Canad. Math. Bull. Vol. 17 (4), 1974

ON LIFTING IDEMPOTENTS

BY
KWANGIL KOH

Let N be an ideal of a ring A. We say that idempotents modulo N can be lifted provided that for every a of A such that $a^2-a \in N$ there exists an element $e^2=e \in A$ such that $e-a \in N$. The technique of lifting idempotents is considered to be a fundamental tool in the classical theory of nonsemiprimitive Artinian rings (refer [2; p. 72]).

One of the important classical results on lifting idempotents is that if every element of N is nilpotent then idempotents modulo N can be lifted (See [2; p. 72] or [1; p. 54]). A standard proof of this fact is usually given by setting $e=a+x(1-2a)$ where $a^2-a \in N$ and solve $x^2-x-z=0$ where $z=(a-a^2)(1-4(a-a^2))^{-1} \in N$ by means of a series $\sum_{n=1}^{\infty} 1/(2n-1)\binom{2n-1}{n}(-z)^n$.

The purpose of this note is to give an elementary proof of this important classical result.

Let A be a ring with 1 and let a be an element of A such that $a-a^2 \in N$ where N is a nil ideal. Let m be a positive integer such that $(a-a^2)^m=0$. Then $(1-a)^m a^m=0$. Write $1=a+(1-a)$. Then

$$1 = (a+(1-a))^{2m} = \sum_{i=0}^{2m} \binom{2m}{i} a^{2m-i}(1-a)^i.$$

Let $e=\sum_{i=0}^{m-1} \binom{2m}{i} a^{2m-i}(1-a)^i$ and let $f=\sum_{j=1}^{m} \binom{2m}{m+j} a^{m-j}(1-a)^{m+j}$. Then $ef=fe=0$ and $1=e+f$. Hence $e^2=e$ and $f^2=f$ and $e \equiv a \bmod N$ and $f \equiv (1-a) \bmod N$.

Remark. The method which we used in the proof also works in the case when a ring may not have an identity. One only needs to embed the ring into a ring with identity in the usual way and factor $a-a^2$.

References

1. N. Jacobson, *Structure of Rings*, American Mathematical Society Colloquium, Vol. 36, Rev. ed. Providence, R.I.: 1964.

2. J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham, Massachusetts: 1966.

North Carolina State University,
Raleigh, North Carolina 27607

https://doi.org/10.4153/CMB-1974-111-5 Published online by Cambridge University Press