \nabla_A(u_n - \hat{u}_n)\overline{\nabla_A w} + \lambda V(x)(u_n - \hat{u}_n)\bar{w} \\
&\qquad\qquad - \lambda\int_{\mathbb{R}^N} K(x)|u_n - \hat{u}_n|^{2^*-2}(u_n - \hat{u}_n)\bar{w} \\
&\qquad\qquad\qquad - \lambda\int_{\mathbb{R}^N} h(x, |u_n - \hat{u}_n|^2)(u_n - \hat{u}_n)\bar{w}\bigg\}
\end{aligned}$$

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$$
\begin{aligned}
&= J'_\lambda(u_n)w - J'_\lambda(\hat u_n)w \\
&\quad + \lambda \operatorname{Re}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*-2}u_n\bar w - |\hat u_n|^{2^*-2}\hat u_n\bar w - |\hat u_n|^{2^*-2}\hat u_n\bar w) \\
&\quad + \lambda \operatorname{Re}\int_{\mathbb{R}^N} (h(x,|u_n|^2)u_n - h(x,|u_n-\hat u_n|^2)(u_n-\hat u_n) - h(x,|\hat u_n|^2)\hat u_n)\bar w.
\end{aligned}
$$

It follows by a standard argument that

$$
\limsup_{n\to\infty} \operatorname{Re}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*-2}u_n\bar w - |\hat u_n|^{2^*-2}\hat u_n\bar w - |\hat u_n|^{2^*-2}\hat u_n\bar w) = 0
$$

uniformly in $\|w\| \leqslant 1$. By Lemma 3.6 we have

$$
\limsup_{n\to\infty} \operatorname{Re}\int_{\mathbb{R}^N} (h(x,|u_n|^2)u_n - h(x,|u_n-\hat u_n|^2)(u_n-\hat u_n) - h(x,|\hat u_n|^2)\hat u_n)\bar w = 0
$$

uniformly in $\|w\| \leqslant 1$. This completes the proof of Lemma 3.7 (ii). □

Let

$$
v_n := u_n - \hat u_n.
$$

Then $u_n - u = v_n + (\hat u_n - u)$, and by (3.6), $u_n \to u$ if and only if $v_n \to 0$. By Lemma 3.7 we have $J_\lambda(v_n) \to c - J_\lambda(u)$ and $J'_\lambda(v_n) \to 0$. Note that

$$
J_\lambda(v_n) - \tfrac{1}{2}J'_\lambda(v_n)v_n \geqslant \frac{\lambda}{N}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*} \geqslant \frac{\lambda m}{N}\int_{\mathbb{R}^N} |v_n|^{2^*}.
$$

Hence,

$$
|v_n|_{2^*}^{2^*} \leqslant \frac{N(c - J_\lambda(u))}{\lambda m} + o(1). \tag{3.7}
$$

Let

$$
\nu_b := \max\{V(x), b\}, \tag{3.8}
$$

where b is the positive constant from assumption (V). Since V^b has finite measure and $v_n \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we see that

$$
\int_{\mathbb{R}^N} V(x)|v_n|^2 = \int_{\mathbb{R}^N} \nu_b|v_n|^2 + o(1).
$$

It follows from assumptions (h_1)–(h_3) that there exists a constant $c_b > 0$ such that

$$
\int_{\mathbb{R}^N} K(x)|u|^{2^*} + \int_{\mathbb{R}^N} h(x,|u|^2)|u|^2 \leqslant b|u|_2^2 + c_b|u|_{2^*}^{2^*}. \tag{3.9}
$$

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Proof of Proposition 3.1. Assume that $u_n \not\to u$. Then $\liminf_{n\to\infty} \|v_n\| > 0$ and $c - J_\lambda(u) > 0$. By (2.2), (3.8) and (3.9) we have

$$\begin{aligned}\frac{1}{c_s}|v_n|_{2^*}^2 &\leqslant \int_{\mathbb{R}^N} |\nabla_A v_n|^2 + \lambda V(x)|v_n|^2 - \lambda \int_{\mathbb{R}^N} V(x)|v_n|^2 \\ &= \lambda \int_{\mathbb{R}^N} K(x)|u|^{2^*} + \lambda \int_{\mathbb{R}^N} h(x,|u|^2)|u|^2 - \lambda \int_{\mathbb{R}^N} \nu_b |v_n|^2 \\ &\leqslant \lambda \int_{\mathbb{R}^N} K(x)|u|^{2^*} + \lambda \int_{\mathbb{R}^N} h(x,|u|^2)|u|^2 - \lambda b \int_{\mathbb{R}^N} |v_n|^2 + o(1) \\ &\leqslant \lambda c_b |v_n|_{2^*}^{2^*} + o(1).\end{aligned}$$

By (3.7) we have

$$\begin{aligned}\frac{1}{c_s} &\leqslant \lambda c_b |v_n|_{2^*}^{2^*-2} + o(1) \\ &\leqslant \lambda c_b \biggl(\frac{N(c - J_\lambda(u))}{\lambda m}\biggr)^{2/N} + o(1) \\ &= \lambda^{1-2/N} c_b \biggl(\frac{N}{m}\biggr)^{2/N} (c - J_\lambda(u))^{2/N} + o(1).\end{aligned}$$

Therefore,

$$\alpha_0 \lambda^{1-N/2} \leqslant c - J_\lambda(u) + o(1),$$

where

$$\alpha_0 = \frac{m}{c_s^{N/2} c_b N}.$$

This completes the proof of Proposition 3.1. □

4. Proof of Theorem 2.3

In the following we always consider $\lambda \geqslant 1$. By the assumptions (V), (A), (K) and (h_1)–(h_3), one can see that $J_\lambda(u)$ have mountain pass geometry.

Lemma 4.1. *Assume that (V), (A), (K) and (h_1)–(h_3) hold. There exist $\alpha_\lambda, \rho_\lambda > 0$ such that $J_\lambda(u) > 0$ if $u \in B_{\rho_\lambda} \setminus \{0\}$ and $J_\lambda(u) \geqslant \alpha_\lambda$ if $u \in \partial B_{\rho_\lambda}$, where $B_{\rho_\lambda} = \{u \in E\colon \|u\| \leqslant \rho_\lambda\}$.*

Proof. By (K) and (h_1)–(h_3), for $\delta \leqslant (4\lambda c_s)^{-1}$ there is $C_\delta > 0$ such that

$$\frac{1}{2^*} \int_{\mathbb{R}^N} K(x)|u|^{2^*} + \tfrac{1}{2} H(x,|u|^2) \leqslant \delta |u|^2 + C_\delta |u|^{2^*}.$$

So, from (A) and (V) it follows that

$$\begin{aligned}J_\lambda(u) &\geqslant \tfrac{1}{2}\|u\|_\lambda^2 - \lambda\delta |u|_2^2 - \lambda C_\delta |u|_{2^*}^{2^*} \\ &\geqslant \tfrac{1}{4}\|u\|_\lambda^2 - \lambda C_\delta |u|_{2^*}^{2^*}.\end{aligned}$$

By (2.2) and $2^* > 2$ we know that the conclusion of Lemma 4.1 holds. This completes the proof of Lemma 4.1. □

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Lemma 4.2. *Under the assumptions of Lemma 4.1, for any finite-dimensional subspace $F \subset E$,*

$$J_\lambda(u) \to -\infty \quad \text{as } u \in F,\ \|u\| \to \infty.$$

Proof. Using conditions (A), (V), (K) and (h_1)–(h_3), we deduce

$$J_\lambda(u) \leqslant \tfrac{1}{2}\|u\|_\lambda^2 - \lambda a_0 |u|_p^p$$

for all $u \in E$ since all norms in a finite-dimensional space are equivalent and $p > 2$. This completes the proof of Lemma 4.2. □

Since $J_\lambda(u)$ does not satisfy the $(\mathrm{PS})_c$ condition for all $c > 0$, in the following we will find special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that the assumption (V) implies that there exists $x_0 \in \mathbb{R}^N$ such that

$$V(x_0) = \min_{x \in \mathbb{R}^N} V(x) = 0.$$

Without loss of generality we assume from now on that $x_0 = 0$.

Observe that, by (h_3),

$$\frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*} + \frac{\lambda}{2}\int_{\mathbb{R}^N} H(x,|u|^2) \geqslant a_0\lambda \int_{\mathbb{R}^N} |u|^p.$$

Define the function $I_\lambda \in C^1(E,R)$ by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + \lambda V(x)|u|^2) - a_0\lambda \int_{\mathbb{R}^N} |u|^p.$$

Then $J_\lambda(u) \leqslant I_\lambda(u)$ for all $u \in E$ and it suffices to construct small minimax levels for I_λ.

Note that

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\phi|^2 \colon \phi \in C_0^\infty(\mathbb{R}^N),\ |\phi|_p = 1\right\} = 0.$$

For any $\delta > 0$ one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\delta|_p = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla\phi_\delta|_2^2 < \delta$. Let

$$f_\lambda = \phi_\delta(\lambda^{1/2}x). \tag{4.1}$$

Then

$$\operatorname{supp} f_\lambda \subset B_{\lambda^{-1/2}r_\delta}(0).$$

Thus, for $t \geqslant 0$,

$$\begin{aligned} I_\lambda(tf_\lambda) &= \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla_A f_\lambda|^2 + \lambda V(x)|f_\lambda|^2) - t^p a_0\lambda \int_{\mathbb{R}^N} |f_\lambda|^p \\ &= \lambda^{1-N/2}\left(\frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla_A \phi_\delta|^2 + V(\lambda^{-1/2}x)|\phi_\delta|^2) - t^p a_0 \int_{\mathbb{R}^N} |\phi_\delta|^p\right) \\ &= \lambda^{1-N/2}\Psi_\lambda(t\phi_\delta), \end{aligned}$$

where $\Psi_\lambda \in C^1(E, R)$ defined by

$$\Psi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V(\lambda^{-1/2}x)|u|^2) - a_0 \int_{\mathbb{R}^N} |u|^p.$$

Obviously,

$$\max_{t\geqslant 0} \Psi_\lambda(t\phi_\delta) = \frac{p-2}{2p(pa_0)^{2/(p-2)}}\left(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^2 + V(\lambda^{-1/2}x)|\phi_\delta|^2\right)^{p/(p-2)}.$$

On the one hand, since $V(0) = 0$ and support $\phi_\delta \subset B_{r_\delta}(0)$, there exists $\Lambda_{\delta_1} > 0$ such that

$$V(\lambda^{-1/2}x) \leqslant \frac{\delta}{|\phi_\delta|_2^2} \quad \text{for all } |x| \leqslant r_\delta \text{ and } \lambda \geqslant \Lambda_{\delta_1}.$$

On the other hand, by Hölder's inequality, we have

$$\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^2 \leqslant \int_{\mathbb{R}^N} 2|\nabla \phi_\delta|^2 + 2|A(\lambda^{-1/2}x)\phi_\delta|^2. \tag{4.2}$$

Since $A(x)$ is continuous on $\mathbb{R}^N$ and $A(0) = 0$, there exists $\Lambda_{\delta_2} > 0$ such that

$$|A(\lambda^{-1/2}x)| \leqslant \sqrt{\frac{\delta}{|\phi_\delta|_2^2}} \quad \text{for all } |x| \leqslant r_\delta \text{ and } \lambda \geqslant \Lambda_{\delta_2}. \tag{4.3}$$

Without loss of generality we can take $\Lambda_\delta := \{\Lambda_{\delta_1}, \Lambda_{\delta_2}\}$. So, by (4.2) and (4.3) we deduce

$$\max_{t\geqslant 0} \Psi_\lambda(t\phi_\delta) \leqslant \frac{p-2}{2p(pa_0)^{2/(p-2)}}(5\delta)^{p/(p-2)}. \tag{4.4}$$

Therefore, for all $\lambda \geqslant \Lambda_\delta$,

$$\max_{t\geqslant 0} J_\lambda(t\phi_\delta) \leqslant \frac{p-2}{2p(pa_0)^{2/(p-2)}}(5\delta)^{p/(p-2)}\lambda^{1-N/2}. \tag{4.5}$$

Thus, we have the following lemma.

Lemma 4.3. *Under the assumptions of Lemma 4.1, for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geqslant \Lambda_\sigma$, there is $\hat{f}_\lambda \in E$ with $\|\hat{f}_\lambda\| > \rho_\lambda$, $J_\lambda(\hat{f}_\lambda) \leqslant 0$ and*

$$\max_{t\in[0,1]} J_\lambda(t\hat{f}_\lambda) \leqslant \sigma\lambda^{1-N/2}. \tag{4.6}$$

Proof. Choose $\delta > 0$ so small that

$$\frac{p-2}{2p(pa_0)^{2/(p-2)}}(5\delta)^{p/(p-2)} \leqslant \sigma$$

and let $f_\lambda \in E$ be the function defined by (4.1). We take $\Lambda_\sigma = \Lambda_\delta$. Let $\hat{t}_\lambda > 0$ be such that $\hat{t}_\lambda\|f_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(tf_\lambda) \leqslant 0$ for all $t \geqslant \hat{t}_\lambda$. Let $\hat{f}_\lambda = \hat{t}_\lambda f_\lambda$. We know that the conclusion of Lemma 4.3 holds. □

For any $m^* \in N$, one can choose m^* functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^i \cap \operatorname{supp}\phi_\delta^k = \emptyset$, $i \neq k$, $|\phi_\delta^i|_p = 1$ and $|\nabla\phi_\delta^i|_2^2 < \delta$. Let $r_\delta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\delta^i \subset B_{r_\delta}^i(0)$ for $i = 1, 2, \dots, m^*$. Let

$$f_\lambda^i(x) = \phi_\delta^i(\lambda^{1/2}x) \quad \text{for } j = 1, 2, \dots, m^*$$

and

$$H_{\lambda\delta}^{m^*} = \operatorname{span}\{f_\lambda^1, f_\lambda^2, \dots, f_\lambda^{m^*}\}.$$

Observe that, for each

$$u = \sum_{i=1}^{m^*} c_i f_\lambda^i \in H_{\lambda\delta}^{m^*},$$

we have

$$\int_{\mathbb{R}^N} |\nabla_A u|^2 = \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} |\nabla_A f_\lambda^i|^2,$$

$$\int_{\mathbb{R}^N} V(x)|u|^2 = \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} V(x)|f_\lambda^i|^2,$$

$$\frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*} = \frac{1}{2^*}\sum_{i=1}^{m^*} |c_i|^{2^*} \int_{\mathbb{R}^N} K(x)|u|^{2^*}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N} H(x,|u|^2) = \frac{1}{2}\sum_{i=1}^{m^*} \int_{\mathbb{R}^N} H(x,|c_i f_\lambda^i|^2).$$

Thus,

$$J_\lambda(u) = \sum_{i=1}^{m^*} J_\lambda(c_i f_\lambda^i)$$

and, as before,

$$J_\lambda(c_i f_\lambda^i) \leqslant \lambda^{1-N/2}\Psi(|c_i| f_\lambda^i).$$

Set

$$\beta_\delta := \max\{|\phi_\delta^i|_2^2 \colon j = 1, 2, \dots, m^*\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V(\lambda^{1-N/2}x) \leqslant \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \leqslant r_\delta^{m^*} \text{ and } \lambda \geqslant \Lambda_{m^*\delta}.$$

As before, we can obtain the following inequality

$$\max_{u \in H_{\lambda\delta}^{m^*}} J_\lambda(u) \leqslant \frac{m^*(p-2)}{2p(pa_0)^{2/(p-2)}}(5\delta)^{p/(p-2)}\lambda^{1-N/2} \tag{4.7}$$

for all $\lambda \geqslant \Lambda_{m^*\delta}$.

Using this estimate we have the following.

Lemma 4.4. *Under the assumptions of Lemma 4.1, for any $m^* \in N$ and $\sigma > 0$ there exists $\Lambda_{m^*\sigma} > 0$ such that, for each $\lambda \geqslant \Lambda_{m^*\sigma}$, there exists an m^*-dimensional subspace $F_{\lambda m^*}$ satisfying*

$$\max_{u \in F_{\lambda\delta}} J_\lambda(u) \leqslant \sigma\lambda^{1-N/2}.$$

Proof. Choose $\delta > 0$ so small that

$$\frac{m^*(p-2)}{2p(pa_0)^{2/(p-2)}}(5\delta)^{p/(p-2)} \leqslant \sigma$$

and take $F_{\lambda\delta} = H_{\lambda\delta}^{m^*}$. By (4.5) we know that the conclusion of Lemma 4.4 holds. □

We now establish the existence and multiplicity results.

Proof of Theorem 2.3. Using Lemma 4.3 we choose $\Lambda_\sigma > 0$ and define, for $\lambda \geqslant \Lambda_\sigma$, the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t\hat{f}_\lambda),$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1], E) \colon \gamma(0) = 0 \text{ and } \gamma(1) = \hat{f}_\lambda\}.$$

By Lemma 4.1 we have $\alpha_\lambda \leqslant c_\lambda \leqslant \sigma\lambda^{1-N/2}$. By virtue of Proposition 3.1, we know that J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition; there exists $u_\lambda \in E$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$; hence, the existence is proved.

Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let $\mathrm{gen}(Z)$ be the Krasnoselski genus and let

$$i(Z) := \min_{h \in \Gamma_{m^*}} \mathrm{gen}(h(Z) \cap \partial B_{\rho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $h \in C(E, E)$ and ρ_λ is the number from Lemma 4.1. Then i is a version of Benci's pseudoindex [**5**]. Let

$$c_{\lambda i} := \inf_{i(Z) \geqslant i} \sup_{u \in Z} J_\lambda(u), \quad 1 \leqslant i \leqslant m^*.$$

Since $J_\lambda(u) \geqslant \alpha_\lambda$ for all $u \in \partial B_{\rho\lambda}^+$ and since $i(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$,

$$\alpha_\lambda \leqslant c_{\lambda 1} \leqslant \cdots \leqslant c_{\lambda m^*} \leqslant \sup_{u \in H_{\lambda m^*}} J_\lambda(u) \leqslant \sigma\lambda^{1-N/2}.$$

It follows from Proposition 3.1 that J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition at all levels c_i. By the usual critical-point theory, all c_i are critical levels and J_λ has at least m^* pairs of non-trivial critical points. □

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