ON THE TOTAL AREA OF THE FACES OF A FOUR-DIMENSIONAL POLYTOPE

L. FEJES TÓTH

Let L be the total length of the edges of a convex polyhedron containing a unit sphere. If the number of edges is small, the edges must be, on the average, comparatively long. If, on the other hand, the edges are short, their number must be great. So the problem arises to find a polyhedron with a possibly small value of L.

By a simple argument the author **(5; 6)** proved that $L > 20$ and announced the conjecture that $L \geqslant 24$ with equality only for the cube (of in-radius 1). The same argument shows that for trigonal-faced polyhedra $L > 28$. This supports the conjecture that for such polyhedra $L \geqslant 12\sqrt{6} = 29.4\ldots$, with equality only for the tetrahedron and octahedron.

These results started some further investigations. Besicovitch and Eggleston **(1)** proved the above conjecture concerning the cube. Thus, of all convex polyhedra containing a sphere, the circumscribed cube has the least possible sum of the edges. Coxeter **(2)** considered the analogous problems in non-Euclidean spaces, pointing out several interesting results to be expected. The case of non-Euclidean polyhedra with triangular faces was investigated by Coxeter and Fejes Tóth **(4)**, who solved the problem for certain particular values of the radius of the sphere. For instance, in hyperbolic space, of all convex polyhedra with triangular faces containing a sphere of radius $0.828\ldots$, the circumscribed regular icosahedron turned out to have the least possible total edge-length. Further results concerning this problem may be found in **(8)** and **(9)**.

In the present paper we shall deal with the analogous problems in four-dimensional Euclidean space which arise by considering the total area of the faces, F, instead of the total length of the edges, L. The following table shows the exact and the approximate values of L and F for the six regular polytopes **(3)** with in-radius 1. The number $\tau = \frac{1}{2}(5^{\frac{1}{2}} + 1)$ used in this table gives the ratio of the golden section.

	$\{3, 3, 3\}$	$\{3, 3, 4\}$	$\{4, 3, 3\}$	$\{3, 4, 3\}$	$\{3, 3, 5\}$	$\{5, 3, 3\}$
L	$20\cdot 10^{\frac{1}{2}}$	$48\cdot 2^{\frac{1}{2}}$	64	$96\cdot 2^{\frac{1}{2}}$	$1440\cdot 2^{\frac{1}{2}}\tau^{-3}$	$2400\,\tau^{-4}$
	63	68	64	136	481	391
F	$100\cdot 3^{\frac{1}{2}}$	$64\cdot 3^{\frac{1}{2}}$	96	$48\cdot 3^{\frac{1}{2}}$	$2400\cdot 3^{\frac{1}{2}}\tau^{-6}$	$720\cdot 5^{3/4}\tau^{-13/2}$
	173	110.85	96	83.14	231	106

Received August 8, 1963.

Our table shows that in the problem concerning L the measure polytope $\{4, 3, 3\}$ is superseded by the simplex $\{3, 3, 3\}$ and in the problem concerning F by the 24-cell $\{3, 4, 3\}$. In the problem concerning F among the regular polytopes with tetrahedral cells the cross-polytope $\{3, 3, 4\}$ is the best one.

Our results read as follows*:

If F is the total area of the faces of a four-dimensional convex Euclidean polytope containing a unit sphere, then

$$F > 67.5.$$

If the polytope has tetrahedral cells only, then

$$F > 110.4$$

and if it is bounded by octahedral cells only, then

$$F > 81.6.$$

Here an "octahedral cell" means a polyhedron topologically isomorphic to a regular octahedron.

We denote the surface area of a 3-dimensional body b by $||b||$, its central projection onto the unit sphere S by b' and its volume simply by b. If b lies in a hyperplane p touching S at the point A, we can represent the volume b in the form

$$b = \int_{b'} f(AP)dv,$$

where dv is the volume element (surface element of S) at the point P and $f(x)$ is an increasing function (independent of b'), namely

$$f(x) = \operatorname{cosec}^4 x.$$

Let s' be a sphere in the spherical 3-space of S centred at A and having the same volume as b' and let s be the projection of s' onto p. Omitting $f(AP)dv$ after the integral signs, we have, in view of the monotony of $f(x)$,

$$b = \int_{b'} = \int_{b's'} + \int_{b'-b's'} \geqslant \int_{b's'} + \int_{s'-b's'} = \int_{s'} = s.$$

Thus, making use of the isoperimetric property of the sphere, $||b|| \geqslant ||s||$. Expressing s' and $||s||$ in terms of the radius r of s', we have

$$b' = s' = 2\pi(r - \tfrac{1}{2}\sin 2r)$$

and

$$||s|| = 4\pi \tan^2 r.$$

This enables us to write the inequality $||b|| \geqslant ||s||$ in the form

$$||b|| \geqslant G(b'),$$

where the function $G(x)$ is defined by

*For the numerical computations I am indebted to G. Krammer.

$$G(x) = 4\pi \tan^2 r, \qquad x = 2\pi(r - \tfrac{1}{2}\sin 2r), \quad 0 < r < \tfrac{1}{2}\pi.$$

Obviously, this inequality continues to hold when b lies in any hyperplane not intersecting S. Equality holds only if b is a sphere touching S at its centre.

The curve $y = G(x)$ is exhibited in Figure 1. It lies above its tangent through

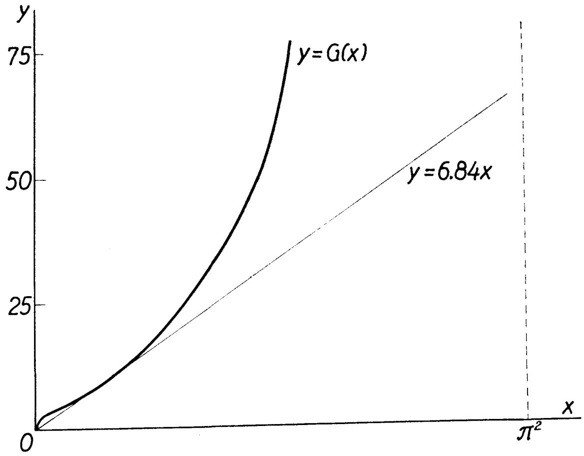

FIGURE 1

the origin (different from the y-axis). In order to find this tangent, we have to solve the equation $G(x)/x = G'(x)$, which is equivalent to

$$\frac{4\pi\tan^2 r}{2\pi(r - \frac{1}{2}\sin 2r)} = \frac{dG/dr}{dx/dr} = \frac{8\pi \tan r \sec^2 r}{4\pi \sin^2 r},$$

i.e., to

$$2r = \sin 2r\,(1 + \sin^2 r).$$

This equation has for $0 < r < \frac{1}{2}\pi$ a single root $r_0 \approx 0.69$. Writing $x_0 = 2\pi(r_0 - \frac{1}{2}\sin 2r_0)$, we have

$$G(x)/x \geqslant G(x_0)/x_0 > 6.84.$$

Thus we have, for each cell c of a convex polytope containing S,

$$||c|| > 6.84\, c'.$$

Summing up these inequalities for all cells, we obtain the desired inequality

$$2F = \sum ||c|| > 6.84 \sum c' = 6.84\cdot 2\pi^2 > 135, \qquad F > 67.5.$$

Turning now to the case of polytopes with tetrahedral cells, suppose that b is a tetrahedron lying in the hyperplane p. Instead of $b \geqslant s$, we have now the sharper inequality

$$b \geqslant t,$$

where t is a regular tetrahedron centred at A such that $t' = b'$. This means that of the spherical tetrahedra with given volume the regular tetrahedron with centre A has the least possible projection onto p. Instead of this it is easier to prove the proposition, equivalent to the original one, that among the Euclidean tetrahedra of constant volume lying in p the regular tetrahedron t with centre A has the greatest projection onto S.

We can represent the volume of the projection b' of any body b lying in p in the form

$$b' = \int_b g(AP)dv,$$

where dv is the volume element of b at the point P and $g(x)$ is a decreasing function (independent of b), namely

$$g(x) = (1 + x^2)^{-2}.$$

Let b be a tetrahedron of prescribed volume varying in p. If the distance of a vertex of b from A is great, b' will be small. Therefore, in order to show the existence of a tetrahedron with minimal projection, we can restrict ourselves to tetrahedra lying in a sufficiently large sphere around A. Since, furthermore, b' varies continuously with the vertices of b, the existence of a best tetrahedron follows from the theorem of Weierstrass.

Suppose now that b is either not regular or it is regular but its centre is not A. Then there is a 2-dimensional plane q orthogonal to an edge of b such that b is not symmetric with respect to q. Translate each chord c of b in its own line into a position $\bar{c}$ symmetric with respect to q. This process, called Steiner symmetrization, transfers b into a new tetrahedron $\bar{b}$ of the same volume. Observing that any inner point of the segment $\bar{c} - c\bar{c}$ lies nearer to A than any point of $c - c\bar{c}$, we have a volume-preserving transformation of b into $\bar{b}$ leaving the points of $b\bar{b}$ invariant and carrying the other ones nearer to A. Thus, in view of the above integral representation of b', $\bar{b}' > b'$, showing that b cannot be extremal. This completes the proof of the inequality $b \geqslant t$.

In view of the isoperimetric property of the regular tetrahedron, this inequality involves

$$||b|| \geqslant ||t||.$$

We proceed to evaluate $||t||$ in terms of t'.

It is known (7) that in spherical 3-space (of curvature 1) the volume V of a regular polyhedron of in-radius r is given by

$$V = 2e \int_0^{\alpha} \left\{ r - \frac{\cos\phi}{\sqrt{k^2 - \sin^2\phi}} \arctan\left(\frac{\sqrt{k^2 - \sin^2\phi}}{\cos\phi} \tan r \right) \right\} d\phi,$$

where

$$k = \frac{\sin(\pi f/2e)}{\cos(\pi v/2e)}, \qquad \alpha = \pi f/2e,$$

and f, e, and v are the numbers of the faces, edges, and vertices.

It will be convenient to deduce an alternative formula for V. For this purpose we use the abbreviation

$$k^2 - 1 = \omega^2$$

and observe that the derivative $dV/dr = h(r)$ can be expressed in terms of elementary functions:

$$h(r) = 2e\omega^2 \sin^2 r \int_0^\alpha \frac{d\phi}{\cos^2\phi + \omega^2 \sin^2 r}$$

$$= 2e \frac{\omega \sin r}{\sqrt{(1 + \omega^2 \sin^2 r)}} \text{arc tan}\left(\frac{\omega \sin r}{\sqrt{(1 + \omega^2 \sin^2 r)}} \tan \alpha\right).$$

Hence

$$V = \int_0^r h(x)dx.$$

For a tetrahedron,

$$h(r) = 12\sqrt{2} \frac{\sin r}{\sqrt{(1 + 2 \sin^2 r)}} \text{arc tan} \frac{\sqrt{6} \sin r}{\sqrt{(1 + 2 \sin^2 r)}}.$$

Thus, denoting the in-radius of t' by r, we have on the one hand

$$||t|| = 24\sqrt{3} \tan^2 r,$$

and on the other hand

$$b' = t' = \int_0^r h(x)dx.$$

Therefore, the inequality $||b|| \geqslant ||t||$ turns out to be equivalent to

$$||b|| \geqslant H(b'),$$

where the function $H(x)$ is defined by

$$H(x) = 24\sqrt{3} \tan^2 r, \qquad x = \int_0^r h(z)dz, \qquad 0 < r < \tfrac{1}{2}\pi.$$

In order to determine the minimum of $H(x)/x$, we have to solve the equation

$$H(x) = xH'(x),$$

i.e., in terms of r, the equation

$$h(r) \sin 2r = 4 \int_0^r h(z)dz.$$

This equation has, for $0 < r < \frac{1}{2}\pi$, a unique root $r_1 \approx 0.487$, showing that

$$H(x)/x \geqslant H(x_1)/x_1 > 11.19,$$

where

$$x_1 = \int_0^{r_1} h(z)dz.$$

Consequently, we have for each cell c of our polytope

$$||c|| > 11.19\, c',$$

whence, in agreement with our statement,

$$2F = \sum ||c|| > 11.19 \sum c' = 2\pi^2 \cdot 11.19 > 220.88, \qquad F > 110.44.$$

Now we suppose that b is an octahedron of given volume lying in p. Among these octahedra we want to find the particular one that has the greatest projection b'. By three successive symmetrizations in planes through A orthogonal to its diagonals, b can be transformed into the convex hull of three segments mutually perpendicular to each other and bisecting one another at A. Since by this operation b' increases, we can restrict ourselves to octahedra of this type. But among such octahedra there is, obviously, a best one and a further symmetrization in a plane through A perpendicular to an edge shows that it must be regular.

It follows that $b \geqslant o$, where o is a regular octahedron centred at A such that $b' = o'$. Now we can refer to the isoperimetric property of the regular octahedron (proved by Steiner) according to which, of the octahedra of constant volume, the regular one has the least possible surface area. Thus

$$||b|| \geqslant ||o||,$$

i.e.

$$||b|| \geqslant K(b'),$$

where

$$K(x) = 12\sqrt{3}\tan^2 r, \qquad x = \int_0^r h(z)dz,$$

$$h(z) = 24 \frac{\sin z}{\sqrt{(2+\sin^2 z)}} \arc\tan \frac{3\sin z}{\sqrt{(2+\sin^2 z)}}.$$

Expressing the equation $K(x) = xK'(x)$ in terms of r, we have again

$$h(r)\sin 2r = 4\int_0^r h(z)dz.$$

The single root of this equation being $r_2 \approx 0.59$, we deduce that

$$\frac{K(x)}{x} \geqslant \frac{K(x_2)}{x_2} = K'(x_2) > 8.268, \qquad \text{where } x_2 = \int_0^{r_2} h(z)dz.$$

Thus, we obtain the desired inequality

$$2F = \sum ||c|| \geqslant 8.268 \sum c' = 2\pi^2 \cdot 8.268 > 163.2, \qquad F > 81.6.$$

The analogous problems in non-Euclidean 4-spaces deserve special attention. Here, of course, the solutions depend on the radius R of the in-sphere S of the polytope. Considering, for example, polytopes with tetrahedral cells, the corresponding root x_1 varies continuously with R, and in hyperbolic 4-space there is a value of R such that $x_1 = 2\pi^2/600$. In this case the above method yields an exact inequality for F expressing an extremum property of the 600-cell $\{3, 3, 5\}$. The details will be discussed in a sequel to this paper.

REFERENCES

1. A. S. Besicovitch and H. G. Eggleston, *The total length of the edges of a polyhedron*, Quart. J. Math., Oxford Ser., *8* (1957), 172–190.
2. H. S. M. Coxeter, *The total length of the edges of a non-Euclidean polyhedron*, Studies in Mathematical Analysis and Related Topics, Essays in honor of George Pólya (Stanford, 1962).
3. ——— *Regular polytopes* (New York, 1963).
4. H. S. M. Coxeter and L. Fejes Tóth, *The total length of the edges of a non-Euclidean polyhedron with triangular faces*, Quart. J. Math., Oxford Ser., *14* (1963), 273–284.
5. L. Fejes Tóth, *On the total length of the edges of a polyhedron*, Norske Vid. Selsk. Forhdl., *21* (1948), 32–34.
6. ——— *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin, 1953).
7. ——— *On the volume of a polyhedron in non-Euclidean spaces*, Publ. Math., 4 (1956), 256–261.
8. J. M. Hammersly, *The total length of the edges of a polyhedron*. Compositio Math., *9* (1951), 239–240.
9. B. Tomor, *Szabályos testek szélsöértéktulajdonságai állandó görbületü terekben* (Manuscript).

University of Veszprém, Hungary

https://doi.org/10.4153/CJM-1965-009-x Published online by Cambridge University Press