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A SEPARATION PRINCIPLE FOR THE STABILISATION OF A CLASS OF FRACTIONAL ORDER TIME DELAY NONLINEAR SYSTEMS

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Abstract

We establish a separation principle for a class of fractional order time-delay nonlinear differential systems. We show that a nonlinear time-delay observer is globally convergent and give sufficient conditions under which the observer-based controller stabilises the system.

2010 *Mathematics subject classification*: primary 93C10; secondary 93D15, 93D20.

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1. Introduction

Fractional order differential systems have found applications in many areas, such as electromagnetic systems, dielectric polarisation, economics and image processing. For an introduction to the theory and applications, see, for example, [13]. However, the stability of fractional nonlinear systems, of central importance in control theory, remains an open problem.

Time delay is frequently a source of instability. Necessary and sufficient conditions for existence and uniqueness of the solutions to a class of nonlinear fractional order systems with delay are derived in [20]. Stability and asymptotic stability of delayed systems under various conditions are studied in [5, 6, 15, 16].

In this paper we investigate the stabilisation problem for a class of uncertain time delay fractional differential systems with a nominal part written in triangular form. Motivated by similar approaches to time-delay first-order differential systems [3, 9] as well as [15], we design a state feedback controller to stabilise the origin of the system and give sufficient conditions for the stabilisation of nonlinear systems with time-varying delays as linear matrix inequalities.

The paper is organised as follows. In Section 2 we introduce the basic definitions and lemmas about fractional order systems. In Section 3, we describe the fractional order system and explain our assumptions. The separation principle and the main results on the observer design and stabilisation for a class of nonlinear time-delay fractional differential systems in triangular form are given in Section 4. In Section 5, we illustrate our results by a physical model. Finally, in Section 6 we extend the results to time-varying delay systems.

2. Preliminaries and definitions

In this section, we state definitions and results related to the fractional calculus.

Definition 2.1 [17]. The Riemann–Liouville fractional integral is defined by

$$ {}_{t_0}I_t^q(x(t)) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)\,ds \quad (q>0), $$

where $\Gamma(q)=\int_0^{\infty} e^{-t}t^{q-1}\,dt$ is the usual gamma function, provided the integral exists.

Definition 2.2 [18, 21]. The Riemann–Liouville and Caputo fractional derivatives respectively of order q on $[t_0,t]$ are defined by

$$ {}_{t_0}D_t^q x(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{q+1-n}}\,ds \quad (n-1\le q<n), $$

$$ {}_{t_0}^{C}D_t^q x(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}\frac{x^{(n)}(s)}{(t-s)^{q+1-n}}\,ds \quad (n-1\le q<n), $$

where n is the integer such that $q<n\le q+1$ and $x(t)\in\mathbb{R}^n$ is differentiable as many times as required, provided the integrals exist.

Property 2.3 [1]. *When* $0<q<1$,

$$ {}_{t_0}^{C}D_t^q x(t) = {}_{t_0}D_t^q x(t) - \frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}. $$

In particular, if $x(t_0)=0$, *then* ${}_{t_0}^{C}D_t^q x(t) = {}_{t_0}D_t^q x(t)$.

Property 2.4 [13]. *If* $p>q>0$, *then the formula*

$$ {}_{t_0}D_t^q({}_{t_0}D_t^{-p}x(t)) = {}_{t_0}D_t^{q-p}x(t), $$

holds for all 'sufficiently good' functions $x(t)$. *In particular, it holds if* $x(t)$ *is integrable.*

Lemma 2.5 [7]. *Let* $x(t)\in\mathbb{R}^n$ *be a vector of differentiable functions and suppose* $0\le q\le 1$. *Then, for any time instant* $t\ge t_0$,

$$ \tfrac{1}{2}\,{}_{t_0}D_t^q x^T(t)Px(t) \le x^T(t)P\,{}_{t_0}D_t^q x(t), $$

where $P\in\mathbb{R}^{n\times n}$ *is a constant, symmetric, positive definite matrix.*

Lemma 2.6 [14]. *Suppose* $0<q<1$ *and* $x(0)\ge 0$. *Then*

$$ {}_{t_0}^{C}D_t^q x(t) \le {}_{t_0}D_t^q x(t). $$

LEMMA 2.7. *For any* $x, y \in \mathbb{R}^n$ *and* $\epsilon > 0$,

$$2x^T y \le \epsilon x^T x + \frac{1}{\epsilon} y^T y.$$

LEMMA 2.8 (Schur complement, [4]). *If* T_1, T_2 *and* T_3 *are matrices and* $T_3 > 0$, *then*

$$\begin{bmatrix} T_1 & T_2^T \\ T_2 & -T_3 \end{bmatrix} < 0 \iff T_1 + T_2^T T_3^{-1} T_2 < 0.$$

REMARK 2.9. For a matrix M, we write M^T for the transpose of M, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximal and minimal eigenvalues of M, and $M > 0$ means that M is symmetric and positive definite. Also, I is an appropriately dimensioned identity matrix.

3. System description

Suppose $0 < q < 1$. Consider the fractional time-delay system

$${}_{t_0}D_t^q x(t) = f(x(t), x(t-\tau)), \tag{3.1}$$

where $\tau > 0$ is the delay time. The knowledge of x at time $t = 0$ does not allow us to deduce x at time t. Thus, an initial condition is specified by a continuous function $\varphi : [-\tau, 0] \to \mathbb{R}^n$.

The state of x at time t determined by (3.1) and the initial condition can be described as a function segment x_t defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0].$$

Fractional time-delay systems form a special class of fractional differential equations

$${}_{t_0}D_t^q x(t) = F(t, x_t), \tag{3.2}$$

where, $F : \mathbb{R}_+ \times C \to \mathbb{R}^n$ and C denotes the Banach space of continuous functions mapping the interval $[-\tau, 0] \to \mathbb{R}^n$ and equipped with the supremum norm

$$\|\varphi\|_\infty = \max_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|, \quad \varphi \in C,$$

where $\|\cdot\|$ is the usual Euclidean norm.

We recall the definition of various forms of stability for the system (3.2).

DEFINITION 3.1. For the system described by (3.2), the trivial solution is called:

- *stable*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\varphi\|_\infty < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \text{for all } t \ge 0;$$

- *attractive*, if there exists $\sigma > 0$ such that

$$\|\varphi\|_\infty < \sigma \Rightarrow \lim_{t \to +\infty} x(t) = 0; \tag{3.3}$$

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- *asymptotically stable*, if it is stable and attractive;
- *globally asymptotically stable*, if it is stable and δ can be chosen arbitrarily large for sufficiently large ε, and (3.3) is satisfied for all $\sigma > 0$.

For a locally Lipschitz functional $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$ we can define a generalisation of the fractional derivative of V along the solutions of (3.2).

DEFINITION 3.2 [19]. Suppose $0 < q < 1$. Let $V(t, \varphi)$ be differentiable and let $x_t(t_0, \varphi)$ be the solution of (3.2) at time t with initial condition $x_{t_0} = \varphi$. Then the fractional derivative of $V(t, x_t)$ with respect to t and evaluated at $t = t_0$ is defined by

$$ {}_{t_0}D_t^q V(t_0, \varphi) = {}_{t_0}D_{t_0}^q V(t, x_t(t_0, \varphi))|_{t=t_0, x_t=\varphi} = \frac{1}{\Gamma(1-q)} \frac{d}{dt}\bigg(\int_{t_0}^t \frac{V(s, x_s)}{(t-s)^q}\, ds \bigg)\bigg|_{t=t_0, x_t=\varphi}. $$

The aim of this paper is to design a nonlinear observer-based controller to stabilise the origin of the following fractional time delay nonlinear system

$$ \begin{cases} {}_{t_0}D_t^q x(t) = Ax(t) + Bu(t) + f(x(t), x(t-\tau), u(t)) \\ \qquad y(t) = Cx(t), \end{cases} \tag{3.4} $$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input of the system, $y \in \mathbb{R}$ is the measured output and τ is a positive known scalar that denotes the time delay affecting the state variables. The matrices A, B and C are given by

$$ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0 \quad \cdots \quad 0 \quad 0]. $$

We suppose that f satisfies the following assumption.

ASSUMPTION A. *The nonlinearity $f(x(t), x(t-\tau), u(t))$ is smooth, globally Lipschitz with respect to x and $x(t-\tau)$ uniformly with respect to u, well defined for all $x(t) \in \mathbb{R}^n$ and satisfies $f(0, 0, u) = 0$.*

In the rest of the paper, the time argument is omitted and the delayed state vector $x(t-\tau)$ is noted by x_τ.

REMARK 3.3. We will derive a separation principle for the class of systems given by (3.4). The high-gain observer design framework established in [11] for free delay systems can be properly extended to this class of time-delay fractional differential systems. For the same class of systems (3.4) with $q = 1$, a separation principle and observer-based stabilisation were studied in [2] and [8, 12] respectively.

4. Separation principle

4.1. Observer design. In this subsection we will design an observer for system (3.4). The dynamics of the observer error $e = \hat{x} - x$ is given by

$$ {}_{t_0}D_t^q \hat{x}(t) = A\hat{x}(t) + Bu(t) + f(\hat{x}, \hat{x}_\tau, u) + L(C\hat{x} - y), \tag{4.1}$$

where $L = [l_1, \ldots, l_n]^T$ is selected such that $A_L = A + LC$ is Hurwitz. The state estimation error dynamic is given by

$$ {}_{t_0}D_t^q e = {}_{t_0} D_t^q \hat{x} - {}_{t_0}D_t^q x = (A + LC)e + f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u). \tag{4.2}$$

The following inequalities hold thanks to Assumption A:

$$ \|f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)\| \le k_1\|\hat{x} - x\| + k_2\|\hat{x}_\tau - x_\tau\| \le k\|e\| + k\|e_\tau\|, \tag{4.3}$$

where k_1 and k_2 are the Lipschitz constants for f with respect to x and x_τ and $k = \max(k_1,\ k_2)$.

The following theorem gives a suitable delay-independent condition that ensures the asymptotic stability of the observer (4.1).

THEOREM 4.1. *Consider the time-delay fractional differential system (3.4), under Assumption A. Let ϵ_1 be a positive scalar. The error system (4.2) is asymptotically stable if there exist symmetric positive definite matrices P and Q such that*

$$ PA_L + A_L^T P + Q + \epsilon_1 P^2 + \frac{2k^2}{\epsilon_1} I < 0, \tag{4.4}$$
$$ \frac{2k^2}{\epsilon_1} I - Q < 0. $$

PROOF. We consider the Lyapunov–Krasovskii functional candidate

$$ V(e_t) = {}_{t_0}D_t^{q-1} e^T P e + \int_{t-\tau}^{t} e^T(s) Q e(s)\, ds. \tag{4.5}$$

From Property 2.4, the derivative of (4.5) along the trajectories of (4.2) is

$$ \dot{V}(e_t) = {}_{t_0}D_t^q (e^T P e) + e^T Q e - e_\tau^T Q e_\tau. \tag{4.6}$$

By Lemma (2.5), taking the derivative of (4.5) yields the estimate

$$\begin{aligned} \dot{V}(e_t) &\le 2e^T P {}_{t_0}D_t^q e + e^T Q e - e_\tau^T Q e_\tau \\ &\le 2e^T P((A + LC)e + f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)) + e^T Q e - e_\tau^T Q e_\tau \\ &\le e^T(PA_L + A_L^T P)e + 2e^T P(f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)) + e^T Q e - e_\tau^T Q e_\tau \\ &\le e^T(PA_L + A_L^T P + Q)e + 2e^T P(f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)) - e_\tau^T Q e_\tau. \end{aligned} \tag{4.7}$$

From Lemma (2.7),

$$ 2e^T P(f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)) \le \epsilon_1 e^T P P e + \frac{1}{\epsilon_1}\|f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)\|^2. $$

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From (4.3),

$$\|f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)\|^2 \le k^2\|e\|^2 + k^2\|e_\tau\|^2 + 2k^2\|e\|\|e_\tau\|$$

and so (4.7) yields the following upper bound for $\dot{V}(e_t)$:

$$e^T(PA_L + A_L^T P + Q)e + \frac{k^2}{\epsilon_1}\|e\|^2 - e_\tau^T Q e_\tau + \frac{k^2}{\epsilon_1}\|e_\tau\|^2 + \frac{2k^2}{\epsilon_1}\|e\|\|e_\tau\| + e^T(\epsilon_1 PP)e.$$

Using the fact that $2\|e\|\|e_\tau\| \le \|e\|^2 + \|e_\tau\|^2$, we deduce that

$$\begin{aligned}\dot{V}(e_t) &\le e^T(PA_L + A_L^T P + Q + \epsilon_1 PP)e + \frac{2k^2}{\epsilon_1}\|e\|^2 + \frac{2k^2}{\epsilon_1}\|e_\tau\|^2 - e_\tau^T Q e_\tau \\ &\le e^T\Big\{PA_L + A_L^T P + Q + \epsilon_1 PP + \frac{2k^2}{\epsilon_1}I\Big\}e + e_\tau^T\Big\{\frac{2k^2}{\epsilon_1}I - Q\Big\}e_\tau.\end{aligned}$$

Therefore, $\dot{V}(e_t)$ is negative definite, which implies that the observation error of the time-delay fractional differential system (3.4) is asymptotically stable. □

Remark 4.2. To check whether the algebraic Riccati inequality (4.4) can be solved, it suffices to determine a positive definite solution of an associated Lyapunov matrix inequality (LMI), and then find sufficient conditions in terms of the solution of the Lyapunov matrix inequality.

Lemma 4.3. *Let ϵ be a positive scalar. Assume that P is a positive definite matrix solution of the Lyapunov inequality*

$$A_L^T P + PA_L + 2Q + \frac{2k^2}{\epsilon}I < 0.$$

Then P is also a solution of the algebraic Riccati inequality (4.4) provided that

$$\epsilon < \frac{\lambda_{\min}(Q)}{\lambda_{\max}^2(P)}. \tag{4.8}$$

Proof. Since Q is symmetric positive definite, then for all $e \in \mathbb{R}^n$,

$$\lambda_{\min}(Q)\|e\|^2 \le e^T Q e \le \lambda_{\max}(Q)\|e\|^2. \tag{4.9}$$

From (4.9), for any vector $e \in \mathbb{R}^n$,

$$e^T\Big(PA_L^T + A_L P + Q + \epsilon P^2 + \frac{2k^2}{\epsilon}I\Big)e \le e^T(-Q + \epsilon P^2)e \le e^T(-\lambda_{\min}(Q) + \epsilon\|P\|^2)e,$$

and it is easy to see from (4.8) that the last quantity is < 0, that is, (4.4) holds. □

Theorem 4.4. *Suppose that Assumption A is satisfied and that there exist symmetric, positive definite matrices P, Q and a positive constant ϵ_1 such that the LMI*

$$\begin{bmatrix} A_L^T P + PA_L + Q + (2k^2/\epsilon_1)I & \sqrt{\epsilon_1}P & 0 \\ \sqrt{\epsilon_1}P & -I & 0 \\ 0 & 0 & -(2k^2/\epsilon_1)I + Q \end{bmatrix} < 0$$

holds. Then (4.1) is a global asymptotic observer for the time-delay fractional differential system (3.4).

Proof. From the proof of Theorem 4.1 and Lemma 2.8 (Schur complement), we conclude that the observation error is asymptotically stable. □

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4.2. Global stabilisation by state feedback. In this subsection, we establish a delay-independent condition for the asymptotic state feedback stabilisation of the nonlinear system (3.4). The state feedback controller is given by

$$u = Kx, \tag{4.10}$$

where $K = [k_1, \ldots, k_n]$ is selected such that $A_K := A + BK$ is Hurwitz.

THEOREM 4.5. *Suppose that Assumption A is satisfied and that there are symmetric, positive definite matrices S, R and a positive constant ϵ_2 such that the LMI*

$$\begin{bmatrix} A_K^T S + SA_K + R + (2k^2/\epsilon_2)I & \sqrt{\epsilon_2}S & 0 \\ \sqrt{\epsilon_2}S & -I & 0 \\ 0 & 0 & -(2k^2/\epsilon_2)I + R \end{bmatrix} < 0 \tag{4.11}$$

holds. Then the origin of the closed-loop time-delay fractional differential system given by (3.4) and (4.10) is globally asymptotically stable.

PROOF. The closed-loop system is given by

$$_{t_0}D_t^q x(t) = (A + BK)x + f(x, x_\tau, u).$$

Choose a Lyapunov–Krasovskii functional candidate of the form

$$W(x_t) = {}_{t_0}D_t^{q-1} x^T S x + \int_{t-\tau}^{t} x^T(s)Rx(s)\,ds. \tag{4.12}$$

By differentiating (4.12) and using Property 2.4,

$$\begin{aligned} \dot{W}(x_t) &= {}_{t_0}D_t^q(x^T S x) + x^T Rx - x_\tau^T Rx_\tau \\ &\leq 2x^T S D^q x + x^T Rx - x_\tau^T Rx_\tau \\ &\leq x^T(A_K^T S + SA_K + R)x + 2x^T S f(x, x_\tau, u) - x_\tau^T Rx_\tau. \end{aligned} \tag{4.13}$$

From Lemma (2.7),

$$2x^T S f(x, x_\tau, u) \leq \epsilon_2 x^T S S x + \frac{1}{\epsilon_2}\|f(x, x_\tau, u)\|^2.$$

Since $f(0, 0, u) = 0$, (4.3) implies that $\|f(x, x_\tau, u)\| \leq k(\|x\| + \|x_\tau\|)$ and so (4.13) gives

$$\begin{aligned} \dot{W}(x_t) \leq x^T(A_K^T S + SA_K + R)x + x^T(\epsilon SS)x + \frac{k^2}{\epsilon_2}\|x\|^2 \\ + \frac{k^2}{\epsilon_2}\|x_\tau\|^2 + \frac{2k^2}{\epsilon_2}\|x\|\|x_\tau\| - x_\tau^T Rx_\tau. \end{aligned}$$

Using the fact that $2\|x\|\|x_\tau\| \leq \|x\|^2 + \|x_\tau\|^2$, we deduce that

$$\begin{aligned} \dot{W}(x_t) &\leq x^T(A_K^T S + SA_K + R + \epsilon_2 SS)x + \frac{2k^2}{\epsilon_2}\|x\|^2 + \frac{2k^2}{\epsilon}\|x_\tau\|^2 - x_\tau^T Rx_\tau \\ &\leq x^T\Big(A_K^T S + SA_K + R + \epsilon SS + \frac{2k^2}{\epsilon}I\Big)x + x_\tau^T\Big(\frac{2k^2}{\epsilon}I - R\Big)x_\tau. \end{aligned}$$

Therefore, $\dot{W}(x_t)$ is negative definite by Lemma 2.8 (Schur complement). We conclude that the closed-loop time-delay fractional differential system (3.4) and (4.10) is globally asymptotically stable if (4.11) holds. □

4.3. Observer-based control stabilisation. In this subsection, we implement the control law with estimated states. We consider the system controlled by the linear feedback law

$$u = K\hat{x}, \tag{4.14}$$

where the estimate $\hat{x}$ is provided by the observer (4.1).

THEOREM 4.6. *Under Assumption A, the origin of the closed-loop time-delay fractional differential system (3.4) and (4.14) is globally asymptotically stable if there exist two positive constants ϵ_1 and ϵ_2 and four symmetric, positive definite matrices P, Q, S and R such that the LMIs*

$$\begin{bmatrix} A_L^T P + PA_L + Q + (2k^2/\epsilon_1)I & \sqrt{\epsilon_1}P & K & 0 \\ \sqrt{\epsilon_1}P & -I & 0 & 0 \\ K^T & 0 & -I & 0 \\ 0 & 0 & 0 & (2k^2/\epsilon_1)I - Q \end{bmatrix} < 0 \tag{4.15}$$

and

$$\begin{bmatrix} A_K^T S + SA_K + R + (2k^2/\epsilon_2)I & \sqrt{\epsilon_2}S & SB & 0 \\ \sqrt{\epsilon_2}S & -I & 0 & 0 \\ B^T S & 0 & -I & 0 \\ 0 & 0 & 0 & (2k^2/\epsilon_2)I - R \end{bmatrix} < 0 \tag{4.16}$$

hold.

PROOF. The closed-loop system in the (U, V) coordinates can be represented by

$$\begin{aligned} {}_{t_0}D_t^q x &= A_K x + BKe + f(x, x_\tau, u), \\ {}_{t_0}D_t^q e &= A_L e + f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u). \end{aligned} \tag{4.17}$$

Let

$$U(e_t, x_t) = V(e_t) + W(x_t),$$

where V and W are given by (4.5) and (4.12) respectively. From the proof of Theorems 4.1 and 4.5,

$$\begin{aligned} \dot{U}(e_t, x_t) &\le 2x^T S\, {}_{t_0}D_t^q x + x^T Rx - x_\tau^T Rx_\tau + 2e^T P\, {}_{t_0}D_t^q e + e^T Qe - e_\tau^T Qe_\tau \\ &\le x^T (A_K^T S + SA_K + R)x + 2x^T SBKe + 2x^T Sf(x, x_\tau, u) - x_\tau^T Rx_\tau \\ &\quad + e^T (A_L^T P + PA_L + Q)e + 2e^T P(f(\hat{x}, \hat{x}_\tau, u) - f(x, x_\tau, u)). \end{aligned}$$

By Lemma 2.6,

$$2x^T SBKe \le x^T SBB^T Sx + e^T K^T Ke. \tag{4.18}$$

Substituting in the previous estimate for $\dot{U}(e_t, x_t)$ gives

$$\begin{aligned} \dot{U}(e_t, x_t) \le\ & x^T\Big(A_K^T S + SA_K + R + SBB^T S + \epsilon_2 SS + \frac{2k^2}{\epsilon_2}I\Big)x \\ & + e^T\Big\{A_L^T P + PA_L + Q + K^T K + \epsilon_1 PP + \frac{2k^2}{\epsilon_1}I\Big\}e \\ & + x_\tau^T\Big\{\frac{2k^2}{\epsilon_2}I - R\Big\}x_\tau + e_\tau^T\Big\{\frac{2k^2}{\epsilon_1}I - Q\Big\}e_\tau. \end{aligned}$$

From Lemma 2.8 (Schur complement), we conclude that the origin of the time-delay fractional differential system (4.17) is globally asymptotically stable if (4.15) and (4.16) hold. □

REMARK 4.7. There are many effective optimisation algorithms to solve (4.15) and (4.16) easily, for example, by means of the Hamiltonian matrix [4] or using MATLAB (LMIs) Control Toolbox [10].

REMARK 4.8. From Property 2.4, we can replace the Riemann–Liouville derivative in the fractional time-delay nonlinear system (3.4) by the Caputo fractional derivative and, with the same Assumption A, the conclusions of Theorems 4.4–4.6 still hold.

5. Numerical example

This section presents an experimental result based on the orientational motion of polar molecules acted on by an external perturbation. The physical model corresponds to a slow relaxation process described by an anomalous exponent α with $0 < \alpha < 1$. Such a model is described by the following system, with the fractional order $q = 0.5$, where $x(t)$ is an augmented state vector containing the plant state vector, u denotes the orientational potential energy and τ is the Debye relaxation time, assumed to be constant:

$$\begin{cases} {}_{t_0}D_t^q x(t) = Ax(t) + Bu(t) + f(x(t), x(t-\tau), u(t)), \\ \quad y(t) = Cx(t), \end{cases}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$f(x(t), x(t-\tau), u(t)) = \begin{bmatrix} 5\sqrt{2}(\sin x_3 + x_2(t-\tau)\cos u) \\ 5\sqrt{2}(\cos x_2 + \sin u) \\ 0 \end{bmatrix}.$$

Select $L = [-2 \;\; -4 \;\; -2]^T$ and $K = [-20 \;\; -15 \;\; -20]$, so that A_L and A_K are Hurwitz, and $\varepsilon_1 = 3.6880$, $\varepsilon_2 = 88.8160$. We find the symmetric positive definite matrices

$$P = \begin{bmatrix} 2.4543 & -0.6269 & -0.0198 \\ -0.6269 & 1.0081 & -0.6633 \\ -0.0198 & -0.6633 & 1.0895 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.8535 & -0.1221 & 0.0095 \\ -0.1221 & 0.2138 & -0.1627 \\ 0.0095 & -0.1627 & 0.2441 \end{bmatrix},$$

$$R = \begin{bmatrix} 36.7479 & 23.4524 & 36.6960 \\ 23.4524 & 19.1780 & 23.8417 \\ 36.6960 & 23.8417 & 38.2399 \end{bmatrix}, \quad S = \begin{bmatrix} 32.3290 & 11.2620 & 2.9435 \\ 11.2620 & 35.6031 & 3.1391 \\ 2.9435 & 3.1391 & 2.9052 \end{bmatrix}.$$

For our numerical simulation, we choose the delay $\tau = 1$. The corresponding numerical simulation results are shown in Figure 1.

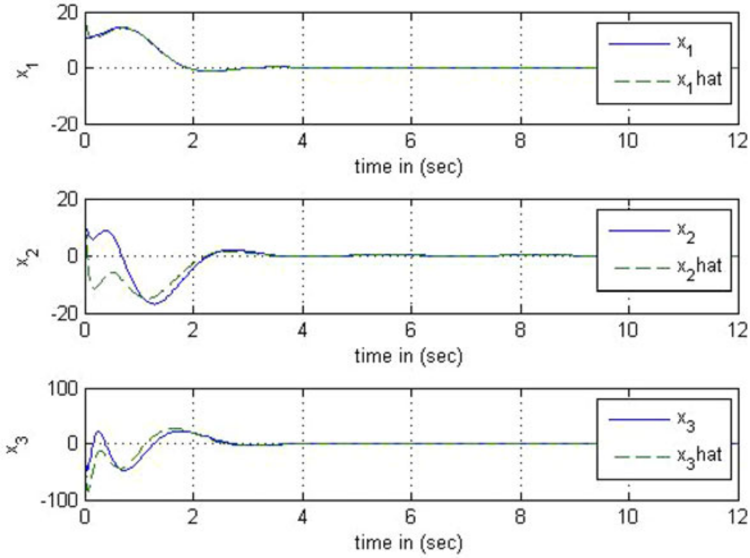

FIGURE 1. State trajectories x_i and their estimates $\hat{x}_i$.

6. Extension to time-varying delay systems

In this section, we extend our results to nonlinear systems of the form (3.4) with time-varying delays. We consider the system

$$\begin{cases} {}_{t_0}D_t^q x(t) = Ax(t) + Bu(t) + f(x(t), x(t-h(t)), u(t)) \\ \quad y(t) = Cx(t), \end{cases} \tag{6.1}$$

where $h(t)$ denotes the time-varying delay. We require an additional assumption to control h and its derivative.

ASSUMPTION B. *The time-varying delay satisfies the following conditions.*

(i) *There exists $\tau > 0$ such that $0 \leq h(t) \leq \tau$.*

(ii) *There exists $\beta > 0$ such that $\dot{h}(t) \leq 1 - \beta$.*

The same observer and state feedback controllers as those presented in Sections 4.1 and 4.2 achieve global asymptotically stability of system (6.1).

THEOREM 6.1. *Suppose that Assumptions A and B are fulfilled and that there exist two positive constants ϵ_1 and ϵ_2 and four symmetric, positive definite matrices P, Q, S and R such that the following LMIs hold:*

$$\begin{bmatrix} A_L^T P + PA_L + Q + (2k^2/\epsilon_1)I & \sqrt{\epsilon_1}P & K & 0 \\ \sqrt{\epsilon_1}P & -I & 0 & 0 \\ K^T & 0 & -I & 0 \\ 0 & 0 & 0 & (2k^2/\epsilon_1)I - \beta Q \end{bmatrix} < 0$$

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and

$$\begin{bmatrix} A_K^T S + SA_K + R + (2k^2/\epsilon_2)I & \sqrt{\epsilon_2}S & SB & 0 \\ \sqrt{\epsilon_2}S & -I & 0 & 0 \\ B^T S & 0 & -I & 0 \\ 0 & 0 & 0 & (2k^2/\epsilon_2)I - \beta R \end{bmatrix} < 0.$$

Then, the system (6.1) is globally asymptotically stable under the observer-based feedback (4.14).

Proof. We consider the Lyapunov–Krasovskii functional candidate

$$V(t, e_t) = {}_{t_0}D_t^{q-1} e^T Pe + \int_{t-h(t)}^{t} e^T(s)Qe(s)\,ds.$$

Denote the observation error by $e = \hat{x} - x$. Following the proof of Theorem 4.1, equation (4.6) becomes

$$\dot{V}(t, e_t) = D^q(e^T Pe) + e^T Qe - (1 - \dot{h}(t))e_{h(t)}^T Qe_{h(t)}.$$

Using this result and Assumption B,

$$\begin{aligned} \dot{V}(t, e_t) &\le 2e^T PD^q e + e^T Qe - (1 - \dot{h}(t))e_{h(t)}^T Qe_{h(t)} \\ &\le e^T(A_L^T P + PA_L + Q)e \\ &\quad + 2e^T P(f(\hat{x}, \hat{x}_{h(t)}, u) - f(x, x_{h(t)}, u)) - \beta e_{h(t)}^T Qe_{h(t)}. \end{aligned}$$

Therefore,

$$\dot{V}(t, e_t) \le e^T\Big\{A_L^T P + PA_L + Q + \epsilon_1 PP + \frac{2k^2}{\epsilon_1}I\Big\}e + e_{h(t)}^T\Big\{\frac{2k^2}{\epsilon_1}I - \beta Q\Big\}e_{h(t)}.$$

We aim to prove that, under the observer-based controller, the closed-loop system is input-to-state-stable with respect to the observation error. The closed-loop system is given by

$${}_{t_0}D_t^q x(t) = A_K x + BKe + f(x, x_{h(t)}, u).$$

Choose a Lyapunov–Krasovskii functional candidate of the form

$$W(t, x_t) = {}_{t_0}D_t^{q-1} x^T Sx + \int_{t-h(t)}^{t} x^T(s)Rx(s)\,ds.$$

Taking into account (4.18) and using Assumption B, we obtain the estimate

$$\begin{aligned} \dot{W}(t, x_t) &\le 2x^T S\, {}_{t_0}D_t^q x + x^T Rx - x_{h(t)}^T Rx_{h(t)} + 2e^T P\, {}_{t_0}D_t^q e + e^T Qe - e_{h(t)}^T Qe_{h(t)} \\ &\le x^T(A_K^T S + SA_K + R)x + 2x^T SBKe + 2x^T Sf(x, x_{h(t)}, u) - x_{h(t)}^T Rx_{h(t)} \\ &\quad + e^T(A_L^T P + PA_L + Q)e + 2e^T P(f(\hat{x}, \hat{x}_{h(t)}, u) - f(x, x_{h(t)}, u)) \\ &\le x^T\Big\{A_K^T S + SA_K + R + SBB^T S + \epsilon_2 SS + \frac{2k^2}{\epsilon_2}I\Big\}x + x_{h(t)}^T\Big\{\frac{2k^2}{\epsilon_2}I - \beta R\Big\}x_{h(t)} \\ &\quad + e^T\Big\{A_L^T P + PA_L + Q + K^T K + \epsilon_1 PP + \frac{2k^2}{\epsilon_1}I\Big\}e + e_{h(t)}^T\Big\{\frac{2k^2}{\epsilon_1}I - \beta Q\Big\}e_{h(t)}. \end{aligned}$$

By the Lemma 2.8 (Schur complement), the system (6.1) is globally asymptotically stable under the observer-based feedback (4.14). □

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REMARK 6.2. In [15], the Lyapunov method is used to give sufficient conditions for the asymptotic stability of Riemann–Liouville fractional systems with time-varying delays. These conditions are generalised in [16] for systems with multiple time-varying delays. However, the conditions are strong and it is difficult to engineer the design to achieve the conditions. Our criteria overcome some of the main sources of conservatism although using the same Lyapunov–Krasovskii functional as in [15, 16].

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