14
Mean field theory

We have seen that confinement arises naturally in the strong coupling limit of the lattice theory, whereas the continuum limit with asymptotic freedom drives us toward the weak coupling regime. Desiring the qualitative features of confinement to persist in the continuum limit, we would like to be able to pass smoothly from high to low temperature in our statistical analog. This leads to the hope that $SU(3)$ lattice gauge theory has no phase transitions separating the strong and weak coupling domains.

Do we expect phase transitions in lattice gauge theory? In the chapter on discrete groups, we will show that indeed deconfining transitions do exist in some toy models. In this chapter we will present some non-rigorous arguments based on mean field theory which suggests that any gauge group potentially displays phase transitions in enough space-time dimensions. The approximation in mean field theory requires each variable to interact directly with a large number of neighbors; consequently, it is effectively a large dimension simplification.

The application of mean field theory to gauge systems has had a somewhat murky history. In its simplest form it ignores Elitzur's theorem, discussed in chapter 9. A link variable is assumed to have an expectation value, which is then calculated with a self-consistency condition. Rigorously, however, this expectation must vanish because the link is a gauge-variant object. Recent formulations (Drouffe, 1981; Flyvbjerg, Lautrup and Zuber, 1982) present mean field theory as a saddle point approximation which gives rise to a consistent expansion in inverse dimension. Gauge rotations appear as zero point modes in the first-order corrections and restore Elitzur's theorem. As our goal here is to motivate possible phase transitions in lattice gauge theory, we will present this approximation in a simple and heuristic form (Balian, Drouffe and Itzykson, 1974). In general, the mean field approach underestimates thermal fluctuations. For the second-order transitions in magnetic systems, the results overestimate critical temperatures, possibly by an infinite factor.

To emphasize the different predictions for spin and gauge models, we first illustrate the technique for the Ising model. Placing a spin s_i from the set $Z_2 = \{1, -1\}$ on each site of a d-dimensional hypercubic lattice, we consider the partition function

$$Z = \sum_{\{s\}} \exp\left(\beta \sum_{\{ij\}} s_i s_j\right), \tag{14.1}$$

where each spin is summed over and the sum in the exponent is over all nearest-neighbor pairs of sites $\{ij\}$.

We wish to find an approximate expression for the magnetization, the expectation value for any given spin

$$M = \langle s_i \rangle. \tag{14.2}$$

We begin by considering a particular site i and replacing the spins on all neighboring sites with their average value M. Then the Boltzmann probability for the spin on site i to have value s_i becomes

$$P(s_i) = \exp(2d\beta M s_i)/(2\cosh(2d\beta M)), \tag{14.3}$$

where the factor $2d$ counts the number of neighbors to site i, and the denominator normalizes the probability. Requiring that the average value of s_i is also M, we obtain the self-consistency condition

$$M = \tanh(2d\beta M). \tag{14.4}$$

For small β, this equation has the unique solution $M = 0$. Mean field theory correctly predicts that the magnetization vanishes at high temperatures. In contrast, whenever

$$\beta > \beta_{mf} = 1/(2d) \tag{14.5}$$

eq. (14.4) also has a non-trivial solution with $M > 0$ (as well as a symmetric one at $M < 0$). A graphical solution of eq. (14.4) is illustrated in figure 14.1. To see that the latter solution is the favored one, consider eq. (14.4) iteratively. If initially M is slightly positive, the expectation of s_i will be increased and driven towards the non-vanishing solution. Thus mean field theory predicts a phase transition at β_{mf}. For larger β the system is predicted to spontaneously magnetize. In table 14.1 we compare this prediction with the known critical temperatures β_c for the Ising model in 1, 2, 3, and 4 dimensions (Fisher and Gaunt, 1964).

Table 14.1.

d	β_{mf}	β_c
1	0.500	∞
2	0.250	0.441
3	0.167	0.222
4	0.125	0.150

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Note that the critical temperatures are always overestimated, for $d = 1$ by an infinite factor. The approximation does, however, improve as d increases.

This analysis predicts a continuous transition for the Ising model in any number of dimensions. As β decreases to β_{mf}, the non-trivial solution to eq. (14.4) decreases smoothly to zero. This will contrast sharply with the gauge theory, where all transitions are predicted to be first order; thermodynamics changes discontinuously at the phase transition.

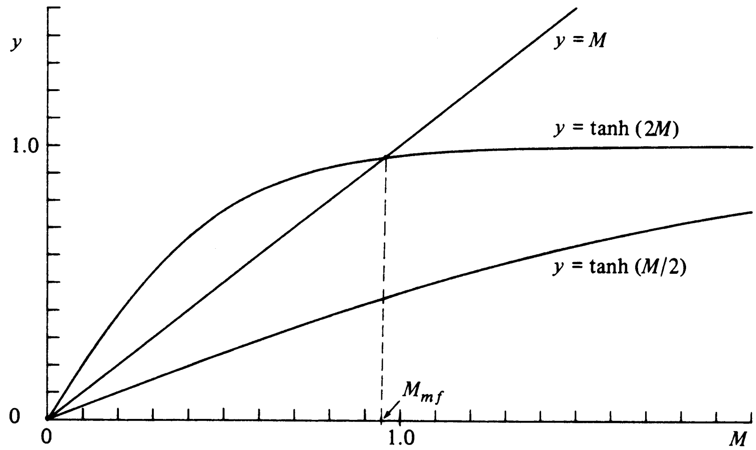

Fig. 14.1. Graphical solution of eq. (14.4) above and below the mean-field critical point.

In this example it was relatively easy to use physical arguments to determine which solution of eq. (14.4) was the relevant one. For the generalization to gauge theories it is useful to reformulate the technique in variational form (Peierls, 1938). For this purpose we use a convexity inequality on the exponential function. Given any function f over some set $X = \{x\}$ and a normalized measure $\rho(x)$ such that

$$\int_X \rho(x)\,\mathrm{d}x = 1, \qquad (14.6)$$

then, because the exponential function is convex, we have

$$\langle \mathrm{e}^f \rangle \geqslant \mathrm{e}^{\langle f \rangle}. \qquad (14.7)$$

Here the averages are with respect to the measure $\rho(x)$

$$\langle f \rangle = \int_X f(x)\,\rho(x)\,\mathrm{d}x. \qquad (14.8)$$

For the application of this inequality to the Ising system, we first add and subtract a source term to the action

$$Z = \sum_{\{s\}} \exp\,(\beta \sum_{\{ij\}} s_i s_j + H \sum_i s_i - H \sum_i s_i), \tag{14.9}$$

where H will become a variational parameter. For our measure $\rho(x)$ we take

$$\int_X \rho(x) f(x)\,\mathrm{d}x \to \sum_s \exp\,(H \sum_i s_i) f(s) / (\sum_s \exp\,(H \sum_i s_i)). \tag{14.10}$$

Applying the convexity inequality with this measure gives

$$Z \geqslant \exp\,(N^d(d\beta \tanh^2(H) - H \tanh(H) + \log(2\cosh(H)))), \tag{14.11}$$

where N^d represents the number of sites on our lattice. This translates into a bound on the free energy

$$\beta F = -N^{-d} \log(Z) \leqslant \beta F_{mf}(H)$$
$$= -d\beta \tanh^2(H) + H \tanh(H) - \log(2\cosh(H)). \tag{14.12}$$

Minimizing the right hand side with respect to the parameter H optimizes the bound and gives rise to the relation

$$(\mathrm{d}/\mathrm{d}H)\beta F_{mf} = 0 = \operatorname{sech}^2 H(-2d\beta \tanh H + H). \tag{14.13}$$

Note that this is equivalent to eq. (14.4) with the identification $H = 2d\beta M$. For low temperatures it is the non-zero root of this equation which minimizes the bound in eq. (14.12). In figure 14.2 we plot the mean-field free energy as a function of H for the cases $\beta = 1.1\beta_{mf}$, β_{mf} and $\beta_{mf}/1.1$.

The terms in the mean-field free energy have a simple thermodynamical interpretation. The piece $-d\beta \tanh^2 H$ is a potential energy driving H to non-zero values. The remainder $H \tanh H - \log(2\cosh H)$ represents an approximation to an entropy factor trying to disorder the system. Which term wins depends on the value of β.

With this formalism in hand, we can proceed directly to the gauge theory. We consider the pure $SU(n)$ gauge theory normalized as in chapter 10 and study the partition function

$$Z = \int (\mathrm{d}U) \exp\,((\beta/n) \sum_\square \operatorname{Tr} U_\square). \tag{14.14}$$

Adding and subtracting $(H/n) \sum_{\{ij\}} \operatorname{Re}\operatorname{Tr} U_{ij}$ from the action allows us to use the measure

$$\int_X \rho(x) f(x)\,\mathrm{d}x \to \frac{\int (\mathrm{d}U)\, \mathrm{e}^{(H/n)\sum \operatorname{Re}\operatorname{Tr} U} f(U)}{\int (\mathrm{d}U)\, \mathrm{e}^{(H/n)\sum \operatorname{Re}\operatorname{Tr} U}} \tag{14.15}$$

in the convexity inequality. This gives the bound

$$\beta F = -N^{-d} \log Z \leqslant \beta F_{mf}(H)$$
$$= -\tfrac{1}{2}d(d-1)\beta t(H)^4 + dHt(H) - d\log(c(H)). \qquad (14.16)$$

Here we have generalized the hyperbolic functions to arbitrary groups

$$c(H) = \int dH\, e^{(H/n)\,\mathrm{Re\,Tr}\,U}, \qquad (14.17)$$

$$t(H) = c(H)^{-1} \int dU \quad n^{-1}\,\mathrm{Re\,Tr}\,U\, e^{(H/n)\,\mathrm{Re\,Tr}\,U}. \qquad (14.18)$$

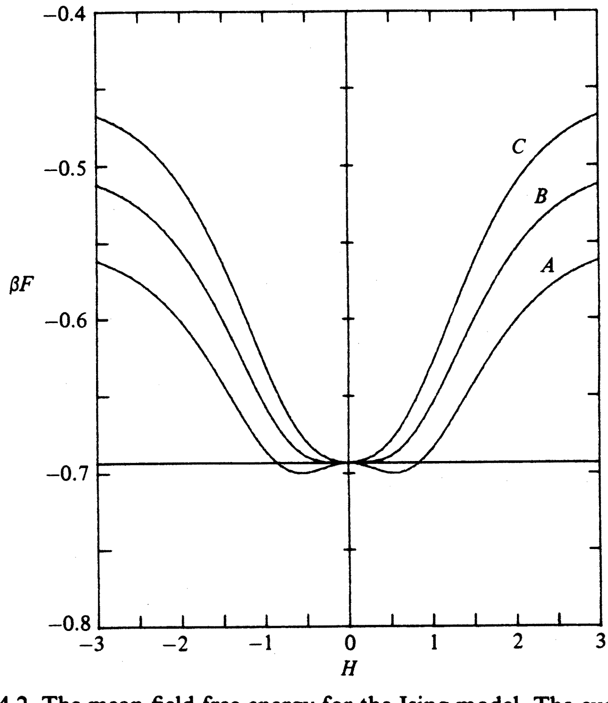

Fig. 14.2. The mean-field free energy for the Ising model. The curves A, B, and C are for β 10% above, exactly on, and 10% below β_{mf}, respectively.

The factor $\frac{1}{2}d(d-1)$ in eq. (14.16) is the number of plaquettes per site on a d-dimensional lattice. Note that for $SU(3)$ $t(H)$ is simply the parameter $b_3/3$ of chapter 10, evaluated for $\beta = H$. Differentiating F_{mf} with respect to H to find the extrema gives the consistency condition

$$(-2d(d-1)\beta t(H)^3 + dH)\,dt(H)/dH = 0 \qquad (14.19)$$

or

$$H = 2(d-1)\beta t(H)^3. \qquad (14.20)$$

The function $t(H)$ vanishes at $H = 0$. Thus $H = 0$ is always a solution of eq. (14.20). At high temperatures this is the only solution. At low temperatures further roots develop; however, in contrast to the Ising case, the root at the origin always represents a local minimum of F_{mf}. The potential term begins quarticly in H and thus the entropy piece will always dominate for small enough H. As β is increased another minimum develops at positive H. If β is large enough these new minima can be lower than

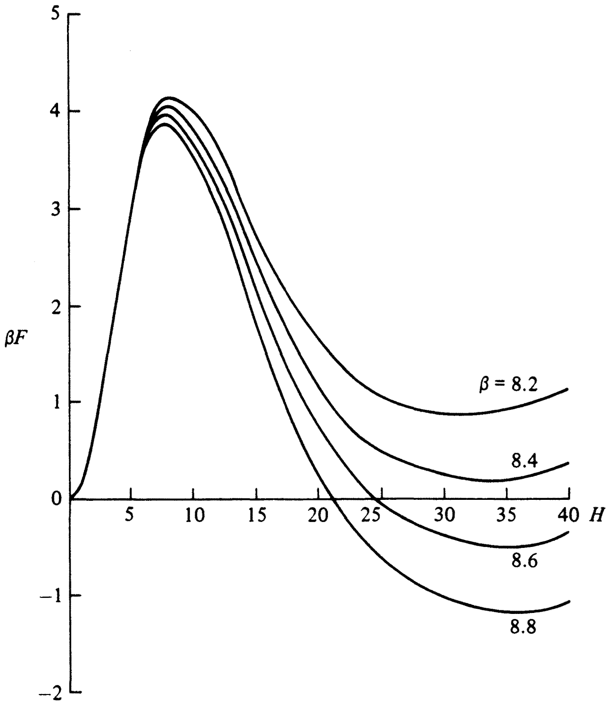

Fig. 14.3. The mean-field free energy as a function of H for several values of β with the group $SU(3)$ and $d = 4$.

the one at the origin. Here we see one dramatic difference from the Ising system; the transitions are predicted to be first order, with a discontinuous jump in the parameter H from zero to non-zero values. In figure 14.3 we plot the mean-field free energy as a function of H for several values of β for $SU(3)$ in four space-time dimensions.

Mean field theory predicts first-order phase transitions for all gauge groups. This prediction should only be trusted for large space-time dimensionality. If strong coupling arguments for confinement are to be relevant to the asymptotically-free continuum limit of lattice gauge theory,

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four dimensions must be inadequate for the mean field analysis to apply when the gauge group is $SU(3)$. In future chapters we will argue that this is the case. However, in four dimensions many gauge groups do give rise to phase transitions. Improved versions of mean field theory give rather accurate estimates for the transition temperatures (Flyvbjerg *et al.*, 1982).

Problems

1. Consider the partition function of eq. (14.1) with the variables s_i in the group $Z_3 = \{1, \exp(\pm 2\pi i)\}$ and a real part taken under the sum in the exponent. Show that mean field theory predicts a first-order transition for this model.
2. Prove the convexity inequality, eq. (14.7).