ON A THEOREM OF KOROUS

by J. A. ANDERSON

(Received 2nd April 1965; revised MS. received 17th July 1965)

1. Introduction

Let

$$l(z)=(z-l_0)\prod_{\nu=1}^{\infty}\left(1-\frac{z}{l_\nu}\right)\left(1-\frac{z}{l_{-\nu}}\right), \tag{1}$$

where the $\{l_\nu\}$ are numbers near to an arithmetic progression of common difference unity. Let

$$\rho(z)=-\frac{l(0)}{\pi z}+\frac{z}{\pi}\sum_{\nu\neq 0}\frac{l(\nu)}{\nu(\nu-z)}+b, \tag{2}$$

b being a constant. Write

$$k(z)=l(z)\cot \pi z+\rho(z), \tag{3}$$

so that $k(z)$ is an integral function, and

$$Q(z)=\frac{k(z)}{2l(z)} \tag{4}$$

is meromorphic with simple poles at $\{l_\nu\}$.

Let $f\in BV(\alpha, \alpha+\pi)$, and suppose that C_p is a circle, centre the origin, not passing through any ν or l_ν, and containing in its interior those l_ν for which $|\nu|\leqq N_p$. Then,

$$\left.\begin{aligned} s_p(x) &= \frac{1}{2\pi i}\int_{C_p} Q(z)dz\int_{\alpha}^{\alpha+\pi} f(t)e^{iz(x-t)}dt \\ &= \sum_{|\nu|\leqq N_p} c_\nu e^{il_\nu x}, \end{aligned}\right\} \tag{5}$$

where

$$c_\nu=\frac{k(l_\nu)}{2l'(l_\nu)}\int_{\alpha}^{\alpha+\pi} f(t)e^{-il_\nu t}dt, \tag{6}$$

is the N_pth partial sum of the Cauchy Exponential Series (CES) of f with respect to $Q(z)$. If

$$S_p(x)=\frac{1}{4\pi i}\int_{C_p}\cot \pi z\,dz\int_{\alpha}^{\alpha+\pi} f(t)e^{iz(x-t)}dt, \tag{7}$$

then it is easily seen that $S_p(x)$ is the N_pth partial sum of the Fourier series (FS) of g, given by

$$g(x)=\begin{cases} f(x) & \alpha\leqq x\leqq \alpha+\pi \\ 0 & \alpha-\pi\leqq x<\alpha \end{cases}. \tag{8}$$

We have

$$s_p(x)-S_p(x) = \frac{1}{4\pi i}\int_{Cp}\frac{\rho(z)}{l(z)}\,dz\int_{\alpha}^{\alpha+\pi} f(t)e^{iz(x-t)}dt, \tag{9}$$

and so, if the right-hand side of (9) is $o(1)$, we have the CES equiconvergent with a FS. We shall give some sufficient conditions for this equiconvergence.

Korous (**1**) takes $\{l_\nu\}$ real, and satisfying

$$l_{-1}<0 \leqq l_0, \quad l_\nu<l_{\nu+1}, \quad l_\nu = \nu+a+\lambda_\nu,$$

where $a = 0$ or $\pm\frac{1}{2}$. He considers partial sums $s_p(x; f)$, $S_p(x; f)$ [see (**1**), p. 3], where f, b are real, and which are given, in terms of (5) and (7) above, by

$$s_p(x; f) = \text{Re}\, s_p(x), \quad S_p(x; f) = \text{Re}\, S_p(x).$$

(9) becomes, therefore,

$$s_p(x;f)-S_p(x;f) = \frac{1}{4\pi i}\int_{Cp}\frac{\rho(z)}{l(z)}\,dz\int_{\alpha}^{\alpha+\pi} f(t)\cos z(x-t)dt,$$

which is (**1**), equation (2.7). He proves (Theorem A) that if $\limsup|\lambda_\nu|<\frac{1}{12}$, and $f\in BV(\alpha, \alpha+\pi)$, then $s_p(x; f)-S_p(x; f) = o(1)$ uniformly in any closed interval interior to $(\alpha, \alpha+\pi)$.

In this note, we suppose that the numbers $\{l_\nu\}$ are complex, say $l_\nu = \alpha_\nu+i\beta_\nu$, where

$$\left.\begin{array}{l}\alpha_\nu = \nu+\lambda_\nu \\ |\beta_\nu|\leqq M\end{array}\right\}, \tag{10}$$

M being a constant. We prove the following generalisation of (**1**), Theorem A:

Theorem 1. *Let* $f\in BV(\alpha, \alpha+\pi)$. *Suppose that the numbers* $\{l_\nu\}$ *satisfy* (10) *and the condition*

$$\limsup|\lambda_\nu|<\tfrac{1}{8}. \tag{11}$$

Then, the CES of f *is uniformly equiconvergent, in any closed interval interior to* $(\alpha, \alpha+\pi)$, *with the FS of the function* g *given by* (8). *Further, the coefficients* c_ν *tend to zero as* $|\nu|\to\infty$.

2. Proof of the theorem

Let $E = \{z\colon |z-l_\nu| \geqq \frac{1}{4}, \,|r-|\nu|| \geqq \frac{1}{4}\}$. Let C_p be the circle

$$|z| = r = p+\tfrac{1}{2},$$

which satisfies the condition $z\in E$, if p is sufficiently large. Write

$$s_p(x)-S_p(x) = \int_{\alpha}^{\alpha+\pi} f(t)\phi_p(x-t)dt,$$

where

$$\phi_p(u) = \frac{1}{4\pi i}\int_{Cp}\frac{\rho(z)}{l(z)}\,e^{izu}dz.$$

If

$$J_p(u) = -\frac{1}{4\pi}\int_{Cp}\frac{\rho(z)e^{izu}}{zl(z)}\,dz, \tag{12}$$

https://doi.org/10.1017/S0013091500011391 Published online by Cambridge University Press

then

$$\int_{\alpha}^{\alpha+\pi} f(t)\phi_p(x-t)dt = -f(\alpha+\pi)J_p(x-\alpha-\pi)+f(\alpha)J_p(x-\alpha)+\int_{\alpha}^{\alpha+\pi} J_p(x-t)df(t).$$

It will therefore suffice to prove that $J_p(u)$ tends to zero uniformly in

$$[-\pi+\eta,\ \pi-\eta].$$

In fact, it is enough to consider the first quadrant, and the function

$$K_p(u) = \int_0^{\pi/2} \frac{\rho(z)}{l(z)} e^{izu} d\theta$$
$$= O\left(\int_0^{\pi/2} \left|\frac{\rho(z)}{l(z)}\right| e^{|y|u} d\theta\right). \qquad (13)$$

We employ Theorem 1 of (**2**): there is a number $L<\frac{1}{8}$ such that for $z \in C_p$,

$$|l(z)| > Ae^{\pi|y|}G^{-2L},$$

where

$$G = \frac{|z|^2+1}{|y|+1},$$
$$=O(r^2)$$

on C_p, whence

$$\left|\frac{1}{l(z)}\right| = O(e^{-\pi|y|}r^{4L}).$$

Also by Theorem 1 of (**2**), we have, for all z,

$$|l(z)| < Ae^{\pi|y|}G^{2L}.$$

Hence, for $v \neq 0$,

$$|l(v)| = O(|v|^{4L}),$$

and so

$$|\rho(z)| = O\left(\sum_{v\neq 0} \frac{r}{|v|^{1-4L}\,\|v|-r|}\right)+O(1).$$

To estimate this, we split up the sum as follows:

$$\sum_{|v|\geqq 1} = \sum_{1\leqq|v|<r} + \sum_{r<|v|\leqq 2r} + \sum_{|v|>2r} = \sum_1 + \sum_2 + \sum_3.$$

Then,

$$\sum_1 < r^{4L} \sum_{1\leqq v<r} \frac{r}{v(r-v)}$$
$$\leqq r^{4L}(2\log r+O(1)).$$

Next,

$$\sum_2 = O\left(r \sum_{r<v\leqq 2r} \frac{v^{4L-1}}{v-r}\right)$$
$$= O(r^{4L}\log r).$$

https://doi.org/10.1017/S0013091500011391 Published online by Cambridge University Press

and finally,

$$\sum_3 = O\left(r \sum_{\nu > 2r} \frac{\nu^{4L-1}}{\nu - r}\right)$$
$$= O(r \sum_{\nu > 2r} \nu^{4L-2})$$
$$= O(r^{4L}).$$

It follows, therefore, that

$$\rho(z) = O(r^{4L} \log r).$$

Hence,

$$K_p(u) = O\left(\int_0^{\pi/2} e^{-\eta y} r^{8L} \log r d\theta\right),$$

where $u \in [-\pi+\eta, \pi-\eta]$,

$$= O(r^{8L-1} \log r)$$
$$= o(1),$$

uniformly, since $L < \frac{1}{8}$.

Finally, to prove that $c_\nu = o(1)$, let C_p^* denote the contour obtained from C_p by replacing the minor arc formed by $\operatorname{Re} z = p - \frac{1}{2}$ by the chord. If

$$J_p^*(u) = -\frac{1}{4\pi} \int_{C_p^*} \frac{\rho(z) e^{izu}}{z l(z)} dz,$$

then $J_p^*(u) \to 0$ uniformly, by the same argument as for $J_p(u)$. Let $\Sigma d_\nu e^{i\nu x}$ be the FS of g. Since, as $p \to \infty$, $J_p \to 0$, we have

$$\sum_{|\nu| \leqq p} \{c_\nu e^{il_\nu x} - d_\nu e^{i\nu x}\} \to 0.$$

Since $J_p^* \to 0$,

$$\sum_{-p \leqq \nu < p} \{c_\nu e^{il_\nu x} - d_\nu e^{i\nu x}\} \to 0.$$

Thus, $c_p e^{il_p x} - d_p e^{ipx} \to 0$ as $p \to \infty$. But $d_p e^{ipx} \to 0$; hence $c_p \to 0$. Similarly, $c_{-p} \to 0$. This completes the proof.

3. By adding the condition

$$\sum_{|\nu| \leqq p} \frac{\lambda_\nu}{\nu + \frac{1}{2}} = O(1), \quad p = 1, 2, 3, \ldots$$

and using (**2**), Theorem 2, we can, in the theorem above, replace $\frac{1}{8}$ by $\frac{1}{4}$.

I am indebted to the referee for several helpful suggestions.

REFERENCES

(**1**) J. Korous, On a generalisation of Fourier series, *Časopis Pešt. Mat.* **71** (1946), 1-15.

(**2**) S. Verblunsky, On a class of infinite products, *Proc. Cambridge Phil. Soc.* **60** (1964), 847-854.

The University, Nottingham