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# STOCHASTIC MONOTONICITY AND DUALITY OF *k*TH ORDER WITH APPLICATION TO PUT–CALL SYMMETRY OF POWERED OPTIONS

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**Abstract**

We introduce a notion of $k$th order stochastic monotonicity and duality that allows us to unify the notion used in insurance mathematics (sometimes refereed to as Siegmund's duality) for the study of ruin probability and the duality responsible for the so-called put–call symmetries in option pricing. Our general $k$th order duality can be interpreted financially as put–call symmetry for powered options. The main objective of this paper is to develop an effective analytic approach to the analysis of duality that will lead to the full characterization of $k$th order duality of Markov processes in terms of their generators, which is new even for the well-studied case of put–call symmetries.

*Keywords:* Stochastic monotonicity; stochastic duality; generators of dual processes; dual semigroup; put–call symmetry and reversal; powered and digital options; straddle

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## 1. Introduction

### 1.1. Main objectives

A real-valued Markov process $X_t^x$ is called stochastically monotone if $\mathbb{P}(X_t^x \geq y)$ is a nondecreasing function of $x$ for any $y$ and any $t$. Siegmund's theorem (see [35]) states that if $X_t^x$ is stochastically monotone and $\mathbb{P}(X_t^x \geq y)$ is a right-continuous function of $x$ for any $y$, then there exists a Markov process $Y_t^y$, called dual to $X_t^x$ such that

$$\mathbb{P}(Y_t^y \leq x) = \mathbb{P}(X_t^x \geq y)$$

holds. This condition can also be rewritten as

$$\mathbb{E}\theta(x - Y_t^y) = \mathbb{E}\theta(X_t^x - y), \tag{1}$$

where $\theta$ is the step function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

In the theory of option pricing, Markov processes $X_t^x$ and $Y_t^y$ are said to satisfy the put–call symmetry relation if the following holds:

$$\mathbb{E}(x - Y_t^y)_+ = \mathbb{E}(X_t^x - y)_+. \tag{2}$$

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Comparing (1) and (2) suggests to us the introduction of a general notion that includes these two dualities as particular cases. Namely, let us say that a Markov process $Y_t^y$ is dual to $X_t^x$ of order $k$, $k \in \mathbb{R}$, if

$$\mathbb{E}(x - Y_t^y)_+^{k-1} = \mathbb{E}(X_t^x - y)_+^{k-1}. \tag{3}$$

The cases of $k = 1$ and $k = 2$ correspond to (1) and (2), respectively (with a natural convention that $x_+^0 = \theta(x)$). These dualities also have a clear financial interpretation describing symmetries between powered European options, where the case of $k = 1$ stands for a symmetry between digital options.

The aim of this paper is to fully characterize Markov processes that satisfy (3) in terms of their generators, paying special attention to processes that are martingales, as such processes appear in a risk-neutral evaluation. This characterization appears to be new even for the standard put–call symmetry (2), however, the important particular cases of underlying price processes being Lévy processes or processes with a price independent compensator are well studied; see [1], [19]. We shall also extend the theory to time nonhomogeneous processes, which is a related notion of duality referred to as the put–call reversal [1].

We will not pay attention to the positivity of our martingales (which should of course be the case for realistic price processes), as this problem can be handled separately from the discussion of duality, either by ensuring that the origin is not attainable, or by directly working with exponents.

We will also not address the issues arising at boundary points as this development is treated separately in [28], in connection to the problems from insurance mathematics, where this issue becomes crucial (for example, the ruin problem; see [2], [3], [4], [14], [36]), because the absorption rates for an attainable origin become the most important quantities to study.

### 1.2. Plan of the paper

In Section 2 we introduce our analytic approach for the analysis of the duality of Markov processes via their generators. In Section 3 we present some of the simplest examples of duality that arise from our results. In Section 4 we extend the notion of stochastic monotonicity to arbitrary orders and prove the corresponding extension of Siegmund's theorem linking stochastic monotonicity and duality. In Section 5 we obtain our main results for the characterization of the duality of one-dimensional Markov processes via their generators. The final section is devoted to the extension of the theory to time-nonhomogeneous Markov processes. In Appendix A we summarize, in appropriate form, some crucial facts about fractional derivatives used in the main text.

### 1.3. Bibliographical comments

The duality of Markov processes is an important topic in probability; see, e.g. [29] for an extensive introduction to the subject, with a special emphasis on interacting particles, and [12] for a related study of stochastic monotonicity. Siegmund [35] initiated systematic research into the duality of Markov processes based on stochastic order. For crucial applications of duality in superprocesses, see [18] and [31]. The duality for general recursions and the duality for discrete Markov chains are developed in [5] and [24], respectively. For stochastic monotonicity and duality of birth and death processes, see [37].

For a general introduction to intertwining and for many examples related to Lévy processes; see [8], [9], [15], [23], [32], and the references therein.

The research into put–call symmetry was initiated in [6] and [7] and since then has attracted a lot of attention. We refer the reader to [11], [17], and [30] for detailed reviews of recent developments. Let us specifically mention [1] and [19], where put–call symmetry was analyzed for markets based on diffusions with price independent jumps and Lévy processes, respectively. The theory for American options was developed in [10] and the theory for Asian options in [21] and [22]. The symmetry in terms of semimartingale characteristics of general dual semimartingales related by the dual martingale measures was characterized in [17]. An important recent development concerns the study of the quasi self-dual process, which relates the conditional symmetry properties of both their ordinary as well as their stochastic logarithms; see [33], [34].

For the application to insurance mathematics; see [2], [3], [14], [36], and the references therein.

The approach to the study of Siegmund's duality via generators was initiated in [25] and continued in [27]; see also monograph [26].

## 2. Analytic approach to the analysis of duality

### 2.1. Definition of stochastic duality

Let us first recall the standard definition of duality of Markov processes. Let $X_t^x$ and $Y_t^y$ be two Markov processes (superscript $x, y$ denote the initial points) with values in possibly different Borel spaces $X$ and $Y$. Then $Y$ is called dual to $X$ with respect to a Borel function $f$ on $X \times Y$, or $f$-dual in shorthand, if

$$\mathbb{E}f(x, Y_t^y) = \mathbb{E}f(X_t^x, y) \tag{4}$$

for all $x \in X$, $y \in Y$, where $\mathbb{E}$ on the left-hand side and the right-hand side correspond to the distributions of the processes $Y_t^y$ and $X_t^x$, respectively.

An important example is given by the duality equation

$$\mathbb{P}(Y_t^y \leq x) = \mathbb{P}(X_t^x \geq y),$$

where $\geq$ is a partial order. This is a particular case of (4) with $f(x, y) = \mathbf{1}_{\{x \geq y\}}$ (denote by $\mathbf{1}_M$ the indicator function of the set $M$).

From the point of view of the general definition of $f$-duality, the duality of $k$th order given in (3) corresponds to $f_k$-duality for $f_k(x, y) = (x - y)_+^{k-1}$, where $x_\pm = \max(0, \pm x)$.

### 2.2. The analytic counterpart of duality

For a metric space $X$ we denote by $B(X)$, $C(X)$, and $\mathcal{M}(X)$ the Banach spaces of bounded measurable functions, bounded continuous functions, and bounded signed Borel measures, respectively, where the first two spaces are equipped with the sup-norm and the last space with the total variation norm. If $X$ is locally compact then $C_\infty(X)$ denotes the closed subspace of $C(X)$ of functions vanishing at $\infty$. The standard duality between $B(X)$ and $\mathcal{M}(X)$ is given by the integration $(f, \mu) = \int_X f(x)\mu(\mathrm{d}x)$.

By a signed (stochastic) kernel from $X$ to $Y$ we mean, as usual, a function of two variables $p(x, A)$, where $x \in X$ and $A$ are Borel subsets of $Y$ such that $p(x, \cdot)$ is a bounded signed measure on $Y$ for any $x$ and $p(\cdot, A)$ is a Borel function for any Borel set $A$. We say that this kernel is bounded if $\sup_x \|p(x, \cdot)\| < \infty$.

Any bounded kernel specifies an integral operator $B(Y) \to B(X)$ via

$$Ug(x) = \int_Y g(z)p(x, \mathrm{d}z).$$

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The standard dual operator $U'$ is defined as the operator $\mathcal{M}(X) \to \mathcal{M}(Y)$ specified by the duality relation

$$(f, U'\mu) = (Uf, \mu),$$

or explicitly as

$$U'\mu(\mathrm{d}y) = \int_X p(x, \mathrm{d}y)\mu(\mathrm{d}x).$$

A bounded linear operator $U^{D(f)}$ in $B(Y)$ (or $C(Y)$ or $C_\infty(Y)$) is said to be $f$-dual to a bounded linear operator $U$ in $B(X)$ (or $C(X)$ or $C_\infty(X)$) if, for any $x$, $y$,

$$(U^{D(f)} f(x, \cdot))(y) = (Uf(\cdot, y))(x). \tag{5}$$

Let us say that a function $f$ on $X \times Y$ *separates measures on* $X$ if, for any $Q_1, Q_2 \in \mathcal{M}(X)$, there exists $y \in Y$ such that $\int f(x, y)Q_1(\mathrm{d}x) \neq \int f(x, y)Q_2(\mathrm{d}x)$. If this is the case then the integral operator $F = F_f : \mathcal{M}(X) \to B(Y)$ given by

$$(FQ)(y) = \int f(x, y)Q(\mathrm{d}x) \tag{6}$$

is an injective bounded operator so that the linear inverse $F^{-1}$ is defined on the image $F(\mathcal{M}(X))$. Let us say that the function $FQ$ is $f$-generated by $Q$.

Our analysis will be based on the following simple but crucial observation.

**Proposition 1.** *Let $f$ be a bounded measurable function separating the measures on $X$. Let $U$ be an integral operator in $B(X)$ with a bounded signed kernel $p$.*

(i) *Suppose that $U^{D(f)}$ is an integral operator with a bounded kernel $p^{D(f)}(y, \mathrm{d}z)$ satisfying (5). Then the action of $U^{D(f)}$ on $F(\mathcal{M}(X))$ is given by*

$$U^{D(f)} = F \circ U' \circ F^{-1}, \tag{7}$$

*or, equivalently, $U^{D(f)}$ satisfies the intertwining relation*

$$U^{D(f)} \circ F = F \circ U'.$$

(ii) *Let us define an operator $U^{D(f)}$ on $F(\mathcal{M}(X))$ by the linear extension of relation (5), that is, by*

$$(U^{D(f)} F(Q))(y) = \left(U^{D(f)} \int_X f(x, \cdot)Q(\mathrm{d}x)\right)(y) = \int_X (Uf(\cdot, y))(x)Q(\mathrm{d}x). \tag{8}$$

*Then $U^{D(f)}$ is well defined on $F(\mathcal{M}(X))$ and (7) holds.*

*Proof.* (i) Let $g \in F(\mathcal{M}(X))$ be given by $g(y) = \int f(x, y)Q_g(\mathrm{d}x)$. Then

$$U^{D(f)} g(y) = \int_Y g(z)p^{D(f)}(y, \mathrm{d}z),$$

which, by Fubini's theorem, rewrites as

$$U^{D(f)} g(y) = \int_X \left(\int_Y f(x, z)p^{D(f)}(y, \mathrm{d}z)\right)Q_g(\mathrm{d}x),$$

and, consequently, as (8) with $Q = Q_g$. Hence,

$$U^{D(f)}g(y) = \int_X \int_Y f(z, y)p(x, \mathrm{d}z)Q_g(\mathrm{d}x) = \int_Y f(z, y)\tilde{Q}(\mathrm{d}z),$$

with

$$\tilde{Q}(\mathrm{d}z) = \int p(x, \mathrm{d}z)Q_g(\mathrm{d}x).$$

Thus, $U^{D(f)}g$ is $f$-generated by $\tilde{Q} = U'Q_g$, as required.

(ii) Instead of using Fubini's theorem, we start with (8) by definition. The remaining calculations are the same.

**Remark 1.** For discrete Markov chains, Proposition 1 was proved in [24].

### 2.3. Application to semigroups and Markov processes

Representation (7) has the following direct implication for the theory of semigroups.

**Proposition 2.** (i) *Let $f$ be a bounded measurable function separating the measures on $X$. Let $T_t$ be a semigroup of integral operators in $B(X)$ (or $C(X)$, or $C_\infty(X)$) specified by the family of bounded signed kernel $p_t(x, \mathrm{d}z)$ from $X$ to $X$. Then the dual operators $T_t^{D(f)}$ (defined by (8) with $U = T_t$) in $F(\mathcal{M}(X))$ also form a semigroup and*

$$T_t^{D(f)} = F \circ T_t' \circ F^{-1}. \tag{9}$$

(ii) *If the semigroup $T_t$ is generated by an operator $L$ in $C(X)$ that is defined on some invariant (under all $T_t$) domain $D$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T_t^{D(f)}g = F \circ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T' \circ F^{-1}g = F \circ L' \circ F^{-1}g,$$

*that is, the generator of the semigroup $T_t^{D(f)}$ is*

$$L^{D(f)} = F \circ L' \circ F^{-1}, \tag{10}$$

*with the domain containing the image (under $F$) of the domain of $L'$.*

*Proof.* (i) This is straightforward from (7) and the standard, obvious fact that $T_t'$ forms a semigroup in $\mathcal{M}(X)$. Part (ii) follows from (i).

### 2.4. Duality for $f$ depending on the difference of its arguments

The theory essentially simplifies if $f$ is translation invariant, that is $f$ depends only on the difference of its arguments, $f(x, y) = f(y-x)$, with some other function $f$ that we still denote by $f$ (with some ambiguity). In this case, the operator $F$ from (6), applied to a measure $Q$ with density $q$, takes the form

$$g(y) = (FQ)(y) = \int_{\mathbb{R}^d} f(y-x)q(\mathrm{d}x), \tag{11}$$

That is, it becomes a convolution operator. It is well known that under appropriate regularity assumptions, $f$ is the fundamental solution of the pseudodifferential operator $L_f$ with the symbol

$$L_f(p) = \frac{1}{\hat{f}(p)},$$

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where

$$\hat{f}(p) = \int \mathrm{e}^{-\mathrm{i}xp} f(x)\,\mathrm{d}x$$

is the Fourier transform of $f$. Hence, $g(y)$ from (11) solves the equation $L_f g = q$, so that $F^{-1} = L_f$. Multidimensional examples with differential operators $L_f$ are given in [28].

## 3. Simplest examples

Let $X_t^x$ be the stable-like Markov process with the Feller semigroup generated by the operator

$$Lg(x) = \mp a(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}, \qquad k \in (0, 2], \tag{12}$$

with a nonnegative continuously differentiable function $a(x)$ (see Appendix A for the definition of fractional derivatives and integral operators used here and in what follows), where the signs $\mp$ correspond to the cases $k \in (0, 1)$ and $k \in (1, 2]$, respectively (for the trivial case $k = 1$ the sign is nonessential). Then, by (10), the dual generator of order $k$ is given by

$$L^{D_k} = \mp I_k^+ \frac{\mathrm{d}^k}{\mathrm{d}(-x)^k} a(x) \frac{\mathrm{d}^k}{\mathrm{d}(-x)^k} = \mp a(x)\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}$$

(where we used definition (43) and the properties of $I_k$ discussed before (44)), so the dual process to $X_t^x$ of order $k$ is the process $Y_t^y$ generated by

$$L^{D_k} g(x) = \mp a(x)\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}, \tag{13}$$

leading to the following result.

**Proposition 3.** *Let $X_t^x$, $Y_t^y$ be Markov processes generated by (12) and (13), respectively, with $k \in (0, 2]$. Then*

$$\mathbb{E}(x - Y_t^y)_+^{k-1} = \mathbb{E}(X_t^x - y)_+^{k-1}.$$

In financial terms this means that the price of the European powered call option for the initial stock price $x$ and the strike $y$ equals the price of the European powered put option for the initial stock price $y$ and the strike $x$ (discounting is assumed to be already included in the definition of processes $X_t^x$ and $Y_t^y$).

The most important cases are with $k \in (1, 2)$, since the corresponding Markov processes $X_t^x$, $Y_t^y$ are martingales, and, thus, the expectation corresponds to a risk-neutral evaluation.

The case of the diffusions, that is $k = 2$ is well known; see, e.g. [1].

Similarly, if $X_t^x$ is generated by

$$Lg(x) = -a(x)\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^k, \tag{14}$$

with $k \in (0, 2]$ then the dual with respect to the function $f(x, y) = |x - y|^k$ has the generator

$$L^D = -\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^{-k}\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^k a(x)\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^k = -a(x)\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^k,$$

which coincides with $L$. Consequently, $X_t^x$ is self-dual in this sense, leading to the following.

**Proposition 4.** *Let $X_t^x$ be the Markov process generated by (14) with $k \in (0, 2]$. Then*

$$\mathbb{E}|y - X_t^x|^{k-1} = \mathbb{E}|X_t^y - x|^{k-1}.$$

In the financial interpretation this means the self-symmetry of powered straddle spreads. Similarly, one can analyze symmetries linking various option spreads, though the conditions for underlying Markov processes can become rather restrictive. For instance, let us consider a symmetry related to the so-called bull put spread, whose premium has the form (up to a linear equivalence)

$$f_{\alpha,\beta}(x, y) = (x - y + \alpha)_+ - (x - y + \beta)_+$$

(the powered version can be analyzed analogously). The corresponding operator $F$ from (6) can be taken as $F = (T_\alpha - T_\beta) I_2^+$, where $T_c f(x) = f(x + c)$ denotes the shift, so that

$$F^{-1} = \sum_{m=0}^{\infty} T_{\beta-\alpha}^m T_\alpha^{-1} \frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Hence, for $L = a(x)\,\mathrm{d}^2/\mathrm{d}x^2$, we obtain

$$L^D g(x) = a(x + \alpha) \sum_{m=0}^{\infty} T_{\beta-\alpha}^m - a(x + \beta) \sum_{m=1}^{\infty} T_{\beta-\alpha}^m,$$

which equals $L$ if $a(x)$ is a $(\beta - \alpha)$-periodic function. In this case we obtain the duality relation

$$\mathbb{E} f_{\alpha,\beta}(x, X_t^y) = \mathbb{E} f_{\alpha,\beta}(X_t^x, y).$$

## 4. Stochastic monotonicity and duality

Let us say that a Markov process $X_t^x$ with transition probabilities $p_t(x, \mathrm{d}z)$ is stochastically monotone of order $k > 0$, if for any $t > 0$, $y \in \mathbb{R}$, the derivative

$$\frac{\partial^k}{\partial x^k} \mathbb{E}(X_t^x - y)_+^{k-1} = \frac{\partial^k}{\partial x^k} \int_{z \geq y} (z - y)_+^{k-1} p_t(x, \mathrm{d}z) \tag{15}$$

exists in the sense of distribution and is a positive measure (this includes the assumption that $\mathbb{E}(X_t^x - y)_+^{k-1}$ is finite for all $y$). If $k \geq 1$ then an equivalent requirement is that, for any $t > 0$, $y \in \mathbb{R}$, the derivative

$$\frac{\partial^{k-1}}{\partial x^{k-1}} \mathbb{E}(X_t^x - y)_+^{k-1} = \frac{\partial^{k-1}}{\partial x^{k-1}} \int_{z \geq y} (z - y)_+^{k-1} p_t(x, \mathrm{d}z)$$

exists in the sense of distribution and is a nondecreasing function of $x$.

**Remark 2.** (i) If $k \geq 2$, this can be reformulated avoiding generalized functions by saying that the derivative

$$\frac{\partial^{k-2}}{\partial x^{k-2}} \int_{z \geq y} (z - y)_+^{k-1} p_t(x, \mathrm{d}z)$$

exists as an absolutely continuous function such that its first derivative (defined almost surely) is a nondecreasing function of $x$. (ii) We can also formulate the notion of stochastic monotonicity of arbitrary order, avoiding generalized derivatives, in terms of the positivity of the increments of $k$th order of the function $\mathbb{E}(X_t^x - y)_+^{k-1}$ (as a function of $y$).

The usual stochastic monotonicity corresponds to $k = 1$. The following result extends Siegmund's theorem to monotonicity of higher orders.

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**Theorem 1.** *A real-valued Markov process $X_t^x$ with transition probabilities $p_t(x, \mathrm{d}z)$ and semigroup $T_t$ has a Markov dual of order $k \geq 1$ if and only if it is stochastically monotone of order $k$ satisfies the limiting relation*

$$\frac{\partial^{k-1}}{\partial x^{k-1}} \int_{z \geq y} \frac{(z-y)_+^{k-1}}{\Gamma(k)} p_t(x, \mathrm{d}z) \to \begin{cases} 1, & x \to \infty, \\ 0, & x \to -\infty \end{cases} \tag{16}$$

*for all $y$ and, if $k = 1$, the function $\int_{z \geq y} p_t(x, \mathrm{d}z) = \mathbb{P}(X_t^x \geq y)$ is right continuous.*

*Proof.* Let us first analyze Siegmund's case, $k = 1$, from our point of view. Then the mapping $F\colon \mathcal{M}(\mathbb{R}) \to B(\mathbb{R})$ given by the corresponding equation (6) becomes the usual integration, that is,

$$FQ(y) = \int \theta(x-y) Q(\mathrm{d}x) = \int_{x \geq y} Q(\mathrm{d}x),$$

whose image consists of the left-continuous functions (because we defined $\theta$ to be right continuous) of uniformly bounded variation (the total variation of $FQ$ being equal to the total variation norm of $Q$) tending to 0 at $+\infty$. By (9), for a $g = FQ$ with a finite measure $Q = -\mathrm{d}g$, the corresponding dual semigroup becomes

$$T_t^D g(y) = F \circ T_t' \circ F^{-1} g(y) = -\int \biggl( \int_{z \geq y} p_t(w, \mathrm{d}z) \biggr) \mathrm{d}g(w).$$

Integrating by parts (note that here it is crucial that the functions $\int_{z \geq y} p_t(x, \mathrm{d}z)$ and $g$ are right continuous and left continuous, respectively; see (46) with $k = 1$), this can be wrtitten as

$$T_t^D g(y) = \int g(w) \, \mathrm{d}_w \int_{z \geq y} p_t(w, \mathrm{d}z), \tag{17}$$

where $\mathrm{d}_w \int_{z \geq y} p_t(w, \mathrm{d}z) = \mathrm{d}_w \mathbb{P}(X_t^w \geq y)$ is the Stiltjes measure of the increasing function $\mathbb{P}(X_t^w \geq y)$. Equation (17) defines an integral operator with a positive stochastic kernel, which, taking into account assumption (16), is in fact a probability kernel. Hence, this operator extends naturally to a positivity-preserving conservative contraction in $B(\mathbb{R})$ thus defining a Markov transition operator. Finally, the Markov property (which is now equivalent to the Chapman–Kolmogorov equation or to the semigroup property of the operators $T_t^{D(f)}$) follows from Proposition 2.

Now let $k > 1$. The corresponding operator $F$, given by (6), becomes (up to a constant multiplier) the integration operator $I_k^+$ (see (43) from Appendix A) and the corresponding function $f$ specifying duality is $(x-y)_+^{k-1}$. Assuming $g$ belongs to the image $\mathcal{I}_k^+$ of $I_k^+$ (see the discussion after (43)) so that

$$(I_k^+)^{-1} g(y) = \frac{\mathrm{d}^k g(w)}{\mathrm{d}(-w)^k} = Q(\mathrm{d}w)$$

is a measure from $\mathcal{M}_k^+$, we can integrate by parts (using (46)) in the equation

$$\begin{aligned} T_t^{D_k} g(y) &= I_k^+ \circ T_t' \circ (I_k^+)^{-1} g(y) \\ &= \int_{z \geq y} \frac{(z-y)_+^{k-1}}{\Gamma(k)} \biggl( \int_{w \in \mathbb{R}} p_t(w, \mathrm{d}z) \frac{\mathrm{d}^k g(w)}{\mathrm{d}(-w)^k} \biggr) \\ &= \int \biggl( \int_{z \geq y} \frac{(z-y)_+^{k-1}}{\Gamma(k)} p_t(w, \mathrm{d}z) \biggr) \frac{\mathrm{d}^k g(w)}{\mathrm{d}(-w)^k} \end{aligned}$$

(where the corresponding dual operators are marked by the subscript $D_k$) leading to

$$T_t^{D_k} g(y) = \int g(w) \frac{\partial^k}{\partial w^k} \biggl( \int_{z \geq y} \frac{(z-y)_+^{k-1}}{\Gamma(k)} p_t(w, \mathrm{d}z) \biggr).$$

This equation can be used to define a natural extension of (17) (initially defined as a mapping $\mathcal{I}_k^+ \to B(\mathbb{R})$) as a positive integral operator. The proof is now completed as in the $k = 1$ case.

For $k \in (0, 1)$ we assume a bit more regularity on the initial process $X_t^x$, which, on the one hand, is enough for most of the applications and, on the other hand, allows us to avoid rather subtle measure-theoretic problems. Consequently, the next result is obtained by the same proof as for $k > 1$ above.

**Theorem 2.** *Suppose that a real-valued Markov process $X_t^x$ has bounded transition probability densities $p_t(x, z)$ for $t > 0$. Then $X_t^x$ has a Markov dual of order $k \in (0, 1)$ if and only if it is stochastically monotone of order $k$ and measure (15) has the total mass* 1.

## 5. Characterization of duality in terms of generators

In this section we obtain our main results. Namely, using the formula for $f$-dual generators (10) we explicitly calculate the dual for an arbitrary Feller process. Let us first consider diffusions. Moreover, for clarity, we will consider separately the cases of integer and real $k$.

For an integer $k$, we will denote by $C^k$ the space of $k$ times continuously differential functions on $\mathbb{R}$ (with bounded derivatives). In what follows we are not aiming at the weakest possible assumption on the coefficients $a$, $b$, but assume as much regularity as needed to obtain the most transparent equations for dual operators. Also, the convenient assumption of boundedness can be relaxed by using the theory of diffusions with unbounded coefficients.

Let us consider a Feller diffusion $X_t^x$ with the Feller semigroup $T_t$ generated by the operator

$$Lg(x) = a(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x) \frac{\mathrm{d}}{\mathrm{d}x} \tag{18}$$

with a nonnegative function $a(x)$.

The following fact is a particular case of a more general multidimensional result from [28]. If $a, b \in C^1$ then the Markov dual process $Y_t^x$ of order $k = 1$ exists and is a diffusion generated by the operator

$$L^{D_k} g(x) = a(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + [a'(x) - b(x)] \frac{\mathrm{d}}{\mathrm{d}x}.$$

This allows us to exclude $k = 1$ from the following arguments.

**Theorem 3.** *Suppose that $k > 1$ is an integer and $a, b \in C^k$. The diffusion $X_t^x$ generated by (18) is stochastically monotone of order $k$ if and only if the function*

$$\omega_y(x) = \frac{\partial^{k-1}}{\partial x^{k-1}} \biggl[ \frac{(x-y)^{k-2}}{\Gamma(k-1)} b(x) + \frac{(x-y)^{k-3}}{\Gamma(k-2)} a(x) \mathbf{1}_{\{k \neq 2\}} \biggr]$$

*is a nondecreasing function of $x \geq y$ for any $y$. If this is the case and, additionally,*

$$\lim_{x \to +\infty} \omega_y(x) = -(k-1)b'(y) - \tfrac{1}{2}(k-1)(k-2)a''(y) \tag{19}$$

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*for all y, then the kth order Markov dual process $Y_t^y$ exists and is generated by the operator*

$$L^{D_k}g(y) = a(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} - [b(y) + (k-2)a'(y)]\frac{\mathrm{d}}{\mathrm{d}y} + \int_y^\infty (g(x) - g(y))\frac{\partial^k}{\partial x^k}\left[\frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x) + \frac{(x-y)^{k-3}}{\Gamma(k-2)}a(x)\mathbf{1}_{\{k\neq 2\}}\right]\mathrm{d}x. \quad (20)$$

**Remark 3.** If we have the inequality $\leq$ rather than the equality in (19), then the dual $Y_t^y$ exists as a sub-Markov process.

*Proof of Theorem 3.* Let us perform the calculations separately for the drift and diffusion parts of $L$ starting with the drift part. By (10) and (44) we have

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -I_k^+\left(\frac{\mathrm{d}}{\mathrm{d}y}\circ b(y)\right)\circ\frac{\mathrm{d}^k}{\mathrm{d}(-y)^k}g(y) = \int_y^\infty \frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x)\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\,\mathrm{d}x. \quad (21)$$

Integrating by parts, first $k-2$ times (where boundary terms cancel) and then twice more yields

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = \int_y^\infty \frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}\left[\frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x)\right]\frac{\mathrm{d}^2 g(x)}{\mathrm{d}x^2}\,\mathrm{d}x = -b(y)g'(y) + (k-1)b'(y)g(y) + \int_y^\infty g(x)\frac{\partial^k}{\partial x^k}\left[\frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x)\right]\mathrm{d}x, \quad (22)$$

which can be written as

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -b(y)g'(y) + \int_y^\infty (g(x)-g(y))\frac{\partial^k}{\partial x^k}\left[\frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x)\right]\mathrm{d}x + g(y)\left[(k-1)b'(y) + \lim_{x\to\infty}\frac{\partial^{k-1}}{\partial x^{k-1}}\left[\frac{(x-y)^{k-2}}{\Gamma(k-1)}b(x)\right]\right],$$

if the last limit is finite.

Similarly, if $k \neq 2$, it follows that

$$\left(a(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)^{D_k} g(y) = I_k^+\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}\circ a(y)\right)\circ\frac{\mathrm{d}^k}{\mathrm{d}(-y)^k}g(y) = \int_y^\infty \frac{(x-y)^{k-3}}{\Gamma(k-2)}a(x)\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\,\mathrm{d}x, \quad (23)$$

which, by the integration by parts, can be rewritten as

$$-\int_y^\infty \frac{\mathrm{d}^{k-3}}{\mathrm{d}x^{k-3}}\left[\frac{(x-y)^{k-3}}{\Gamma(k-2)}a(x)\right]\frac{\mathrm{d}^3 g(x)}{\mathrm{d}x^3}\,\mathrm{d}x = a(y)g''(y) - (k-2)a'(y)g'(y) + \frac{1}{2}(k-1)(k-2)a''(y)g(y) + \int_y^\infty g(x)\frac{\partial^k}{\partial x^k}\left[\frac{(x-y)^{k-3}}{\Gamma(k-2)}a(x)\right]\mathrm{d}x,$$

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or, finally, as

$$a(y)g''(y) - (k-2)a'(y)g'(y) + \int_y^\infty (g(x) - g(y)) \frac{\partial^k}{\partial x^k} A_k(x,y)\,\mathrm{d}x$$
$$+ g(y)\bigg[\tfrac{1}{2}(k-1)(k-2)a''(y) + \lim_{x\to\infty} \frac{\partial^{k-1}}{\partial x_{k-1}} \bigg[\frac{(x-y)^{k-3}}{\Gamma(k-2)} a(x)\bigg]\bigg],$$

if the limit on the right-hand side exists. Summing up the equations for the dual operators to the drift and diffusive parts leads to (20) under condition (19).

The extension to noninteger $k$ is as follows.

**Theorem 4.** *Suppose that $k > 0$ and $k \neq 1, 2$, and $a \in C^{[k]+2}(\mathbb{R})$, $b \in C^{[k]+1}(\mathbb{R})$ (here $[k]$ denotes the integer part of a number $k$). Let us define, for $x \geq y$, the functions*

$$B_k(x,y) = \frac{1}{\Gamma(k-1)} [b(x) - b(y) - b'(y)(x-y)](x-y)^{k-2},$$
$$A_k(x,y) = \frac{1}{\Gamma(k-2)} [a(x) - a(y) - a'(y)(x-y) - \tfrac{1}{2}a''(y)(x-y)^2](x-y)^{k-3}.$$

*The diffusion $X_t^x$ generated by (20) is stochastically monotone of order $k$ if and only if the function*

$$\frac{\partial^{k-1}}{\partial x^{k-1}} (B_k(x,y) + A_k(x,y))$$

*is a nondecreasing function of $x \geq y$ for any $y$. If this is the case and, additionally,*

$$\lim_{x\to+\infty} \frac{\partial^{k-1}}{\partial x^{k-1}} (B_k(x,y) + A_k(x,y)) = -(k-1)b'(y) - \tfrac{1}{2}(k-1)(k-2)a''(y)$$

*for all $y$, then the $k$th order Markov dual process $Y_t^y$ exists and is generated by the operator*

$$L^{D_k} g(y) = a(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} - [b(y) + (k-2)a'(y)]\frac{\mathrm{d}}{\mathrm{d}y}$$
$$+ \int_y^\infty (g(x) - g(y)) \frac{\partial^k}{\partial x^k} (B_k(x,y) + A_k(x,y))\,\mathrm{d}x.$$

*Proof.* If $k > 1$, (21) remains valid, but we rewrite it now as

$$\int_y^\infty \bigg[\frac{(x-y)^{k-2}}{\Gamma(k-1)} (b(y) + b'(y)(x-y)) + B_k(x,y)\bigg] \frac{\mathrm{d}^k}{\mathrm{d}(-x)^k} g(x)\,\mathrm{d}x.$$

Using fractional integration by parts, (46) and (45) yields,

$$\bigg(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\bigg)^{D_k} g(y) = -b(y)g'(y) + (k-1)b'(y)g(y) + \int_y^\infty g(x) \frac{\partial^k}{\partial x^k} B_k(x,y)\,\mathrm{d}x. \qquad (24)$$

Note that the measure in the last integral does not have an atom at $y$ (in fact the function $B_k(x,y)$ was introduced specifically in order to be able to single out such a measure, corresponding

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boundary terms being written explicitly). Equation (24) can also be written as

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -b(y)g'(y) + \int_y^\infty (g(x)-g(y))\frac{\partial^k}{\partial x^k}B_k(x,y)\,\mathrm{d}x + g(y)\left[(k-1)b'(y) + \lim_{x\to\infty}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}B_k(x,y)\right], \tag{25}$$

if the last limit is finite.

If $k \in (0,1)$, a small modification is required. Namely, in this case, instead of (21), we obtain

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -\int_y^\infty \frac{(x-y)^{k-1}}{\Gamma(k)}\frac{\partial}{\partial x}\left(b(x)\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\right)\mathrm{d}x.$$

Before integration by parts we need to add and subtract $b(y)$ from $b(x)$, leading to

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -b(y)g'(y) - \int_y^\infty \frac{(x-y)^{k-1}}{\Gamma(k)}\frac{\partial}{\partial x}\left[(b(x)-b(y))\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\right]\mathrm{d}x.$$

Now we can integrate by parts, yielding

$$\left(b(y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^{D_k} g(y) = -b(y)g'(y) + \int_y^\infty \frac{(x-y)^{k-2}}{\Gamma(k-1)}\left[(b(x)-b(y))\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\right]\mathrm{d}x,$$

which again turns to (24) and consequently to (25).

Similarly, (23) remains valid for $k > 2$, and we rewrite it now as

$$\int_y^\infty \left[\frac{(x-y)^{k-3}}{\Gamma(k-2)}(a(y) + a'(y)(x-y) + \tfrac{1}{2}a''(y)(x-y)^2) + A_k(x,y)\right]\frac{\mathrm{d}^k}{\mathrm{d}(-x)^k}g(x)\,\mathrm{d}x,$$

or using fractional integration by parts formulas (46) and (45) as

$$\begin{aligned} a(y)g''(y) - (k-2)a'(y)g'(y) + \tfrac{1}{2}(k-1)(k-2)a''(y)g(y) + \int_y^\infty g(x)\frac{\partial^k}{\partial x^k}A_k(x,y)\,\mathrm{d}x \\ = a(y)g''(y) - (k-2)a'(y)g'(y) + \int_y^\infty (g(x)-g(y))\frac{\partial^k}{\partial x^k}A_k(x,y)\,\mathrm{d}x \\ + g(y)\left[\tfrac{1}{2}(k-1)(k-2)a''(y) + \lim_{x\to\infty}\frac{\partial^{k-1}}{\partial x_{k-1}}A_k(x,y)\right], \end{aligned}$$

if the last limit is finite. The modifications needed for $k < 2$ are similar to those used above when dealing with the drift term. The remaining part is the same as in Theorem 3.

Let us now turn to processes with jumps starting with the generator

$$Lg(x) = \int (g(y)-g(x))\nu(x,\mathrm{d}y) \tag{26}$$

with a finite stochastic kernel $\nu(x, dy)$. Since

$$L'\phi(\mathrm{d}y) = \int_{z\in\mathbb{R}} \phi(\mathrm{d}y)\nu(y,\mathrm{d}z) - \phi(\mathrm{d}y)\int_{z\in\mathbb{R}} \nu(y,\mathrm{d}z)$$

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we obtain, for the dual of order $k > 0$, the expression

$$\begin{aligned}
L^{D_k} g(y) &= I_k^+ \circ L' \circ \frac{\mathrm{d}^k}{\mathrm{d}(-y)^k} g(y) \\
&= \int_y^\infty \frac{(z-y)^{k-1}}{\Gamma(k)} \bigg[ \int \mathrm{d}w \frac{\mathrm{d}^k}{\mathrm{d}(-w)^k} g(w) \nu(w, \mathrm{d}z) - \frac{\mathrm{d}^k}{\mathrm{d}(-z)^k} g(z) \, \mathrm{d}z \int \nu(z, \mathrm{d}w) \bigg] \\
&= \int \int \frac{\mathrm{d}^k}{\mathrm{d}(-z)^k} g(z) \nu(z, \mathrm{d}w) \bigg[ \mathbf{1}_{\{w \geq y\}} \frac{(w-y)^{k-1}}{\Gamma(k)} - \mathbf{1}_{\{z \geq y\}} \frac{(z-y)^{k-1}}{\Gamma(k)} \bigg] \mathrm{d}z.
\end{aligned}$$

We will now use the same trick as when analyzing the diffusion, by separating the part of the expression in the square brackets that contributed to the boundary terms after the $k$th order differentiation (everything is, of course, simpler for integer $k$). Thus, we write

$$\begin{aligned}
L^{D_k} g(y) = \int \int & \frac{\mathrm{d}^k}{\mathrm{d}(-z)^k} g(z) \\
& \times \bigg[ \nu(z, \mathrm{d}w) \mathbf{1}_{\{w \geq y\}} \frac{(w-y)^{k-1}}{\Gamma(k)} + (\nu(y, \mathrm{d}w) - \nu(z, \mathrm{d}w)) \mathbf{1}_{\{z \geq y\}} \frac{(z-y)^{k-1}}{\Gamma(k)} \\
& \qquad - \nu(y, \mathrm{d}w) \mathbf{1}_{\{z \geq y\}} \frac{(z-y)^{k-1}}{\Gamma(k)} \bigg] \mathrm{d}z.
\end{aligned}$$

Integration by parts using (46), assuming that the kernel $\nu(x, \mathrm{d}y)$ is $k$ times differentiable with respect to $x$ as a measure, yields

$$\begin{aligned}
L^{D_k} g(y) = & -g(y) \int \nu(y, \mathrm{d}w) \\
& + \int g(z) \frac{\partial^k}{\partial z^k} \int \bigg[ \nu(z, \mathrm{d}w) \mathbf{1}_{\{w \geq y\}} \frac{(w-y)^{k-1}}{\Gamma(k)} \\
& \qquad + (\nu(y, \mathrm{d}w) - \nu(z, \mathrm{d}w)) \mathbf{1}_{\{z \geq y\}} \frac{(z-y)^{k-1}}{\Gamma(k)} \bigg] \mathrm{d}z. \qquad (27)
\end{aligned}$$

For integer $k$, the term containing $\nu(y, \mathrm{d}w)$ in the square brackets becomes superfluous. In particular, for $k = 1$ and $k = 2$ this simplifies to

$$\begin{aligned}
L^{D_1} g(y) = & \int g(z) \bigg[ \mathbf{1}_{\{z < y\}} \int_{w \geq y} \frac{\partial \nu}{\partial z}(z, \mathrm{d}w) - \mathbf{1}_{\{z \geq y\}} \int_{w < y} \frac{\partial \nu}{\partial z}(z, \mathrm{d}w) \bigg] \mathrm{d}z \\
& - g(y) \int \nu(y, \mathrm{d}w), \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
L^{D_2} g(y) = & \int g(z) \, \mathrm{d}z \bigg[ \mathbf{1}_{\{z < y\}} \int_{w \geq y} (w-y) \frac{\partial^2 \nu}{\partial z^2}(z, \mathrm{d}w) \\
& \qquad + \mathbf{1}_{\{z \geq y\}} \bigg( \int_{w < y} (y-w) \frac{\partial^2 \nu}{\partial z^2}(z, \mathrm{d}w) + \frac{\partial^2}{\partial z^2} \int (z-w) \nu(z, \mathrm{d}w) \bigg) \bigg] \\
& - g(y) \int \nu(y, \mathrm{d}w), \qquad (29)
\end{aligned}$$

respectively.

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For operator (27) to be conservative and conditionally positive, the function

$$L(z,y)=\frac{\partial^{k-1}}{\partial z^{k-1}}\int\bigg[\nu(z,\mathrm{d}w)\mathbf{1}_{\{w\ge y\}}\frac{(w-y)^{k-1}}{\Gamma(k)}+(\nu(y,\mathrm{d}w)-\nu(z,\mathrm{d}w))\mathbf{1}_{\{z\ge y\}}\frac{(z-y)^{k-1}}{\Gamma(k)}\bigg] \tag{30}$$

has to be positive nondecreasing and has to satisfy the boundary conditions

$$L(z,y)|_{z=-\infty}^{\infty}=\int\nu(y,\mathrm{d}w).$$

The simplest natural conditions ensuring the latter can be taken as follows:

$$\lim_{z\to-\infty}\frac{\partial^{k-1}}{\partial z^{k-1}}\int_y^\infty(w-y)^{k-1}\nu(z,\mathrm{d}w)=0, \tag{31}$$

$$\lim_{z\to+\infty}\frac{\partial^{k-1}}{\partial z^{k-1}}\int(\mathbf{1}_{\{w\ge y\}}(w-y)^{k-1}-(z-y)^{k-1})\nu(z,\mathrm{d}w)=0. \tag{32}$$

Summarizing, we obtain the following.

**Theorem 5.** *Suppose that a Feller process $X_t^x$ is generated by operator (26) with a bounded positive kernel $\nu(x,\mathrm{d}y)$ such that its derivatives with respect to $x$ up to and including order $k$ exists as (possibly signed) stochastic kernels and (31) holds. Then a Markov dual of order $k$ exists if and only if function (30) is nondecreasing and condition (32) holds. If this is the case then the generator of the dual process is given by*

$$L^{D_k}g(y)=\frac{1}{\Gamma(k)}\int(g(z)-g(y))\frac{\partial^k}{\partial z^k}\int[\nu(z,\mathrm{d}w)\mathbf{1}_{\{w\ge y\}}(w-y)^{k-1}+(\nu(y,\mathrm{d}w)-\nu(z,\mathrm{d}w))\mathbf{1}_{\{z\ge y\}}(z-y)^{k-1}]\,\mathrm{d}z.$$

*If we have inequality '$\le$' in (32), rather than equality, then the dual exists as a sub-Markov process, the generator being given by (27).*

**Remark 4.** As we assumed maximum regularity, the measure of jumps of the dual process turns out to be absolutely continuous with respect to the Lebesgue measure. More generally, the dual generator would look like

$$L^{D_k}g(y)=\frac{1}{\Gamma(k)}\int(g(z)-g(y))\,\mathrm{d}_z\frac{\partial^{k-1}}{\partial z^{k-1}}\int[\nu(z,\mathrm{d}w)\mathbf{1}_{\{w\ge y\}}(w-y)^{k-1}+(\nu(y,\mathrm{d}w)-\nu(z,\mathrm{d}w))\mathbf{1}_{\{z\ge y\}}(z-y)^{k-1}].$$

We have constructed dual generators separately for diffusive and jump parts of the original generators. For an arbitrary Feller process with a pseudodifferential generator the dual is constructed by putting these parts together. As an example let us consider Markov processes that are martingales, that is, they have generators of the form

$$Lg(x)=a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\int[g(y)-g(x)-g'(x)(y-x)]\nu(x,\mathrm{d}y) \tag{33}$$

with

$$\int\min(|y-x|,(y-x)^2)\nu(x,\mathrm{d}y)<\infty.$$

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As the duality of all orders reverses the sign of the drift, for self-duality we have necessarily the condition

$$\int (y-x)\nu(x,\mathrm{d}y) = 0, \tag{34}$$

where the integral can be understood in the sense of the main value. Taking these into account and looking at (28) and (29) we arrive at the following.

**Theorem 6.** *Let $k = 1$ or $k = 2$, and let a Feller process $X_t^x$ be generated by an operator of type (33) with a continuously differentiable nonnegative function $a(x)$ and continuously differentiable in $x$ Lévy kernel $\nu$. The process $X_t^x$ is self-dual of order* 1 *or* 2, *if (34) holds and*

$$\nu(y,\mathrm{d}z) = \mathbf{1}_{\{z<y\}}\,\mathrm{d}_z \int_{w\geq y} \nu(z,\mathrm{d}w) - \mathbf{1}_{\{z\geq y\}}\,\mathrm{d}_z \int_{w<y} \nu(z,\mathrm{d}w) \tag{35}$$

*or*

$$\nu(y,\mathrm{d}z) = \mathbf{1}_{\{z<y\}}\,\mathrm{d}_z \int_{w\geq y} (w-y)\frac{\partial \nu}{\partial z}(z,\mathrm{d}w) + \mathbf{1}_{\{z\geq y\}}\,\mathrm{d}_z \int_{w<y} (y-w)\frac{\partial \nu}{\partial z}(z,\mathrm{d}w), \tag{36}$$

*respectively.*

Finally, let us note that the relation between dual generators becomes more transparent in differential form (even though in this way some information of boundary behavior is lost). For instance, differentiating (35) and (36) with respect to $y$ once or twice, respectively, yields the differential relations

$$\mathrm{d}_y \nu(y,\mathrm{d}z) = \mathrm{d}_z \nu(z,\mathrm{d}y) \quad \text{and} \quad \mathrm{d}_y \frac{\partial \nu}{\partial y}(y,\mathrm{d}z) = \mathrm{d}_z \frac{\partial \nu}{\partial z}\nu(z,\mathrm{d}y),$$

respectively. These equations take an especially simple form

$$\frac{\partial \nu}{\partial y}\nu(y,z) = \frac{\partial \nu}{\partial z}\nu(z,y), \qquad \frac{\partial^2 \nu}{\partial y^2}\nu(y,z) = \frac{\partial^2 \nu}{\partial z^2}\nu(z,y),$$

respectively, if $\nu(x,\mathrm{d}y)$ has a density, $\nu(x,y)$, with respect to the Lebesgue measure.

## 6. Time-nonhomogeneous extension

Equation (9) suggests the necessity to include time reversion when studying a time-nonhomogeneous situation. We mean just a simple time reversion around a deterministic time, not a more sophisticated general time reversion, as developed, say, in [16] or [13].

Let us recall that a family $U_{s,t}$, $0 \leq s \leq t$, of transformations in $B(X)$ (or $C(X)$ or $C_\infty(X)$) for locally compact spaces $X$, $Y$ is called a (backward) propagator if $U_{t,t}$ is the identity operator and the chain rule, or the propagator equation, holds for $t \leq s \leq r$: $U^{t,s}U^{s,r} = U^{t,r}$.

Suppose that $U^{t,r}$ is a strongly continuous backward propagator of bounded linear operators in $C_\infty(X)$ with a common dense invariant domain $D$. Let $A_t$, $t \geq 0$, be a family of linear operators $D \mapsto B$ that are strongly continuous in $t$. Let us say that the family $A_t$ *generates* $U^{t,r}$ on $D$ if, for any $f \in D$, the equations

$$\frac{\mathrm{d}}{\mathrm{d}s}U^{t,s}f = U^{t,s}A_s f, \qquad \frac{\mathrm{d}}{\mathrm{d}s}U^{s,r}f = -A_s U^{s,r} f, \qquad 0 \leq t \leq s \leq r,$$

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hold for all $s$ with the derivatives taken in the topology of $B$, where for $s = t$ (respectively $s = r$) it is assumed to be only a right (respectively left) derivative. The second equation (which in fact follows from the first one under mild natural conditions) implies by duality that, for any $T$,

$$\frac{\mathrm{d}}{\mathrm{d}s}U'_{T-t,T-s} = -A'_{T-s}U'_{T-t,T-s}, \qquad 0 \le s \le t \le T,$$

in the weak sense.

The time-nonhomogeneous counterpart of Proposition 2 (with the same proof) reads as follows.

**Proposition 5.** (i) *Let $f$ be a bounded measurable function separating the measures on $X$ and $U_{s,t}$, $s \le t$, a (backward) propagator of integral operators in $B(X)$ specified by the family of bounded signed kernel $p_{s,t}(x, \mathrm{d}z)$ from $X$ to $X$. Then, for any $T > 0$, the operators $U_{s,t}^{D(f,T)}$ in $F(\mathcal{M}(X))$, $f$-dual to $U_{T-t,T-s}$, also form a propagator and*

$$U_{s,t}^{D(f,T)} = F \circ U'_{T-t,T-s} \circ F^{-1}. \tag{37}$$

(ii) *If the propagator $\{U_{s,t}\}$ is strongly continuous in $C_\infty(X)$ with an invariant domain $D$ and is generated by a family of operator $A_t\colon D \to C_\infty$, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}U_{s,t}^{D(f,T)}g = -F \circ A'_{T-s} \circ F^{-1}U_{s,t}^{D(f,T)}g,$$

*that is, the generator of the propagator $\{U_{s,t}^{D(f,T)}\}$ is*

$$A_s^{D(f,T)} = F \circ A'_{T-s} \circ F^{-1}. \tag{38}$$

The propagator $\{U_{s,t}^{D(f,T)}\}$ will be called $(f, T)$-dual to $\{U_{s,t}\}$.

Let us describe the probabilistic analog of this duality. We will write $X_t^{x,s}$ for a Markov process at time $t > s$ with initial position $x$ at time $s$. For a function $f$ on $X \times Y$ with two metric (or measurable) spaces $X, Y$ and a number $T$, we say that the Markov processes $Y_t^{y,s}$ in $Y$ is $(f, T)$-dual to the Markov process $X_t^{x,s}$ in $X$, if

$$\mathbb{E}f(x, Y_t^{y,s}) = \mathbb{E}f(X_{T-s}^{x,T-t}, y)$$

for all $x \in X$, $y \in Y$ and $s \le t \le T$, where $\mathbb{E}$ on the left-hand side and the right-hand side correspond to the distributions of processes $Y_t$ and $X_t$n, respectively. In the particular case of $X = Y$ and $f(x, y) = \mathbf{1}_{\{x \ge y\}}$ (where $\ge$ is any measurable partial order on $X$) the above equation reduces to

$$\mathbb{P}(Y_t^{y,s} \le x) = \mathbb{P}(X_{T-s}^{x,T-t} \ge y).$$

Thus, the duality of Markov processes is equivalent to the duality of their propagators. Proposition 5 implies that dual distributions to a Markov process automatically form a Markov family, as their transition operators form a propagator and, hence, satisfy the chain rule (or the Chapman–Kolmogorov equation).

It is now clear that all our results have a natural counterpart for time-dependent generators. Namely, let us say that a Markov process $Y_t^y$ is dual to $X_t^x$ of order $k$, $k \in \mathbb{R}$, if

$$\mathbb{E}(x - Y_t^{y,s})_+^{k-1} = \mathbb{E}(X_{T-s}^{x,T-t} - y)_+^{k-1}.$$

The characterization in terms of generators or stochastic monotonicity remains the same, once the time dependence is adjusted appropriately, that is, via (37) and (38).

## Appendix A

For completeness, we deduce the fundamental solutions of the generators of Lévy stable motions and fractional derivative operators, as well as the related integration by parts equations.

Recall that the characteristic function of a $\beta$-stable Lévy motion for $\beta \in (0,1) \cup (1,2)$ equals

$$\exp\{-t\sigma|p|^{\beta}\mathrm{e}^{\mathrm{i}\pi\gamma\operatorname{sgn}p/2}\},$$

where $\sigma > 0$ is the scale and $\gamma$ is the skewness parameter satisfying the conditions $|\gamma| \le \beta$ or $|\gamma| \le 2-\beta$ for $\beta \in (0,1)$ or $\beta \in (1,2)$, respectively. For simplicity, we omit the discussion of a more complicated general case $\beta = 1$, and for $\beta = 1$ will only deal with the symmetric case $\gamma = 0$, for which the above equations remain valid.

Thus, the generator $L_{\beta,\gamma,\sigma}$ of this Lévy motion is the pseudodifferential operator with the symbol (denoted with some abuse of notation by the same letter)

$$L_{\beta,\gamma,\sigma}(p) = -\sigma|p|^{\beta}\mathrm{e}^{\mathrm{i}\pi\gamma\operatorname{sgn}p/2},$$

meaning that $L_{\beta,\gamma,\sigma}$ acts on the Fourier transform $\mathcal{F}(f)(p) = \hat{f}(p) = \int \mathrm{e}^{-\mathrm{i}xp} f(x)\,\mathrm{d}x$ of a function $f$ as the multiplication by $L_{\beta,\gamma,\sigma}(p)$.

The fundamental solution of the operator $L_{\beta,\gamma,\sigma}$ equals

$$f(x) = \bigg[\mathcal{F}^{-1}\frac{1}{L_{\beta,\gamma,\sigma}(p)}\bigg](x) = -\frac{1}{2\pi}\int_{\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}p}{\sigma|p|^{\beta}\mathrm{e}^{\mathrm{i}\pi\gamma\operatorname{sgn}p/2}}.$$

In other words,

$$f = -\frac{1}{\sigma}\mathcal{F}^{-1}[\mathrm{e}^{-\mathrm{i}\pi\gamma/2}p_+^{-\beta} + \mathrm{e}^{\mathrm{i}\pi\gamma/2}p_-^{-\beta}].$$

Using known equations for the Fourier transforms (in the sense of distributions) of one-sided powers (see, e.g. [20, page 176], that is,

$$(\mathcal{F}^{-1}p_{\pm}^{\lambda})(x) = \frac{\pm\mathrm{i}}{2\pi}\mathrm{e}^{\pm\mathrm{i}\lambda\pi/2}\Gamma(\lambda+1)(x\pm\mathrm{i}0)^{-\lambda-1}$$

(slight deviations in our notation from the Fourier transform used in [20] are taken into account), where

$$(x\pm\mathrm{i}0)^{\lambda} = x_+^{\lambda} + \mathrm{e}^{\pm\mathrm{i}\lambda\pi}x_-^{\lambda},$$

we find that

$$f(x) = -\frac{\Gamma(1-\beta)}{2\pi\sigma}[\mathrm{i}\mathrm{e}^{-\mathrm{i}(\beta+\gamma)\pi/2}(x+\mathrm{i}0)^{\beta-1} - \mathrm{i}\mathrm{e}^{\mathrm{i}(\beta+\gamma)\pi/2}(x-\mathrm{i}0)^{\beta-1}]$$

leading to

$$f(x) = -\frac{\Gamma(1-\beta)}{\sigma\pi}[\sin(\tfrac{1}{2}\pi(\beta+\gamma))x_+^{\beta-1} + \sin(\tfrac{1}{2}\pi(\beta-\gamma))x_-^{\beta-1}]. \tag{39}$$

The most important cases are the fully skewed motions with $\gamma = \pm\beta$ or $\gamma = \pm(2-\beta)$ for $\beta \in (0,1)$ or $\beta \in (1,2)$, respectively, and the symmetric motions with $\gamma = 0$. In these cases, for $\beta \in (0,1)$, the generators are negations of fractional derivatives, that is,

$$L_{\beta,\pm\beta,1}f = -\frac{\mathrm{d}^{\beta}}{\mathrm{d}(\pm x)^{\beta}}f(x) = -\frac{1}{\Gamma(-\beta)}\int_0^{\infty}(f(x\mp y) - f(x))\frac{\mathrm{d}y}{y^{1+\beta}}, \tag{40}$$

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and

$$L_{\beta,0,1} f = -\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|^{\beta} = -\frac{1}{2\cos(\pi\beta/2)}\left(\frac{\mathrm{d}^{\beta}}{\mathrm{d}x^{\beta}} + \frac{\mathrm{d}^{\beta}}{\mathrm{d}(-x)^{\beta}}\right); \tag{41}$$

see, e.g. [26, Section 1.8], where the fractional derivatives can be defined either by the corresponding expressions on the right-hand side of these equations, or, equivalently, via the following Fourier transforms:

$$\mathcal{F}\left(\frac{\mathrm{d}^{\beta}}{\mathrm{d}(\pm x)^{\beta}} f\right)(p) = \exp\{\pm\tfrac{1}{2}\mathrm{i}\pi\beta \operatorname{sgn} p\}|p|^{\beta}\mathcal{F}(f)(p),$$
$$\mathcal{F}\left(\frac{\mathrm{d}^{\beta}}{\mathrm{d}x^{\beta}} + \frac{\mathrm{d}^{\beta}}{\mathrm{d}(-x)^{\beta}}\right)(p) = 2\cos\left(\tfrac{1}{2}\pi\beta\right)|p|^{\beta}\mathcal{F}(f)(p).$$

The processes generated by $-\mathrm{d}^{\beta}/\mathrm{d}(\pm x)^{\beta}$ with $\beta \in (0, 1)$ are called the stable Lévy subordinators.

From (39) it follows that the functions

$$\frac{\Gamma(1-\beta)}{\pi}\sin(\pi\beta)x_{\pm}^{\beta-1} = \frac{x_{\pm}^{\beta-1}}{\Gamma(\beta)} \tag{42}$$

(where the equation $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin(\pi\beta)$ was used), represent fundamental solutions for the operators $\mathrm{d}^{\beta}/\mathrm{d}(\pm x)^{\beta}$.

Fractional derivatives of order higher than 1 can be defined by the compositions of the derivatives of order $\beta \in (0, 1)$ with the derivatives of an integer order. Namely, for $\beta \in (n, n+1)$, $n \in \mathbb{N}$, we define

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}(\pm x)^{\beta}} = \frac{\mathrm{d}^{n}}{\mathrm{d}(\pm x)^{n}}\frac{\mathrm{d}^{\beta-n}}{\mathrm{d}(\pm x)^{\beta-n}}$$

with the second component given by (40). It is then easy to check that (42) for the fundamental solutions remain valid for all $\beta = k+1 > 0$, $\beta \notin \mathbb{N}$. Equation (41) remains valid for $\beta \in (1, 2)$, but in (40) the sign has to be changed, leading to

$$L_{\beta,\pm(2-\beta),1} f = \frac{\mathrm{d}^{\beta}}{\mathrm{d}(\pm x)^{\beta}} f(x) = \frac{1}{\Gamma(1-\beta)}\int_0^{\infty}((\pm f')(x \mp y) - (\pm f)'(x))\frac{\mathrm{d}y}{y^{\beta}}.$$

Turning to the integration by parts, let us define, for $k \geq 1$, the fractional integration operator $I_k^{\pm}\colon \mathcal{M}_k^{\pm}(\mathbb{R}) \to B(\mathbb{R})$, by the equation

$$(I_k^{\pm}Q)(y) = \int \frac{(x-y)_{\pm}^{k-1}}{\Gamma(k)} Q(\mathrm{d}x), \tag{43}$$

where

$$\mathcal{M}_k^{\pm} = \left\{Q \in \mathcal{M}(\mathbb{R})\colon \int x_{\pm}^{k-1}|Q|(\mathrm{d}x) < \infty\right\},$$

and

$$x_{+}^{k} = \begin{cases} x^{k}, & x \geq 0, \\ 0, & x < 0. \end{cases} \qquad x_{-}^{k}(x) = (-x)_{+}^{k}.$$

The image $\mathcal{I}_1^{\pm}$ of $I_1^{\pm}$ (defined on $\mathcal{M}_1^{\pm} = \mathcal{M}(\mathbb{R})$) is the set of right- (respectively left-) continuous functions of finite total variation, tending to 0 at $\pm\infty$. Moreover, $(I_1^{\pm} Q)' = \mp Q$ in the sense of distributions. The image $\mathcal{I}_k^{\pm}$ of $I_k^{\pm}$, $k > 1$, consists of continuous functions $g$ tending to 0 at $\pm\infty$ and such that

$$\frac{\mathrm{d}^k}{\mathrm{d}(\mp x)^k} g \in \mathcal{M}_k^{\pm}(\mathbb{R})$$

in the sense of distributions. Moreover,

$$\frac{\mathrm{d}^k}{\mathrm{d}(\mp x)^k} \circ I_k^{\pm}, \qquad I_k^{\pm} \circ \frac{\mathrm{d}^k}{\mathrm{d}(\mp x)^k}$$

are the identity operators in $\mathcal{M}_k^{\pm}(\mathbb{R})$ and $\mathcal{I}_k^{\pm}$, respectively. Other simple equations worth mentioning are

$$I_k^{\pm} \circ \frac{\mathrm{d}}{\mathrm{d}(\mp x)} = \begin{cases} I_{k-1}^{\pm}, & k > 1, \\ \dfrac{\mathrm{d}^{1-k}}{\mathrm{d}(\mp x)^{1-k}}, & k < 1, \end{cases} \tag{44}$$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{(x-a)_+^{k-1}}{\Gamma(k)} = \delta_a(x), \qquad k > 1. \tag{45}$$

Now let $\phi_{\pm} = I_k^{\pm} Q_{\pm}$ with some $Q_{\pm} \in \mathcal{M}_k^{\pm}$. By Fubini's theorem

$$\int_{\mathbb{R}^2} \frac{(x-y)_+^{k-1}}{\Gamma(k)} Q_+(\mathrm{d}x) Q_-(\mathrm{d}y) = \int (I_k^+ Q_+)(y) Q_-(\mathrm{d}y) = \int (I_k^- Q_-)(x) Q_+(\mathrm{d}x).$$

The last equation can be called the integration by parts equation, as it can be rewritten as

$$\int \phi_+(y) \frac{\mathrm{d}^k}{\mathrm{d}y^k} \phi_-(\mathrm{d}y) = \int \phi_-(x) \frac{\mathrm{d}^k}{\mathrm{d}(-x)^k} \phi_+(\mathrm{d}x) \tag{46}$$

(where the derivatives are defined, generally speaking, in the sense of distributions and represent measures, not necessarily functions).

It is important to stress that this equation holds not only for the integration over $\mathbb{R}$, but also for the integration over an interval or a half-line, the corresponding boundary terms being taken into account automatically by the measures $\mathrm{d}^k \phi_{\pm}(x)/\mathrm{d}(\mp x)^k$.

Finally, for $k \in (0, 1)$ we can define the fractional integration (43) on functions $g \in B(\mathbb{R}) \cap L^1(\mathbb{R})$, that is,

$$(I_k^{\pm} g)(y) = \int \frac{(x-y)_{\pm}^{k-1}}{\Gamma(k)} g(x)\, \mathrm{d}x,$$

in which case the image belongs to the set of continuous functions with the sup-norm bounded by $\|g\|/k + \|g\|_{L^1}$ and (46) still holds by the same reasoning.

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