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Casselman's Basis of Iwahori Vectors and Kazhdan–Lusztig Polynomials

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Abstract. A problem in representation theory of p-adic groups is the computation of the *Casselman basis* of Iwahori fixed vectors in the spherical principal series representations, which are dual to the intertwining integrals. We shall express the transition matrix $(m_{u,v})$ of the Casselman basis to another natural basis in terms of certain polynomials that are deformations of the Kazhdan–Lusztig R-polynomials. As an application we will obtain certain new functional equations for these transition matrices under the algebraic involution sending the residue cardinality q to q^{-1}. We will also obtain a new proof of a surprising result of Nakasuji and Naruse that relates the matrix $(m_{u,v})$ to its inverse.

1 Statement of Results

We will state most of our results in this section, with proofs in Section 2. A few more results will be stated in Section 3.

Let q be the residue cardinality of F and let $\mathfrak{o}$ be its ring of integers. Let $\widehat{T}(\mathbb{C})$ be a split maximal torus in the Langlands dual group $\widehat{G}(\mathbb{C})$, a reductive algebraic group over $\mathbb{C}$. Let Φ be the root system of $\widehat{G}$ in the weight lattice $X^*(\widehat{T})$ of rational characters of $\widehat{T}$ that we identify with the group $X_*(T)$ of cocharacters in the maximal torus T of G that is dual to $\widehat{T}$. Let $B = TU$ be the Borel subgroup of G that is positive with respect to a decomposition of Φ into positive and negative roots. Let K be the standard (special) maximal compact subgroup, and let J be the positive Iwahori subgroup. The *Weyl group* is $W = N_G(T(F))/T(F)$. We will choose Weyl group representatives from K.

If $\mathbf{z} \in \widehat{T}$, then $\mathbf{z}$ parametrizes an unramified character $\chi_{\mathbf{z}}$ of $T(F)$. The corresponding principal series module $V_{\mathbf{z}}$ of $G(F)$ consists of smooth functions f on $G(F)$ such that $f(bg) = (\delta^{1/2}\chi_{\mathbf{z}})(b)f(g)$ for $b \in B(F)$. If $w \in W$, then choosing a Weyl group representative from K, and by abuse of notation denoting it also as w, there is an *intertwining integral operator* $\mathcal{A}_w \colon V_{\mathbf{z}} \to V_{w\mathbf{z}}$ defined by the integral

$$\mathcal{A}_w f(g) = \int_{U \cap wU_- w^{-1}} f(w^{-1}xg)\, dx.$$

Here, U_- is the unipotent radical of the Borel subgroup B_- opposite B, and although the integral is only convergent for $\mathbf{z}$ in an open subset of $\widehat{T}(\mathbb{C})$, it extends meromorphically to all of $\widehat{T}(\mathbb{C})$ by analytic continuation. Casselman [9] and Casselman

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and Shalika [10] emphasized the importance of the functionals $f \mapsto \mathcal{A}_w f(1)$ on the $|W|$-dimensional space $V_{\mathbf{z}}^J$ of Iwahori fixed vectors.

The space $V_{\mathbf{z}}^J$ of Iwahori-fixed vectors in $V_{\mathbf{z}}$ then have several important bases parametrized by the Weyl group W. One basis $\{\phi_w\}$ is obtained by restricting the standard spherical vector to the various cells in the Bruhat decomposition. That is, $G(F)$ is the disjoint union over $w \in W$ of cells BwJ, so if $w \in W$, we can define

$$\phi_w(bk) = \begin{cases} (\delta^{1/2}\chi_{\mathbf{z}})(b) & \text{if } k \in JwJ, \\ 0 & \text{otherwise.} \end{cases}$$

For us, a more useful basis is

$$\psi_w = \sum_{u \geqslant w} \phi_u,$$

where $\geqslant$ is the Bruhat order in W.

Another more subtle basis than the $\{\phi_w\}$ or $\{\psi_w\}$ was defined in [9] to be dual to the functionals $f \mapsto \mathcal{A}_w f(1)$. Thus, $\mathcal{A}_w f_{w'}(1) = \delta_{w,w'}$. Casselman wrote:

> It is an unsolved problem and, as far as I can see, a difficult one to express the bases $\{\phi_w\}$ and $\{f_w\}$ in terms of one another.

It seems more natural to ask for the transition function between the bases $\{f_w\}$ and $\{\psi_w\}$, and we will interpret the "Casselman problem" to mean this question.

The difficulty of this problem does not prevent the use of the Casselman basis $\{f_w\}$ in applications, for as Casselman [9] and Casselman-Shalika [10] showed, a small amount of information about the Casselman basis can be used to compute special functions such as the spherical and Whittaker functions. This is an idea that has been used in a great deal of subsequent literature. Because detailed information about the Casselman basis is not needed for these proofs, the Casselman problem has not seemed urgent. Nevertheless, the Casselman problem is very interesting in its own right because of a deep underlying structure similar to Kazhdan–Lusztig theory.

Before continuing, we remark that we will often find functions $(u,v) \mapsto a_{u,v}$ on $W \times W$ such that $a_{u,v}$ vanishes unless $u \leqslant v$. It is convenient to think of $(a_{u,v})_{u,v \in W}$ as a matrix whose index set is the Weyl group. Its product with another such matrix $(b_{u,v})$ is $(c_{u,v})$, where

$$c_{u,v} = \sum_{u \leqslant x \leqslant v} a_{u,x} b_{x,v}.$$

An important special case is the matrix $(a_{u,v})$, where $a_{u,v} = 1$ if $u \leqslant v$ and 0 otherwise. Then a theorem of Verma, which we will often use, is that if $(b_{u,v})$ is the inverse matrix, then $b_{u,v} = (-1)^{l(v)-l(u)}$ when $u \leqslant v$. This is the Möbius function for the Bruhat order; see [23, 25].

Applying Casselman's functionals to the basis $\{\psi_w\}$ gives numbers

$$m_{u,v} = \mathcal{A}_v \psi_u(1),$$

and these are the subject of this paper, as well as [6]. This is zero unless $u \leqslant v$ in the Bruhat order.

We also let $m'_{u,v}$ (denoted $\widetilde{m}_{u,v}$ in [6]) denote the inverse matrix so that

$$\sum_{u \leqslant x \leqslant v} m_{u,x} m'_{x,v} = \delta_{u,v},$$

where δ is the Kronecker delta. Clearly,

$$\psi_u = \sum_{v \geqslant u} m_{u,v} f_v \quad \text{and} \quad f_u = \sum_{v \geqslant u} m'_{u,v} \psi_v,$$

so the essence of the Casselman problem is to understand the $m_{u,v}$ and $m'_{u,v}$. We will give a kind of solution to this problem by showing that the $m_{u,v}$ and $m'_{u,v}$ can be expressed in terms of certain polynomials that are deformations of the Kazhdan–Lusztig R-polynomials.

First, we review two conjectures from our previous paper [6]. Let $P_{u,v}$ be the Kazhdan–Lusztig polynomials for W, defined as in [16]. We will also use the *inverse Kazhdan–Lusztig polynomial* $Q_{u,v} = P_{w_0 v, w_0 u}$, where w_0 is the long Weyl group element. Both $P_{u,v}$ and $Q_{u,v}$ vanish unless $u \leqslant v$.

If $\alpha \in \Phi$, let r_α denote the corresponding reflection in W. Assume that $u \leqslant v$. Define

$$S(u,v) = \{\alpha \in \Phi^+ \mid u \leqslant v.r_\alpha < v\} \quad \text{and} \quad S'(u,v) = \{\alpha \in \Phi^+ \mid u \leqslant u.r_\alpha < v\}.$$

It is a consequence of work of Deodhar [12], Carrell and Peterson [7], Polo [21], Dyer [13], and Jantzen [15] that the sets $S(u,v)$ and $S'(u,v)$ have cardinality $\geqslant l(v) - l(u)$. Moreover, if the inverse Kazhdan–Lusztig polynomial $Q_{u,v} = 1$, then $|S(u,v)| = l(v) - l(u)$, while if $P_{u,v} = 1$ then $|S'(u,v)| = l(v) - l(u)$.

In [6] we conjectured that if Φ is simply-laced and $Q_{u,v} = 1$, then

$$m_{u,v} = m_{u,v}(\mathbf{z}) = \prod_{\alpha \in S(u,v)} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha}. \tag{1.1}$$

This formula generalizes the well-known formula of Gindikin and Karpelevich, which is actually due to Langlands [17] in this nonarchimedean setting. This is the special case where $u = 1$, so that ψ_1 is the K-spherical vector in $V_{\mathbf{z}}$. However, the method commonly used to prove the formula of Gindikin and Karpelevich inductively does not work for general u, and this conjecture still seems difficult. See [19, 20] for recent work on this problem, and Section 3 below for some new results based on the methods of this paper.

Similarly, if $P_{u,v} = 1$, then $|S'(u,v)| = l(v) - l(u)$, and in this case we conjectured that

$$m'_{u,v} = (-1)^{l(v)-l(u)} \prod_{\alpha \in S'(u,v)} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha}. \tag{1.2}$$

It was shown by Nakasuji and Naruse [20] that these two conjectured formulas (1.1) and (1.2) are equivalent. They did this by proving a very interesting fact relating the matrices $(m_{u,v})$ and $(m'_{u,v})$, which we will reprove in this paper as Theorem 1.5.

In this paper we will not prove these conjectures. Instead we will strive to adapt methods of Kazhdan and Lusztig [16] to this situation. For example, the above conjectures can be thought of as closely related to their formula (2.6.b).

Our algebraic results about $m_{u,v}$ are independent of the origin of the problem in p-adic groups. So we can regard q as an indeterminate. If f is a polynomial in q, following Kazhdan and Lusztig, $\overline{f}$ will denote the result of replacing q by q^{-1}. If f

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involves $\mathbf{z}$, then $\mathbf{z}$ is unchanged in $\overline{f}$ unless we explicitly indicate a change. We will also use the notation $\varepsilon_w = (-1)^{l(w)}$ and $q_w = q^{l(w)}$ from [16].

Assume that $Q_{u,v} = 1$, that Φ is simply-laced so that (1.1) is conjectured, and moreover, $|S(u,v)| = l(v) - l(u)$. Observe that $m_{u,v}$ satisfies the functional equation

$$\overline{m_{u,v}}(\mathbf{z}) = q_v q_u^{-1} m_{u,v}(\mathbf{z}^{-1}). \tag{1.3}$$

Theorem 1.1 *Assume that $Q_{u,v} = 1$. Then the functional equation (1.3) is satisfied.*

Note that this does not require Φ to be simply-laced, even though (1.1) has counterexamples already for B_2. Proofs can be found in the next section.

The key to this and other results is to introduce a deformation of the Kazhdan and Lusztig R-polynomials, defined in [16].

Theorem 1.2 *There exist polynomials $r_{u,v}(\mathbf{z})$, depending on $\mathbf{z} \in \widehat{T}(\mathbb{C})$ such that $r_{u,u} = 1$ and $r_{u,v} = 0$ unless $u \leqslant v$. They have the property that $r_{u,v}(\mathbf{z}) \to R_{u,v}$ if $\mathbf{z} \to \infty$ in such a direction that $\mathbf{z}^\alpha \to \infty$ for all positive roots $\alpha \in \Phi^+$. They can be calculated by the following recursion formula. Choose a simple reflection $s = s_\alpha$ corresponding to the simple root α such that $sv < v$. If $su < u$, then*

$$r_{u,v}(\mathbf{z}) = \frac{1-q}{1-\mathbf{z}^{-v^{-1}\alpha}} r_{u,sv}(\mathbf{z}) + r_{su,sv}(\mathbf{z}).$$

If $su > u$, then

$$r_{u,v}(\mathbf{z}) = (1-q)\frac{\mathbf{z}^{-v^{-1}\alpha}}{1-\mathbf{z}^{-v^{-1}\alpha}} r_{u,sv}(\mathbf{z}) + q r_{su,sv}(\mathbf{z}).$$

In the recursion, it is worth noting that since $sv < v$, $-v^{-1}\alpha$ is a positive root. Then the $m_{u,v}$ can be expressed in terms of the $r_{u,v}$ as follows.

Theorem 1.3 *Suppose that $u \leqslant v$. Then*

$$m_{u,v} = \sum_{u \leqslant x \leqslant v} \overline{r_{x,v}}, \tag{1.4}$$

$$r_{u,v} = \sum_{u \leqslant x \leqslant v} \varepsilon_u \varepsilon_x \overline{m_{x,v}}. \tag{1.5}$$

The proof will be given in the next section. We will deduce (1.3) from this result. Moreover, we will prove the following general identity. If $u \leqslant v$, define

$$c_{u,v} = \sum_{u \leqslant x \leqslant y \leqslant z \leqslant v} \varepsilon_x \varepsilon_y q_y^{-1} q_u P_{x,y} \overline{Q_{y,z}} \varepsilon_z \varepsilon_v. \tag{1.6}$$

(Let $c_{u,v} = 0$ if u is not $\leqslant v$.)

Theorem 1.4 *If $u \leqslant v$, then*

$$\overline{m_{u,v}}(\mathbf{z}) = q_v q_u^{-1} \sum_{u \leqslant w \leqslant v} c_{u,w} m_{w,v}(\mathbf{z}^{-1}). \tag{1.7}$$

The proof will be given in the next section. The coefficients $c_{u,v}$ are interesting. If $u = v$, then $c_{u,v} = 1$, but otherwise they are usually zero. The 46 pairs u, v with $c_{u,v} \neq 1$

and $u < v$ for the A_4 Weyl group are tabulated in Figure 1. This includes all 38 pairs of Weyl group elements with $u \prec v$ in the notation of Kazhdan and Lusztig. This means that $l(v) - l(u)$ is odd and $\geqslant 3$, and that the degree of $P_{u,v}$ is $\frac{1}{2}(l(v) - l(u) - 1)$, the largest possible. But there are a few other values for which $c_{u,v} \neq 0$.

u	v	$c_{u,v}$	$u \prec v$				
s_3s_2	$s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-3}$		$s_4s_1s_2$	$s_1s_2s_3s_4s_3s_2$	$q^{-1} - q^{-2}$	✓
s_3s_1	$s_3s_4s_2s_3s_1$	$q^{-1} - q^{-2}$	✓	s_4s_2	$s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓
$s_4s_1s_2s_1$	$s_1s_2s_3s_4s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_3s_1s_2$	$s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓
$s_2s_3s_2$	$s_2s_3s_4s_1s_2s_3$	$q^{-1} - q^{-2}$	✓	$s_1s_2s_3s_2$	$s_3s_4s_1s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓
s_4s_2	$s_2s_3s_4s_3s_1s_2$	$-q^{-1} + q^{-3}$		$s_2s_3s_2$	$s_2s_3s_4s_1s_2s_3s_1s_2$	$q^{-2} - q^{-3}$	✓
$s_3s_4s_1s_2s_1$	$s_1s_2s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_3s_4s_3s_1$	$s_1s_2s_3s_4s_2s_3s_2s_1$	$-q^{-1} + q^{-3}$	
s_4s_2	$s_2s_3s_4s_3s_2$	$q^{-1} - q^{-2}$	✓	$s_3s_4s_3s_1s_2s_1$	$s_1s_2s_3s_4s_2s_3s_1s_2s_1$	$q^{-1} - q^{-2}$	✓
s_3s_1	$s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_4s_2s_3s_1$	$s_2s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-3}$	
$s_3s_4s_1$	$s_1s_2s_3s_4s_2s_1$	$q^{-1} - q^{-2}$	✓	s_2	$s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓
$s_2s_3s_4s_2$	$s_2s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_4s_1s_2s_1$	$s_1s_2s_3s_4s_3s_1s_2s_1$	$-q^{-1} + q^{-3}$	
$s_2s_3s_2$	$s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_4s_2s_3s_2$	$s_2s_3s_4s_1s_2s_3s_2$	$q^{-1} - q^{-2}$	✓
s_4s_1	$s_1s_2s_3s_4s_3s_2s_1$	$q^{-2} - q^{-3}$	✓	$s_4s_2s_3$	$s_2s_3s_4s_1s_2s_3$	$q^{-1} - q^{-2}$	✓
s_3s_1	$s_3s_4s_1s_2s_3$	$q^{-1} - q^{-2}$	✓	s_2s_3	$s_2s_3s_4s_1s_2s_3$	$q^{-1} - q^{-3}$	
$s_4s_2s_3s_2s_1$	$s_2s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_4s_3s_1$	$s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓
$s_1s_2s_3s_4s_3s_1$	$s_1s_2s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_3s_4s_1s_2$	$s_1s_2s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-3}$	
$s_4s_1s_2s_1$	$s_1s_2s_3s_4s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_3s_4s_3s_1$	$s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓
s_4s_2	$s_2s_3s_4s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_3s_4s_3s_1$	$s_1s_2s_3s_4s_2s_3s_1$	$q^{-1} - q^{-2}$	✓
$s_2s_3s_4s_3s_1$	$s_2s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_4s_1s_2s_3s_1$	$s_2s_3s_4s_1s_2s_3s_2s_1$	$q^{-1} - q^{-2}$	✓
$s_2s_3s_2s_1$	$s_2s_3s_4s_1s_2s_3s_1$	$q^{-1} - q^{-2}$	✓	s_3s_1	$s_3s_4s_1s_2s_3s_1$	$-q^{-1} + q^{-3}$	
$s_3s_4s_3s_1s_2$	$s_1s_2s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_2s_3s_1$	$s_2s_3s_4s_1s_2s_3$	$q^{-1} - q^{-2}$	✓
$s_4s_1s_2s_1$	$s_2s_3s_4s_3s_1s_2s_1$	$q^{-1} - q^{-2}$	✓	$s_4s_2s_1$	$s_2s_3s_4s_3s_2s_1$	$q^{-1} - q^{-2}$	✓
$s_3s_4s_3s_1$	$s_1s_2s_3s_4s_3s_2s_1$	$q^{-1} - q^{-2}$	✓	s_3	$s_3s_4s_2s_3$	$q^{-1} - q^{-2}$	✓
$s_3s_4s_2$	$s_3s_4s_2s_3s_1s_2$	$q^{-1} - q^{-2}$	✓	$s_1s_2s_3s_4s_2$	$s_1s_2s_3s_4s_2s_3s_1s_2a$	$q^{-1} - q^{-2}$	✓

Figure 1: The pairs u,v in the A_4 Weyl group with $u < v$ and $c_{u,v} \neq 0$. The simple reflections are s_1, s_2, s_3 and s_4. This list includes all 38 pairs with $u \prec v$ in the notation of Kazhdan and Lusztig (marked with ✓). Note that if $u \prec v$ then $c_{u,v} = q^{-1} - q^{-2}$ but there are a few other pairs u, v with $c_{u,v} \neq 0$.

Finally, we have a striking symmetry of the coefficients $m_{u,v}$. Equation (1.9) in the following theorem was proved previously by Nakasuji and Naruse [20]. We will give another proof based on Theorem 1.2.

Theorem 1.5 (Nakasuji and Naruse [20]) *Suppose that* $u \leqslant v$. *Then*

$$\sum_{u \leqslant x \leqslant v} r_{u,x} \varepsilon_x \varepsilon_v r_{w_0 v, w_0 x} = \delta_{u,v}, \tag{1.8}$$

$$\sum_{u \leqslant x \leqslant v} m_{u,x} \varepsilon_x \varepsilon_v m_{w_0 v, w_0 x} = \delta_{u,v}. \tag{1.9}$$

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The proof will be given in the next section. Because $(m'_{u,v})$ was defined to be the inverse of the matrix $(m_{u,v})$, the last result can be written $m'_{u,v} = \varepsilon_u \varepsilon_v m_{w_0 v, w_0 u}$. This seems a remarkable fact.

We end this section with a conjecture about the poles of $m_{u,v}$. As functions of $\mathbf{z}$, the function $r_{u,v}(\mathbf{z})$ is analytic on the regular set of $\widehat{T}$, that is, the subset of $\mathbf{z}$ such that $\mathbf{z}^\alpha \neq 1$ for all $\alpha \in \Phi$.

Conjecture 1.6 *The functions*

$$\prod_{\beta \in S(u,v)} (1 - \mathbf{z}^\beta) m_{u,v} \quad \textit{and} \quad \prod_{\beta \in S(u,v)} (1 - \mathbf{z}^\beta) r_{u,v}$$

are analytic on all of $\widehat{T}(\mathbb{C})$.

Since $m_{u,v} = \sum_{u \leqslant x \leqslant v} \overline{r_{x,v}}$ and $S(x,v) \subseteq S(u,v)$ when $u \leqslant x \leqslant v$, the statement about $m_{u,v}$ follows from the statement about $r_{u,v}$. Moreover, the recursion in Theorem 1.2 gives a way of trying to prove this recursively. So let us choose a simple reflection s such that $sv < v$. It is sufficient to show that $\prod_{\alpha \in S(u,v)} (1 - \mathbf{z}^\beta)$ cancels the poles of both $r_{su,sv}$ and of $(1 - \mathbf{z}^{-v^{-1}\alpha})^{-1} r_{u,sv}$.

The factor $(1 - \mathbf{z}^{-v^{-1}\alpha})^{-1}$ that appears with $r_{u,sv}$ is cancelled for the following reason. It only appears if $r_{u,sv} \neq 0$, that is, if $u \leqslant sv$. Now if this is so, then the positive root $-v^{-1}\alpha$ is in $S(u,v)$, because $v r_{-v^{-1}\alpha} = sv$, and then $u \leqslant sv$ implies $-v^{-1}\alpha \in S(u,v)$.

So the statement that $\prod_{\beta \in S(u,v)} (1 - \mathbf{z}^\beta)$ cancels the poles of $r_{u,v}$ would follow recursively if we knew that $S(u,sv)$ and $S(su,sv)$ are both contained in $S(u,v)$. Unfortunately, this is not always true, as the following example shows.

Example 1.7 Let Φ be the A_2 root system, with simple roots α_1, α_2 and corresponding simple reflections s_1, s_2. Let $u = s_1$, $v = s_1 s_2 s_1$, and $\beta = \alpha_1 + \alpha_2$. Then if we take $s = s_1$, we have $\beta \in S(u,sv)$ and $\beta \in S(su,sv)$ but $\beta \notin S(u,v)$. This means that the locus of $\mathbf{z}^\beta = 1$ is a pole of both terms in the recursion, but these poles cancel, and it is not a pole of $r_{u,v}(\mathbf{z})$.

At the moment we do not have a proof that such cancellation always occurs, but often it can be proved using a different descent. In Example 1.7 with u, v and β as given, we could take $s = s_2$ instead, and then we find that $\beta \notin S(u,sv)$ and $\beta \notin S(su,sv)$, so $1 - \mathbf{z}^\beta$ does not divide the denominator of $r_{u,v}$.

2 Proofs

Let $\mathcal{H}$ be the Iwahori Hecke algebra of the Coxeter group W, with basis elements T_w for $w \in W$, such that $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$. Thus, if s is a simple reflection, we have $T_s^2 = (q-1)T_s + q$, and the usual braid relations are satisfied. We extend the scalars to the field of meromorphic functions on $\widehat{T}(\mathbb{C})$. Then the Hecke algebra has another basis, which we will now describe. Let $\mathbf{z} \in \widehat{T}(\mathbb{C})$. If $s = s_\alpha$ is a simple reflection, and α is the corresponding simple root, let $\mu_{\mathbf{z}}(s)$ be the element of the Hecke algebra defined by

$$\mu_{\mathbf{z}}(s) = q^{-1}T_s + (1-q^{-1})\frac{\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha} = T_s^{-1} + \frac{1-q^{-1}}{1-\mathbf{z}^\alpha}.$$

It was shown in [6], using ideas of Rogawski [22], that we can extend this definition to $\mu_{\mathbf{z}}(w)$ for $w \in W$ such that if $l(w_1w_2) = l(w_1) + l(w_2)$, then

$$\mu_{\mathbf{z}}(w_1w_2) = \mu_{\mathbf{z}}(w_2)\mu_{w_2\mathbf{z}}(w_1).$$

The Hecke operator $\mu_w(\mathbf{z})$ models the intertwining operator $\mathcal{A}_w\colon V_{\mathbf{z}} \to V_{w\mathbf{z}}$, as is explained in [22] or [6]. It was clarified by Nakasuji and Naruse [20] that the basis μ_w is essentially the "Yang–Baxter basis" of Lascoux, Leclerc, and Thibon [18], and the consistency of the definition follows from the Yang–Baxter equation. The appearance of the Yang–Baxter equation in the context of p-adic intertwining operators is then related to the viewpoint in Brubaker, Buciumas, Bump, and Friedberg [5].

Suppose that $s = s_\alpha$ is a simple reflection. Then it is easy to check by direct computation that

$$\mu_{\mathbf{z}}(s)\mu_{s\mathbf{z}}(s) = \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}\cdot\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}. \tag{2.1}$$

Lemma 2.1 *Let $s = s_\alpha$ be a simple reflection. Then for any $w \in W$, we have $\mu_{\mathbf{z}}(w)\mu_{w\mathbf{z}}(s) = c\cdot\mu_{\mathbf{z}}(sw)$, where the constant*

$$c = \begin{cases} 1 & \text{if } sw > w,\\ \frac{1-q^{-1}\mathbf{z}^{w^{-1}\alpha}}{1-\mathbf{z}^{w^{-1}\alpha}}\cdot\frac{1-q^{-1}\mathbf{z}^{-w^{-1}\alpha}}{1-\mathbf{z}^{-w^{-1}\alpha}} & \text{if } sw < w.\end{cases}$$

Proof If $sw > w$, this follows from the definition of $\mu_{\mathbf{z}}(sw)$. In the other case, we write $\mu_{\mathbf{z}}(w) = \mu_{\mathbf{z}}(sw)\mu_{sw\mathbf{z}}(s)$, then apply (2.1). ∎

Let $\Lambda\colon \mathcal{H} \to \mathbb{C}(q)$ be the functional such that $\Lambda(T_w) = 1$ if $w = 1$, and 0 otherwise. Also, let $\psi_w = \sum_{u\geqslant w} T_u$. We are reusing the notation ψ_w used previously to denote certain Iwahori fixed vectors, but we are leaving the origins of the problem in the p-adic group behind, so this reuse should not cause any confusion. Following Rogawski [22], there is a vector space isomorphism between the Iwahori fixed vectors in the principal series representation and the Hecke algebra $\mathcal{H}$, and in this isomorphism, the Iwahori fixed vectors ψ_w correspond to the Hecke elements ψ_w.

In [6], we proved that

$$m_{u,v} = m_{u,v}(\mathbf{z}) = \Lambda(\psi_u\mu_{\mathbf{z}}(v)).$$

This will be the starting point of our proofs.

Lemma 2.2 *If $u, v \in W$, then*

$$\Lambda(T_uT_v) = \begin{cases} q_u & \text{if } u = v^{-1},\\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Proof Without loss of generality $l(u) \leqslant l(v)$. Assume that $\Lambda(T_uT_v) \neq 0$. We will show that $u = v^{-1}$ and that $\Lambda(T_uT_v) = q_u$. Proof is by induction on $l(u)$, so we assume that $\Lambda(T_{u'}T_v)$ is given by this formula for all $u' < u$ and for all v. The formula

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(2.2) is trivial if $u = 1$, so we can assume that $u > 1$. Let s be a simple reflection such that $us < u$. Let $u' = us$ and $v' = sv$.

Suppose that $v' < v$. Then $T_v = T_s T_{v'}$ and $T_u = T_{u'} T_s$. Thus,

$$T_u T_v = T_{u'} T_s^2 T_{v'} = (q-1) T_{u'} T_s T_{v'} + q T_{u'} T_{v'} = (q-1) T_{u'} T_v + q T_{u'} T_{v'}. \tag{2.3}$$

Thus, either $\Lambda(T_{u'} T_v) \neq 0$ or $\Lambda(T_{u'} T_{v'}) \neq 0$. By induction, we have either $u' = v^{-1}$ or $u' = (v')^{-1}$. The first is not possible, since $l(u') < l(v)$, so $u' = (v')^{-1}$ and $u^{-1} = v^{-1}$. Now applying Λ to (2.3) gives $\Lambda(T_u T_v) = q\Lambda(T_{u'} T_{v'}) = q q_{u'} = q_u$.

The case $v' > v$ is easier. Then $\Lambda(T_u T_v) = \Lambda(T_{u'} T_{s_i} T_v) = \Lambda(T_{u'} T_{v'}) = 0$, since $l(u') < l(v')$. And u cannot equal v^{-1}, since s is a right descent of u but not v^{-1}. ■

We will make use of the Kazhdan–Lusztig involution $f \mapsto \overline{f}$ on functions f of q and $\mathbf{z}$. This is the map that sends q to q^{-1} and $\mathbf{z}$ to $\mathbf{z}$. We recall from [16] that it is also the map $q \mapsto q^{-1}$ extended to an automorphism of the Hecke algebra by the map $T_w \mapsto T_{w^{-1}}^{-1}$.

We define $r_{u,v} = r_{u,v}(\mathbf{z})$ by

$$\mu_{\mathbf{z}}(v) = \sum_{u \leqslant v} q_u^{-1} \overline{r_{u,v}}\, T_{u^{-1}}. \tag{2.4}$$

We will first prove Theorem 1.2 followed by Theorem 1.3 and Theorem 1.1.

Proof of Theorem 1.2 Beginning with (2.4), we can compute $\overline{r_{u,v}}$ by calculating the coefficient of $T_{u^{-1}}$ in

$$\mu_{\mathbf{z}}(v) = \mu_{\mathbf{z}}(sv)\, \mu_{sv\mathbf{z}}(s) = \Big(\sum_{x \leqslant sv} q_x^{-1} \overline{r_{x,sv}}\, T_{x^{-1}} \Big) \Big(q^{-1} T_s + (1-q^{-1}) \frac{(sv\mathbf{z})^{\alpha}}{1-(sv\mathbf{z})^{\alpha}} \Big).$$

Only $x = u$ or su can contribute to the coefficient of $T_{u^{-1}}$. Comparing the coefficients of $T_{u^{-1}}$ and noting that $(sv\mathbf{z})^{\alpha} = \mathbf{z}^{-v^{-1}\alpha}$, the recursion formula is obtained.

Now the Kazhdan–Lusztig R-polynomials satisfy a similar recurrence, at the beginning of [16, Section 2]. So specializing $\mathbf{z} \to \infty$ in such a way that $\mathbf{z}^{\alpha} \to \infty$ for all positive roots, we see that $r_{u,v} \to R_{u,v}$. For this it is important that when s is a left descent of v, the root $-v^{-1}\alpha$ that appears in the recursion is positive. ■

Theorem 1.2 has the following implication for the Yang–Baxter basis $\mu_{\mathbf{z}}(w)$, which was pointed out to us by the referee. Suppose that we specialize $\mathbf{z} \to \infty$ as in Theorem 1.2. Then since $r_{u,v}(\mathbf{z}) \to R_{u,v}$, using [16, (2.0.a)] and the fact that $R_{u,v} = R_{u^{-1},v^{-1}}$,

$$\mu_{\mathbf{z}}(v) \longrightarrow \sum_u q_u^{-1} \overline{R_{u,v}}\, T_{u^{-1}} = T_v^{-1}.$$

Proposition 2.3 *We have $r_{u,v} = 0$ unless $u \leqslant v$, and $r_{v,v} = 1$. Moreover, $\overline{r_{u,v}} = \varepsilon_u \varepsilon_v q_u q_v^{-1} r_{u,v}$.*

Proof Both assertions follow from Theorem 1.2 by induction on $l(v)$. ■

If $u \leqslant v$ in W, we will denote by $[u,v]$ the Bruhat interval $\{x \in W | u \leqslant x \leqslant v\}$.

Proof of Theorem 1.3 By definition

$$m_{u,v} = \Lambda(\psi_u \mu_{\mathbf{z}}(v)) = \sum_{x \geqslant u} \sum_{y \leqslant v} q_y^{-1} \overline{r_{y,v}} \Lambda(T_x T_{y^{-1}}).$$

Equation (1.4) now follows from Lemma 2.2. By Verma's theorem, the Möbius function on the Bruhat interval $[u,v]$ is $(x,y)\mapsto \varepsilon_x\varepsilon_y$. (See [23].) Thus, (1.5) follows from (1.4). ∎

Lemma 2.4 *We have* $\overline{\mu_{\mathbf{z}}(w)} = q_w\mu_{\mathbf{z}^{-1}}(w)$.

Proof This reduces to the case where w is a simple reflection, and this case is easily checked from the definition. ∎

Proposition 2.5 *We have*

$$\sum_{u\leqslant w\leqslant t\leqslant v}\overline{Q_{u,w}}\varepsilon_w\varepsilon_t m_{t,v}(\mathbf{z}^{-1}) = q_u q_v^{-1}\sum_{u\leqslant y\leqslant v} Q_{u,y}r_{y,v}(\mathbf{z}). \tag{2.5}$$

Proof Using Lemma 2.4,

$$q_v\sum_{u\leqslant v} q_u^{-1}\overline{r_{u,v}}(\mathbf{z}^{-1})T_{u^{-1}} = q_v\mu_{\mathbf{z}^{-1}}(v) = \overline{\mu_{\mathbf{z}}(v)} = \sum_{y\leqslant v} q_y r_{y,v}(\mathbf{z})T_y^{-1}.$$

Bearing in mind that $R_{u,v}=R_{u^{-1},v^{-1}}$, [16, (2.0.a)] implies that $T_y^{-1}=\sum_{u\leqslant y}\overline{R_{u,y}}q_u^{-1}T_{u^{-1}}$. Substituting this on the right-hand side, and comparing the coefficients of $T_{u^{-1}}$, gives

$$\overline{r_{u,v}}(\mathbf{z}^{-1}) = q_v^{-1}\sum_{u\leqslant y\leqslant v} q_y\overline{R_{u,y}}r_{y,v}(\mathbf{z}).$$

Then by Theorem 1.3,

$$m_{u,v}(\mathbf{z}^{-1}) = \sum_{u\leqslant x\leqslant v}\overline{r_{x,v}}(\mathbf{z}^{-1}) = \sum_{u\leqslant x\leqslant y\leqslant v} q_v^{-1}q_y\overline{R_{x,y}}r_{y,v}(\mathbf{z}).$$

By Verma's theorem [23, 25],

$$\sum_{u\leqslant t\leqslant v}\varepsilon_u\varepsilon_t m_{t,v}(\mathbf{z}^{-1}) = \sum_{u\leqslant t\leqslant x\leqslant y\leqslant v}\varepsilon_u\varepsilon_t q_v^{-1}q_y\overline{R_{x,y}}r_{y,v}(\mathbf{z}) = \sum_{u\leqslant y\leqslant v} q_v^{-1}q_y\overline{R_{u,y}}r_{y,v}(\mathbf{z}).$$

Thus,

$$\sum_{u\leqslant w\leqslant t\leqslant v}\overline{Q_{u,w}}\varepsilon_w\varepsilon_t m_{t,v}(\mathbf{z}^{-1}) = \sum_{u\leqslant w\leqslant y\leqslant v}\overline{Q_{u,w}}q_v^{-1}q_y\overline{R_{w,y}}r_{y,v}(\mathbf{z}).$$

Now we require the identity

$$Q_{u,y} = q_u^{-1}q_y\sum_{u\leqslant w\leqslant y}\overline{Q_{u,w}}\cdot\overline{R_{w,y}},$$

which can be deduced from [16, (2.2.a)]. Applying this gives (2.5). ∎

The following property of Kazhdan–Lusztig polynomials is due to Carrell and Peterson [7].

$$\text{If } u\leqslant v \text{ and } P_{u,v}=1, \text{ then } P_{x,v}=1 \text{ for } u\leqslant x\leqslant v. \tag{2.6}$$

This is also proved in [1], where the result is stated on page 77, and the proof is contained in the proof of Theorem 6.2.4.

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Proof of Theorem 1.1 Using (2.6), since $Q_{u,v}=1$, we have $Q_{u,x}=1$ for all $x\in[u,v]$. Thus, (2.5) reads

$$\sum_{u\leqslant s\leqslant t\leqslant v}\varepsilon_s\varepsilon_t m_{t,v}(\mathbf{z}^{-1})=q_u q_v^{-1}\sum_{u\leqslant y\leqslant v}r_{y,v}(\mathbf{z})=q_u q_v^{-1}\overline{m_{u,v}}.$$

The result follows from Verma's theorem. ∎

Let $c_{u,v}$ be as in (1.6).

Proof of Theorem 1.4 Using (2.5), write

$$q_v\sum_{x\leqslant y\leqslant z\leqslant t\leqslant v}\varepsilon_x\varepsilon_y q_y^{-1}P_{x,y}\overline{Q_{y,z}}\varepsilon_z\varepsilon_t m_{t,v}(\mathbf{z}^{-1})=\sum_{x\leqslant y\leqslant t\leqslant v}\varepsilon_x\varepsilon_y P_{x,y}Q_{y,t}r_{t,v}(\mathbf{z}).$$

Using the inversion formula [16, Theorem 3.1], the right-hand side is just $r_{x,v}(\mathbf{z})$. Now summing over x in $[u,v]$ and using (1.4) gives (1.7). ∎

In preparation for proving Theorem 1.5, define $r'_{u,v}$ to be the inverse of $(r_{u,v})$ regarded as a matrix on $|W|$. Thus,

$$\sum_{u\leqslant x\leqslant v}r_{u,x}r'_{x,v}=\sum_{u\leqslant x\leqslant v}r'_{u,x}r_{x,v}=\delta_{u,v}.$$

Then, using Verma's theorem, it is easy to see that

$$m'_{u,v}=\sum_{u\leqslant x\leqslant v}\varepsilon_x\varepsilon_v\overline{r'_{u,x}}\quad\text{and}\quad r'_{u,v}=\sum_{u\leqslant x\leqslant v}\overline{m'_{u,x}}.$$

The coefficient $r'_{u,v}(\mathbf{z})$ specializes to $\varepsilon_u\,\varepsilon_v\,R_{u,v}$ as $\mathbf{z}\to\infty$. This is clear from [16, Lemma 2.1(ii)]. Nevertheless, we are not aware of any simple relationship between the coefficients r and r'.

The coefficients $r'_{u,v}$ satisfy a recursion similar to Theorem 1.2.

Proposition 2.6 *Suppose that $su>u$. If $v<sv$,*

$$r'_{u,v}=r'_{su,sv}+\frac{q-1}{1-\mathbf{z}^{u^{-1}\alpha}}r'_{su,v}\,. \tag{2.7}$$

If $v>sv$,

$$r'_{u,v}=\frac{(q-1)\mathbf{z}^{u^{-1}\alpha}}{1-\mathbf{z}^{u^{-1}\alpha}}r'_{su,v}+qr'_{su,sv}\,. \tag{2.8}$$

Note that $su>u$ implies that $u^{-1}\alpha$ is a positive root.

Proof Since $(r'_{u,v})$ is the inverse matrix of $(r_{u,v})$, we have

$$T_{v^{-1}}=\sum_{u\leqslant v}q_v\overline{r'_{w,v}}(\mathbf{z})\mu_{\mathbf{z}}(w). \tag{2.9}$$

First let us consider the case $v>sv$. Then $T_{(sv)^{-1}}=T_{v^{-1}}T_s^{-1}$. Moreover, for any $w\in W$, we can write T_s^{-1} as a linear combination of $\mu_{w\mathbf{z}}(s)$ and 1 to obtain

$$T_{(sv)^{-1}}=\sum_{w\leqslant v}\Big(q_v\overline{r'_{w,v}}(\mathbf{z})\mu_{\mathbf{z}}(w)\Big)\Big(\mu_{w\mathbf{z}}(s)-\frac{1-q^{-1}}{1-\mathbf{z}^{w^{-1}\alpha}}\Big). \tag{2.10}$$

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Then we can use Lemma 2.1 to compare the coefficients of $\mu_{\mathbf{z}}(su)$ in this equation and in (2.9) applied to sv. In (2.10) there are two ways to get a coefficient of su: we can either take $w = u$ or $w = su$. We obtain

$$q_{sv}\overline{r'_{su,sv}} = q_v\overline{r'_{u,v}} - \overline{r'_{su,v}}q_v \cdot \frac{1-q^{-1}}{1-\mathbf{z}^{-u^{-1}\alpha}}.$$

Applying the involution and rearranging gives (2.8).

Now let $sv > v$. Then $T_{(sv)^{-1}} = T_{v^{-1}}T_s$. We can proceed as before, except that now it is T_s that we are expressing as a linear combination of $\mu_{w\mathbf{z}}(s)$ and 1. We obtain

$$T_{(sv)^{-1}} = \sum_{w\leqslant v}\Big(q_v\overline{r'_{w,v}}(\mathbf{z})\mu_{\mathbf{z}}(w)\Big)\Big(q\mu_{w\mathbf{z}}(s) - \frac{(q-1)\mathbf{z}^{w^{-1}\alpha}}{1-\mathbf{z}^{w^{-1}\alpha}}\Big).$$

Now comparing the coefficient of $\mu_{\mathbf{z}}(sw)$ gives the identity

$$q_{sv}\overline{r'_{su,sv}} = qq_v\overline{r'_{u,v}} - q_v\overline{r'_{su,v}}\frac{(q-1)\mathbf{z}^{-u^{-1}\alpha}}{1-\mathbf{z}^{-u^{-1}\alpha}}.$$

Applying the involution and rearranging gives (2.7). ■

Proof of Theorem 1.5 It is sufficient to prove that defining

$$r'(u,v) = \varepsilon(u)\varepsilon(v)r_{w_0v,w_0u} \tag{2.11}$$

makes the recursion of Proposition 2.6 true. Since $w \mapsto w_0w$ is a Bruhat-order reversing bijection of the Weyl group to itself, we can apply Theorem 1.2 with u, v, and s being replaced by w_0v, w_0u, and w_0sw_0. With this substitution, it is easy to see that the definition (2.11) makes the recursion (2.7)–(2.8) true, so this definition must agree with our original one that makes of $(r'_{u,v})$ being the inverse matrix of the matrix $r_{u,v}$. This is equivalent to (1.8). To obtain (1.9), we use equation (1.4) to express $m_{u,v}$ and m_{w_0v,w_0u} in the left-hand side and then use (1.8). ■

3 Descent Properties of $m_{u,v}$

Although we will not prove the conjectured formula (1.1) we now have tools to prove it in many cases.

Proposition 3.1 *Let $u, v \in W$ and assume that s is a simple reflection such that $su < u$ and $sv < v$. Then the following are equivalent:*

(i) $u \leqslant v$,
(ii) $su \leqslant v$,
(iii) $su \leqslant sv$.

Proof This is Property Z in Deodhar [11]. It is sometimes called the *lifting property* of the Bruhat order. See [2, Proposition 2.2.7] for a proof. ■

The next result allows computation of $m_{u,v}$ from $m_{u,sv}$ if a simple reflection s can be found such that $sv < v$ and $su > u$. If this is true, the map $x \mapsto sx$ is a *special matching* in the sense of Brenti [3] and the reduction is reminiscent of the proof in certain cases that that the Kazhdan–Lusztig polynomials are combinatorial invariants of the Bruhat interval poset; see [4].

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Proposition 3.2 *Let $u < v$ and let $s = s_\alpha$ be a simple reflection such that $sv < v$ and $u < su$. Then*

$$S(u,v) = S(u,sv) \cup \{-v^{-1}\alpha\} \qquad \text{(disjoint)},$$

and

$$\overline{m_{u,v}} = \Big(\frac{1 - q\mathbf{z}^{-v^{-1}\alpha}}{1 - \mathbf{z}^{-v^{-1}\alpha}}\Big)\overline{m_{u,sv}}. \tag{3.1}$$

Proof Note that by Proposition 3.1, we have $u \leqslant sv$. If $\beta = -v^{-1}\alpha$, then $vr_\beta = sv$, so $u \leqslant vr_\beta < v$ is true but $u \leqslant svr_\beta < sv$ is not, showing that $-v^{-1}\alpha \in S(u,v)$ but not $S(u,sv)$. If β is a positive root not equal to $-v^{-1}\alpha$, we must show $\beta \in S(u,v)$ if and only if $\beta \in S(u,sv)$. First suppose that $svr_\beta < vr_\beta$. Then this statement is easily deduced from Proposition 3.1. Therefore, let us assume that $vr_\beta < svr_\beta$. If $\beta \in S(u,sv)$, then $u \leqslant svr_\beta < sv$. Proposition 3.1 implies that $u \leqslant vr_\beta$, and $vr_\beta < svr_\beta$ while again by Proposition 3.1, $svr_\beta \leqslant v$. Therefore, $\beta \in S(u,v)$. We are left to check that if $\beta \in S(u,v)$ but $\beta \notin S(u,sv)$ then $\beta = -v^{-1}\alpha$. To do this, we use the Strong Exchange Property for Coxeter groups, which is [14, Theorem 5.8]. Write $v = s_1 \cdots s_N$ where the s_i are simple reflections, and the expression is reduced. Since s is a left descent we can assume that $s_1 = s$. The Strong Exchange Property states that $vr_\beta = s_1 \cdots \widehat{s_i} \cdots s_N$ for some i. Suppose that $i \neq 1$. Then $svr_\beta = s_2 \cdots \widehat{s_i} \cdots s_N < s_2 \cdots s_N = sv$, while by Proposition 3.1, we have $u \leqslant svr_\beta$. This contradicts our assumption that $\beta \notin S(u,sv)$. Therefore, $i = 1$, which implies that $\beta = s_N s_{N-1} \cdots s_2(\alpha) = v^{-1}(-\alpha)$.

We turn to (3.1). Using Proposition 3.1, the fact that $sv < v$ and $su < u$ implies that $u \leqslant x \leqslant v$ if and only if $u \leqslant sx \leqslant v$. Therefore

$$\overline{m_{u,v}} = \sum_{u \leqslant x \leqslant v} r_{x,v} = \sum_{\substack{u \leqslant x \leqslant v \\ sx < x}} (r_{x,v} + r_{sx,v}).$$

We can now use both cases of Theorem 1.2 to rewrite this. The first case of the recursion applies to $r_{x,v}$, and the second applies to $r_{sx,v}$. We have

$$r_{x,v} + r_{sx,v} = \frac{1-q}{1 - \mathbf{z}^{-v^{-1}\alpha}} r_{x,sv}(\mathbf{z}) + r_{sx,sv}(\mathbf{z}) + (1-q)\frac{\mathbf{z}^{-v^{-1}\alpha}}{1 - \mathbf{z}^{-v^{-1}\alpha}} r_{sx,sv}(\mathbf{z}) + q r_{x,sv}(\mathbf{z}).$$

Simplifying, we get

$$r_{x,v} + r_{sx,v} = \Big(\frac{1 - q\mathbf{z}^{-v^{-1}\alpha}}{1 - \mathbf{z}^{-v^{-1}\alpha}}\Big)(r_{x,sv} + r_{sx,sv}).$$

The term $r_{x,sv}$ can be zero, since it is possible that x is not $\leqslant sv$, but we always have $sx \leqslant sv$ by Proposition 3.1. Discarding $r_{x,sv}$ when x is not sv, we get

$$\overline{m_{u,v}} = \Big(\frac{1 - q\mathbf{z}^{-v^{-1}\alpha}}{1 - \mathbf{z}^{-v^{-1}\alpha}}\Big) \sum_{\substack{u \leqslant x \leqslant v \\ sx < x}} (r_{x,sv} + r_{sx,sv}) = \Big(\frac{1 - q\mathbf{z}^{-v^{-1}\alpha}}{1 - \mathbf{z}^{-v^{-1}\alpha}}\Big) \sum_{u \leqslant x \leqslant sv} r_{x,sv},$$

which equals the right-hand side of (3.1). ∎

Here is another type of descent result.

Proposition 3.3 *Assume that $sv < v$ and $su < u$. Assume further that u is not $\leqslant sv$.*

(i) *Then $S(u,v) = S(su,sv)$.*

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(ii) *The map $x \mapsto sx$ is a bijection of the Bruhat interval $[u, v] = \{x | u \leqslant x \leqslant v\}$ to $[su, sv]$. If $u \leqslant x \leqslant v$, then $sx < x$ and x is not $\leqslant sv$, and $S(x, v) = S(sx, sv)$.*
(iii) *We have $m_{u,v} = m_{su,sv}$, $r_{u,v} = r_{su,sv}$.*
(iv) *If, in addition, $Q_{u,v} = 1$, then $Q_{su,sv} = 1$.*

Proof To prove (i), we first show that $S(u, v) \subseteq S(su, sv)$. Let $\beta \in S(u, v)$ and let $r = r_\beta$, so that $u \leqslant vr_\beta$. Let $v = s_1 \cdots s_N$ be a reduced expression with $s_1 = s$. Thus, $vr = s_1 \cdots \widehat{s_i} \cdots s_N$ for some i. We claim that $i \neq 1$. Indeed, if $i = 1$ then $vr = sv$ so $u \leqslant vr$, which contradicts our assumption. Therefore, $i \neq 1$ and $svr = s_2 \cdots \widehat{s_i} \cdots s_N$. Since $sv = s_2 \cdots s_N$ is a reduced expression, we see that $svr < sv$. On the other hand, since $su < u$, Proposition 3.1 implies that $su < svr$ and therefore $\beta \in S(su, sv)$.

On the other hand, let us show that $S(su, sv) \subseteq S(u, v)$. Thus, assume that $r = r_\beta$ where $\beta \in S(su, sv)$ and $su \leqslant svr < sv$. We claim that $svr < vr$. Indeed, if not, then $su \leqslant svr$ implies $u \leqslant svr$ by Proposition 3.1 and so $u \leqslant svr < sv$, contradicting our hypothesis. Now since $vr > svr$, $u > su$, and $su \leqslant svr$, Proposition 3.1 implies that $u \leqslant vr$. On the other hand, since $v > sv$ and $svr < sv$, Proposition 3.1 implies that $vr < v$. (We cannot have $vr = v$, since r is a reflection.) Thus, $u \leqslant vr < v$ and so $\beta \in S(u, v)$. Now (i) is proved.

We prove (ii). First, if $x \in [u, v]$, then we claim that $x > sx$. Indeed, if $sx > x$, then $x < sv$ by Proposition 3.1. Then $u \leqslant x < sv$, contradicting our hypothesis. Now two applications of Proposition 3.1 show that $su \leqslant sx$ and $sx \leqslant sv$. Thus, $x \mapsto sx$ maps $[u, v]$ into $[su, sv]$. The fact that this map is surjective also follows from Proposition 3.1. Finally, since we have shown that $x < sx$ for $x \in [u, v]$, part (i) applies to the pair x, v, implying that $S(x, v) = S(sx, sv)$. Now (ii) is proved.

As for (iii), the fact that $r_{u,v} = r_{su,sv}$ follows from Theorem 1.2, since $r_{u,sv} = 0$ under our assumption that u is not $\leqslant sv$. By (ii), we have similarly $r_{x,v} = r_{sx,sv}$ for $x \in [u, v]$. Summing over x and applying the involution gives $m_{u,v} = m_{su,sv}$.

We prove (iv). A criterion for $P_{x,y} = 1$ due to Kazhdan and Lusztig [16, Lemma 2.6] is that $\sum_{x \leqslant z \leqslant y} R_{x,z} = q_y q_x^{-1}$. (Actually in this lemma this is the condition that $P_{z,y} = 1$ for all $x \leqslant z \leqslant y$, but by (2.6), this is equivalent to $P_{x,y} = 1$.) By [16, Lemma 2.1(iv)] it follows that the criterion for $Q_{x,y} = 1$ is that $\sum_{x \leqslant z \leqslant y} R_{z,y} = q_y q_x^{-1}$. Thus, $Q_{u,v} = 1$ we have $\sum_{u \leqslant x \leqslant v} R_{x,v} = q_v q_u^{-1}$. Moreover, using (ii) and the recurrence [16, (2.0.b)] for R we have $R_{x,v} = R_{sx,sv}$, and it follows that $\sum_{su \leqslant sx \leqslant sv} R_{sx,sv} = q_v q_u^{-1} = q_{sv} q_{su}^{-1}$. Therefore, $Q_{su,sv} = 1$. ■

We make the following conjecture.

Conjecture 3.4 *Assume that Φ is simply-laced and that $u < v$ in W such that $Q_{u,v} = 1$. Then there exists a simple reflection $s \in W$ such that either:*

(i) *$sv < v$ and $su > u$, or*
(ii) *$sv < v$ and $su < u$, and u is not $\leqslant sv$.*

We have checked this (using Sage) for Cartan types A_5 and D_4. For A_5 we find 1346 pairs $u < v$ such that no descent s of v exists satisfying either (i) or (ii), and for each of these, we have $Q_{u,v} \neq 1$. For example, we can take $(u, v) = (s_2, s_2 s_1 s_3 s_2)$ and the

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only descent $s = s_2$ of v does not satisfy either (i) or (ii), but this does not contradict the conjecture, since $Q_{u,v} = 1 + q$.

Theorem 3.5 *Assume that Φ is simply-laced and $Q_{u,v} = 1$. Then Conjecture 3.4 implies (1.1), which is conjectured in [6] under these assumptions.*

Proof We assume that $u < v$ and $Q_{u,v} = 1$. Choose a left descent s of v. If $su > u$, then Proposition 3.2 applies. Note

$$m_{u,sv} = m_{u,sv}(\mathbf{z}) = \prod_{\beta \in S(u,sv)} \frac{1 - q^{-1}\mathbf{z}^\beta}{1 - \mathbf{z}^\beta}.$$

Now Proposition 3.2 implies (1.1) for u, v.

On the other hand, if $su < u$, then on Conjecture 3.4 we have u not $\leqslant sv$, and so Proposition 3.3 applies, and again the result follows. ■

We end with another puzzle. We give the root lattice the usual partial order in which $x \geqslant y$ if $x - y$ lies in the cone generated by the positive roots. Then the set T of reflections has a partial order in which if $\alpha, \beta \in \Phi^+$, then $r_\alpha \geqslant r_\beta$ if and only if $\alpha \geqslant b$. Let $\mathrm{AD}(u, v) = \{r \in T | ru > u, rv < v\}$. We will write $u \lessdot v$ to denote the covering relation in the Bruhat order. Thus, $u \lessdot v$ if $u < v$ and $l(u) = l(v) - 1$.

Theorem 3.6 (Tsukerman and Williams [24], Caselli and Sentinelli [8]) *Suppose that Φ is a simply-laced root system. Suppose $u < v$. Then $\mathrm{AD}(u, v)$ is nonempty and if t is a minimal element, then $u \lessdot tu \leqslant v$ and $u \leqslant tv \lessdot v$. In this case,*

$$R_{u,v} = qR_{tu,tv} + (q - 1)R_{u,tv}.$$

Suppose in the setting of this theorem that $t = r_\alpha$ ($\alpha \in \Phi^+$). Let $\beta = -v^{-1}\alpha$. Then β is a positive root. To generalize Theorem 1.2, it is natural to ask whether

$$r_{u,v} = qr_{tu,tv} + (q - 1)\frac{\mathbf{z}^\beta}{\mathbf{z}^\beta - 1} r_{u,tv}, \qquad \beta = -v^{-1}\alpha. \tag{3.2}$$

This is often, but not always, true. For A_3, it fails in the following cases:

u	v	t	$P_{u,v}$	$Q_{u,v}$
s_1	$s_1s_2s_3s_2s_1$	$s_1s_2s_1$	$1+q$	1
s_3	$s_1s_2s_3s_2s_1$	$s_2s_3s_2$	$1+q$	1
s_1s_3	$s_1s_2s_3s_2s_1$	$s_1s_2s_1$	$1+q$	$1+q$
s_1s_3	$s_1s_2s_3s_2s_1$	$s_2s_3s_2$	$1+q$	$1+q$
s_2	$s_2s_1s_3s_2$	$s_1s_2s_1$	$1+q$	$1+q$
s_2	$s_2s_1s_3s_2$	$s_2s_3s_2$	$1+q$	$1+q$
s_2	$s_3s_2s_1s_3s_2$	$s_1s_2s_1$	1	$1+q$
s_2	$s_1s_2s_1s_3s_2$	$s_2s_3s_2$	1	$1+q$

Except in these cases, we have not only (3.2), but also

$$\overline{m_{u,v}} = \left(\frac{1 - q\mathbf{z}^\beta}{1 - \mathbf{z}^\beta}\right) \overline{m_{u,tv}}. \tag{3.3}$$

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We can conjecture that (3.2) and (3.3) are true if both $P_{u,v} = Q_{u,v} = 1$. (This has been checked for A_4 as well as A_3.) This does not imply the conjecture (1.1), because of the condition $P_{u,v} = 1$.

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