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A NOTE ON THE BANACH-MAZUR PROBLEM

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Abstract. We prove that if X is a real Banach space, with $\dim X \geq 3$, which contains subspace of codimension 1 which is 1-complemented in X and whose group of isometries is almost transitive then X is isometric to a Hilbert space. This partially answers the Banach-Mazur rotation problem and generalizes some recent related results.

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1. Introduction. In 1930's Banach and Mazur [**2**] (see also [**20**, Problem 9.6.2]) posed a problem whether every separable Banach space with transitive group of isometries has to be isometric to a Hilbert space. Here we say that the group of isometries of a Banach space X is *transitive* if for every $x, y \in X$ with $||x|| = ||y|| = 1$ there exists an isometry $T : X \overset{\text{onto}}{\longrightarrow} X$ such that

$$Tx = y.$$

Mazur [**16**] answered this problem positively for finite dimensional Banach spaces X and Pełczyński and Rolewicz [**18**] showed that the answer is negative when X is not assumed to be separable. The case of infinite dimensional separable spaces remains open despite active research in the area, see [**20**] and a recent survey [**9**].

Closely related to the notion of transitivity are the notions of almost transitivity and convex transitivity. We say that a group of isometries of X is *almost transitive* (resp. *convex transitive)* if for every x in the unit sphere of X, $S_X = \{x : ||x|| = 1\}$ the orbit of x i.e. the set $G_x = \{Tx : T \text{ isometry of } X\}$ is dense in S_X (resp. $\text{conv}(G_x)$ is dense in S_X). Sometimes we will abuse language and say that a space X is almost transitive (resp. convex transitive or transitive) provided the group of isometries of X is almost transitive (resp. convex transitive or transitive).

Spaces with almost transitive and convex transitive groups of isometries have been actively studied. It is known that there exist non-Hilbertian separable Banach spaces with almost transitive groups of isometries, for example $L_p[0, 1]$, $1 \leq p < \infty$, are such spaces [**18**], see also [**20**] and see [**10**] for detailed study which function spaces are almost transitive. Several questions have been posed to find additional conditions on a Banach space X which together with almost transitivity, or with just convex transitivity, imply that X is isometric to a Hilbert space. Maybe the most famous conjecture of this type is the conjecture of Wood [**23**] that if $C_0(L)$ is almost transitive in its natural supremum norm then L is a singleton, i.e. $C_0(L)$ is one dimensional. Wood's conjecture is still open despite recent active research in the area, see [**9**].

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The main theorem of the present paper is the following:

THEOREM 1.1. (see Theorem 2.7 below). *Suppose that X is a real Banach space, with $\dim X \geq 3$, which contains a 1-complemented hyperplane and whose group of isometries is almost transitive. Then X is isometric to a Hilbert space.*

Our method of proof relies on the theory of numerical ranges [**6**,**7**]. We postpone the proof to the next section and now we will discuss the connections with existing results in the literature.

Note that spaces $L_p[0, 1]$, $1 \leq p < \infty$, do not have 1-complemented hyperplanes (see e.g. [**12**,**19**]) and, as mentioned above, they are almost transitive.

Theorem 1.1 generalizes a recent result of Skorik and Zaidenberg:

THEOREM 1.2. [**22**] *Suppose that X is a real Banach space which contains an isometric reflection and whose group of isometries is almost transitive. Then X is isometric to a Hilbert space.*

Here we say that an operator T is *a reflection on* X if there exist $e \in X, e^* \in X^*$ with $e^*(e) = 1$ so that

$$T(x) = x - 2e * (x)e.$$

If this happens we write $T = S_{e,e^*}$, and if S_{e,e^*} is an isometry we say that e is an *isometric reflection vector* in X.

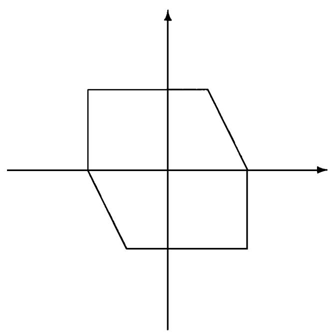
To see that Theorem 1.1 is more general than Theorem 1.2 we observe that if a space X admits an isometric reflection operator then X contains a 1-complemented hyperplane but not vice-versa.

Indeed, assume that for some $e \in S_X$ there exists $e^* \in X^*$ with $e^*(e) = 1$ so that an operator $S_{e,e^*} : x \mapsto x - 2e^*(x)e$ is an isometric reflection in X. Then for all $x \in X$ we have:

$$\|x - e^*(x)e\| = \left\|\frac{1}{2}((x - 2e^*(x)e) + x)\right\| \leq \frac{1}{2}\left(\|x - 2e^*(x)e\| + \|x\|\right) = \|x\|$$

Thus $x \mapsto x - e^*(x)e$ is a contractive projection in X onto the hyperplane $\mathrm{Ker}(e^*) \subset X$.

On the other hand consider the two dimensional real space X whose unit sphere is the convex hull of the points $(1, 0)$, $(1/2, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1/2, -1)$, $(-1, -1)$ as sketched below.



Then X contains no isometric reflections but clearly every hyperplane is 1-complemented in X.

It is not difficult to construct spaces of arbitrary dimension which contain no isometric reflections but which do contain 1-complemented subspaces of co-dimension 1.

Statements similar to Theorem 1.2 have been recently studied by J. Becerra Guerrero, F. Cabello Sanchez and A. Rodriguez Palacios. F. Cabello Sanchez [**8**] showed that Theorem 1.2 is valid in complex Banach spaces. J. Becerra Guerrero and A. Rodriguez Palacios linked this result with the following characterization of Hilbert spaces due to Berkson [**5**] and Kalton and Wood [**13**]:

THEOREM 1.3. *Let X be a complex Banach space. If $x \in X$ is such that* $\operatorname{span}\{x\}$ *is the range of a hermitian projection in X then x is called a hermitian element in X.*

If every nonzero element of X is hermitian in X then X is a Hilbert space.

J. Becerra Guerrero and A. Rodriguez Palacios [**3**] observed that an element $x \in X$ is hermitian in X if and only if x is an isometric reflection vector in X. Thus in the complex case we have the following stronger version of Theorem 1.2:

THEOREM 1.4. [**13**, Theorem 6.4] *Let X be a complex Banach space. If X is convex transitive and X contains an isometric reflection vector then X is a Hilbert space.*

J. Becerra Guerrero and A. Rodriguez Palacios generalized Theorem 1.2 as follows:

THEOREM 1.5. [**4**] *Let X be a real or complex Banach space. If there exists a nonrare set in S_X consisting of isometric reflection vectors then X is a Hilbert space.*

We do not know whether Theorem 1.1 can be generalized for convex transitive spaces (to obtain an analogue of Theorem 1.4). However the following example illustrates that an analogue of Theorem 1.5 for norm-one complemented hyperplanes fails in a very strong way. Namely we have:

EXAMPLE 1.6. For every $\varepsilon > 0$ there exists a 3-dimensional Banach space X_ε which is not isometric to a Hilbert space and such that the set F of functionals $f \in S_{X_\varepsilon^*}$ with $\operatorname{Ker} f$ norm-one complemented in X_ε is open in $S_{X_\varepsilon^*}$ and $\mu\left(S_{X_\varepsilon^*} \setminus F\right) < \varepsilon$, (here μ denotes the Lebesgue measure on $S_{X_\varepsilon^*} \subset \mathbb{K}^3, \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Moreover X_ε can be chosen to be uniformly convex.

Proof. Let $\varepsilon > 0$. Fix $\delta > 0$ so that

$$\mu\left(\{(x_1, x_2, x_3) \in S_{\ell_2} : |x_3| \geq 1 - \delta\}\right) < \varepsilon.$$

Let $\varphi : (\mathbb{R}_+)^3 \to \mathbb{R}$ be a convex continuous function such that

$$\varphi(t_1, t_2, t_3) \begin{cases} = 1 & \text{if } (t_1, t_2, t_3) = (0, 01) \\ = \sqrt{t_1^2 + t_2^2 + t_3^2} & \text{if } t_3 \leq (1 - \delta)\sqrt{t_1^2 + t_2^2 + t_3^2} \\ > \sqrt{t_1^2 + t_2^2 + t_3^2} & \text{if } t_3 > (1 - \delta)\sqrt{t_1^2 + t_2^2 + t_3^2} \end{cases}$$

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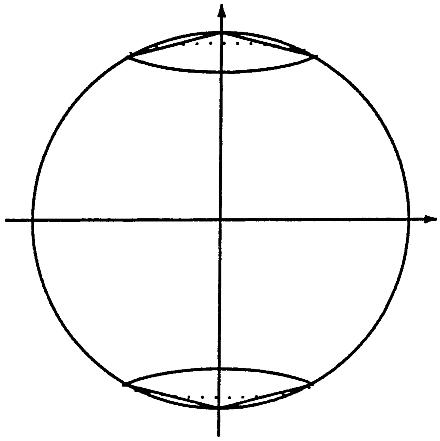
We can additionally require φ to have any desired degree of smoothness.

We define a norm on $\mathbb{K}^3$ using function φ:

$$\|(x_1, x_2, x_3)\|_{X_\varepsilon} = \varphi(|x_1|, |x_2|, |x_3|)$$

Then $S_{X_\varepsilon} \subseteq S_{\ell_2}$ and $S_{X_\varepsilon^*} \cap S_{l_2} \supset \{(x_1, x_2, x_3) \in S_{l_2} : |x_3| \leq 1 - \delta\}$ as illustrated in the figure below.



Thus, if $f \in S_{X_\varepsilon^*} \cap S_{\ell_2}$ then $\operatorname{Ker} f \cap S_{X_\varepsilon} \subset S_{\ell_2}$ and the orthogonal projection P onto $\operatorname{Ker} f$ has norm 1 in X_ε. Hence

$$F = \left\{f \in S_{X_\varepsilon^*} : \operatorname{Ker} f \text{ is 1-complemented in } X_\varepsilon\right\} \supset S_{X_\varepsilon^*} \cap S_{\ell_2}.$$

Thus $\mu\left(S_{X_\varepsilon^*} \setminus F\right) < \varepsilon$, as desired. □

Theorem 1.1 relies on the real analogue of the notion of hermitian elements (see Definitions 1 and 2 below) whose existence in X is equivalent to the existence of norm one complemented hyperplanes in X. We note here that our proof is much shorter than the existing proofs of Theorem 1.2.

2. Proofs of main results. We begin with definitions of real analogues of hermitian elements which were introduced by Kalton and the author [**12**] based on ideas of P. H. Flinn, cf. [**21**].

Definition 1. [**21**] We say that an operator $T : X \to X$ is *numerically positive* if for all $x \in X$ there exists a $x^* \in X^*$ or, equivalently, for all $x^* \in X^*$ with $\|x^*\|^2 = \|x\|^2 = x^*(x)$ and $x^*(Tx) \geq 0$, i.e. the numerical range of T is contained in $\mathbb{R}_+ \cup \{0\}$ (cf. [**6,14**]).

Definition 2. [**12**] (based on ideas of P. H. Flinn [**21**]) We say that $u \in X$ is *a Flinn element* if there exists a numerically positive projection $P : X \overset{\text{onto}}{\longrightarrow} \operatorname{span}\{u\}$ i.e. if there exists $f \in X^*$ with $f(u) = 1$ and such that the map $f \otimes u$, defined by $x \mapsto f(x)u$, is numerically positive. We say then that (u, f) is a *Flinn pair*.

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The set of all Flinn elements of X will be denoted by $\mathcal{F}(X)$.

We list few straightforward properties of Flinn elements which are important for the future use. We include their short proofs for completeness.

PROPOSITION 2.1. [**21**, Lemma 1.4] *A projection $P : X \to X, P \neq I$, is numerically positive if and only if $\|I - P\| = 1$.*

Proof. If $\|I - P\| = 1$ and $x \in X, x^* \in X^*$ are such that $1 = \|x^*\|^2 = \|x\|^2 = x^*(x)$, then

$$1 \geq x^*((I - P)(x)) = 1 - x^*(P(x)).$$

Thus $x^*(P(x)) \geqslant 0$ and P is numerically positive.

To see the implication in the other direction we rely on the result of Lumer and Phillips [**15**] that operator P is numerically positive if and only if $\|\exp(-tP)\| \leq 1$ for all real $t \geq 0$. We have for all real $t \geq 0$:

$$\exp(-tP) = I + \sum_{j=1}^{\infty} \frac{(-t)^j P^j}{j!} = I + \left(\sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \right) = I + \left(e^{-t} - 1\right)P,$$

where the second equality holds because P is a projection. Thus by result of Lumer and Phillips if P is numerically positive we have

$$\|I - P\| = \lim_{t \to \infty} \|\exp(-tP)\| \leq 1$$

Since $P \neq I$ and $I - P$ is a projection we get $\|I - P\| = 1$. □

PROPOSITION 2.2. [**12**, Proposition 3.2] *Suppose that $T : X \to X$ is a surjective isometry. Then $T(\mathcal{F}(X)) = \mathcal{F}(X)$.*

Proof. If $u \in \mathcal{F}(X) \setminus \{0\}$ then there exists $f \in X^*$ such that the projection $P : X \to \text{span}\,\{u\}$ defined by $x \mapsto f(x)u$ is numerically positive. Then the projection $Q : x \mapsto \left((T^*)^{-1} f\right)(x).T(u)$ establishes the fact that $T(u)$ is Flinn. □

PROPOSITION 2.3. [**12**, Proposition 3.1] *The set $\mathcal{F}(X)$ is closed.*

Proof. Suppose $u_n \in \mathcal{F}(X)$ with $\lim\|u_n - u\| = 0$. It suffices to consider the case when $\|u_n\| \neq 0$ and $\|u\| \neq 0$. Then there exist $f_n \in X^*$ so that $f_n \otimes u_n$, is a numerically positive projection. Thus $\|f_n \otimes u_n\| = \|f_n\|\|u_n\| \leq 2$. Thus $\|f_n\| \leq 2\sup(1/\|u_n\|)$. By Alaoglu's theorem (f_n) has a weak*-cluster point f and clearly (u, f) is a Flinn pair. □

PROPOSITION 2.4. *Suppose $u \in \mathcal{F}(X)$ and that Y is a subspace of X such that $u \in Y$. Then $u \in \mathcal{F}(Y)$.*

Proof. Without loss of generality $u \neq 0$. Since $u \in F(X)$ there exists $f \in X^*$ with $f(u) = 1$ and such that the map $x \mapsto f(x)u$ is numerically positive. Consider $g = f|_Y \in Y^*$ and the map $Q : Y \to \text{span}\{u\} \subset Y$ defined by

$$Q(y) = g(y)u$$

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By Hahn-Banach Theorem for every $y \in Y$ and every $y^* \in Y^*$ with $\|y^*\|^2 = \|y\|^2 = y^*(y)$ there exists $\tilde{y}^* \in X^*$ with $\|\tilde{y}^*\| = \|y^*\| = \|y\|$ and $\tilde{y}^*|_Y = y^*$. Thus we get, since (u, f) is a Flinn pair in X:

$$g(y)y^*(u) = f(y)\tilde{y}^*(u) \geq 0$$

and $g(u) = f(u) = 1$. Hence (u, g) is a Flinn pair in Y. □

Now we are ready for the real analogue of Theorem 1.3 ([**5**, Theorem 2.22], [**13**, Corollary 4.4]).

THEOREM 2.5. *Suppose that X is a real Banach space* $\dim X \geq 3$ *and* $(X) = X$. *Then X is isometric to a Hilbert space.*

Proof. Since Hilbert spaces are characterized by the parallelogram identity it is enough to show that the result holds for all 3-dimensional subspaces of X (cf. [**1**, (1.4')]). Suppose that $Y \subset X$ is a real Banach space with $\dim Y = 3$. Then, by Proposition 2.4, $\mathcal{F}(Y) = Y$ i.e. for every $u \in Y$ there exists an $f \in Y^*$ such that $f(u) = 1$ and the map $f \otimes u$ is a numerically positive projection in Y, and also in Y^*.

Hence, by Proposition 2.1, the map $I - f \otimes u$ is a contractive projection of Y^* onto Ker $u \subset Y^*$. Since $\dim Y = 3$, we conclude that every 2-dimensional subspace of Y^* is contractively complemented. Thus Y^* is isometric to a Hilbert space by the following criterion due to Kakutani:

THEOREM 2.6. [**11**] (see also [**1**, p. 99]) *Suppose that Z is a Banach space of dimension at least 3. Then Z is isometric to a Hilbert space if and only if for every 2-dimensional subspace F of Z there is a norm one linear projection $P : Z \to F$.*

Therefore Y is isometric to Hilbert space and the proof is finished. □

Next is our main theorem.

THEOREM 2.7. *Suppose that X is a real Banach space, with* $\dim X \geq 3$, *which contains a 1-complemented hyperplane and whose group of isometries is almost transitive. Then X is isometric to a Hilbert space.*

Proof. When X contains a subspace of codimension 1 which is 1-complemented in X then, by Proposition 2.1, X contains a nonzero Flinn element. Let $u \in S_X$ be a Flinn element. By Proposition 2.2

$$G_u = \left\{ Tu : T \text{ isometry of } X \text{ onto } X \right\} \subset \mathcal{F}(X)$$

and by almost transitivity of X, G_u is dense in S_X. Since, by Proposition 2.3, $\mathcal{F}(X)$ is closed we obtain:

$$F(X) \cap S_X = S_X.$$

Thus, $\mathcal{F}(X) = X$ and the result follows by Theorem 2.5. □

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Added in proof. After this paper has been completed and accepted for publication P.L. Papini has pointed out to me the reference **[17]** which contains a result analogous to Theorem 1.1 with the additional assumptions that X is reflexive and transitive.

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