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The General Structure of G-Graded Contractions of Lie Algebras I. The Classification

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Abstract. We give the general structure of complex (resp., real) G-graded contractions of Lie algebras where G is an arbitrary finite Abelian group. For this purpose, we introduce a number of concepts, such as pseudobasis, higher-order identities, and sign invariants. We characterize the equivalence classes of G-graded contractions by showing that our set of invariants (support, higher-order identities, and sign invariants) is complete, which yields a classification.

1 Introduction

Let G be a finite Abelian group. A G-graded Lie algebra $L = (V, \mu)$ has the structure $V = \bigoplus_{j \in G} V_j$ where $\mu(V_j, V_k) \subset V_{j+k}$. The notion of a graded contraction $L \xrightarrow{\gamma} L_\gamma$ of a graded Lie algebra L was introduced in 1991 [4, 6]. It transforms a G-graded Lie algebra $L = (V, \mu)$ into a G-graded Lie algebra $L_\gamma = (V, \mu_\gamma)$ in a purely algebraic way by defining, with the obvious meaning, $\mu_\gamma(V_j, V_k) = \gamma_{jk}\mu(V_j, V_k)$ where γ is a matrix which is symmetric (so that μ_γ is antisymmetric) and satisfies non-linear "defining equations" (2.3) which enforce the Jacobi identity for μ_γ. By a graded contraction is meant the matrix γ, whose definition depends only on the grading group G and not on L.[1] The process $L \xrightarrow{\gamma} L_\gamma$ is called the graded contraction of the Lie algebra L by γ.

The notion of a contraction of Lie algebras was introduced 40 years earlier where, motivated by physics, it is defined by a limiting process [3, 9, 10, 14]. The reader should note that the standard terminology in the literature violates normal grammatical and mathematical usage. Namely, a graded contraction is not a contraction which is graded (since it is defined algebraically and not by a limiting process). Indeed, a graded contraction is, in general, not even equivalent to a contraction [15].

The notions of equivalence and continuity for graded contractions were defined in [6]. In [13] we presented some general results on non-negative $\mathbb{Z}_N$-graded contractions. In particular, we gave a complete characterization of the continuous ones. Otherwise the rather extensive literature contains virtually no general results. Graded contractions γ have been calculated for specific grading groups such as $\mathbb{Z}_2, \mathbb{Z}_3$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Graded contractions of Lie algebras have been calculated for specific L such

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[1] If one considers a graded associative commutative algebra or a Lie superalgebra instead of a Lie algebra, our results hold verbatim (*cf.* Remark 2.4).

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as the simple complex Lie algebras A_2, A_3, B_2, C_2, C_3, some of their real forms and inhomogeneous versions, and the Kac–Moody algebra $A_1^{(1)}$; see [1, 2, 4, 5, 7, 11, 12].

This is the first of three papers on the general structure of graded contractions of Lie algebras. This paper deals only with the matrices γ. The second paper treats the process $L \xrightarrow{\gamma} L_\gamma$ and especially the general properties of L_γ. The third paper will deal with Conjecture 2.15.

The defining equations for γ reflect the interplay between the Jacobi identity and the finite Abelian grading group G. Our results show that this natural structure does indeed produce a rich internal structure for graded contractions.

Let G be a fixed grading group. The equivalence relation for γ's is that they differ only by a change of basis compatible with the grading (Definition 2.14). It is trivial to see that the support S of γ (*cf.* (2.11)) is an invariant. The possible supports $\mathcal{S}(G)$ follow easily from the defining equations (Remark 6.2). The G-dependent higher-order identities we introduced in [13] yield a second type of invariant. For a given support S, we show that a straightforward calculation yields an integer $Q(S) \geq 0$ with the property that γ can arbitrarily strongly violate precisely $Q(S)$ independent higher-order identities (Definition 4.1, Theorem 6.5). In the complex and non-negative cases, this means that there is a $Q(S)$-parameter family of inequivalent graded contractions with support S. In the real case, one must also take into account the maximal number $J(S)$ of independent sign invariants (Lemmas 6.14, 6.17). The resulting classification in Section 7 is our main result.

Although the γ's with zeroes are the interesting ones, those without zeroes provide us with some necessary insights and tools which play a key role in our analysis of γ's with zeroes. In spite of the nonlinearity of the defining equations, some ideas from linear algebra, such as dependence and basis, can be adapted to the present situation, and play an important role.

We now outline the contents of the paper. Section 2 contains notation and basic definitions. In Section 3 the equivalence classes for G-graded contractions without zeroes are determined. In both the complex and the positive case there is only one equivalence class, but in the real case we can have more (*e.g.*, two in the case $G = \mathbb{Z}_N$ when N is even).

Section 4.1 deals with the invariants associated with the G-dependent higher-order identities which we introduced in [13]. Section 4.2 deals with the real case, where sign invariants come into play. We show that all sign invariants split naturally into two classes.

In Section 5 we define independence, and introduce a pseudobasis as a maximal set of independent elements of a complex G-graded contraction γ without zeroes. We construct all pseudobases, and all γ's which agree on a given pseudobasis. Then we answer the question of when real values of a pseudobasis yield a real γ.

Section 6 contains a number of results on the general structure of G-graded contractions with zeroes. In Theorem 6.5 we produce the $Q(S)$ higher-order identities that a γ with support S can arbitrarily and independently violate. In Theorem 6.7 (resp. Theorem 6.11) we characterize those complex (resp., real) γ's for which $\gamma \sim \pi(\gamma)$ where $\pi(\gamma)$ (Definition 2.8 (ii)) has the same support as γ but all of its non-vanishing elements are equal to 1. Finally, we introduce the notion of independent sign invariants (Definition 6.13, Lemma 6.16). In Lemma 6.17 and Algorithm 6.18 we then determine the maximal number $J(S)$ of independent sign invariants for a real γ with support S.

In Theorem 7.1 we apply these results to show that two inequivalent G-graded contractions must differ on one of our invariants (support, higher-order identities, and sign invariants), thus proving their completeness. This leads to a classification of all G-graded contractions.

An appendix contains all lengthier manipulations of the defining equations which lead to "natural" bases for graded contractions for all G.

Finally, we make a few remarks on our second paper [15], which deals with the general structure of the contracted Lie algebra L_γ. The author's results for continuous non-negative $\mathbb{Z}_N$-graded contractions in [13] will be generalized to complex (resp., real) G-graded contractions. The effect of continuous and discrete G-graded contractions will be compared with each other and with contractions. In addition, the usefullness of graded contractions in mathematical physics will be critically examined.

2 Definitions and Notation

Let G be a finite Abelian group of order $|G| \geq 2$. It is well known that G is isomorphic to a direct product

$$G \simeq \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r} \tag{2.1}$$

where $\mathbb{Z}_N$ is the additive group of $\mathbb{Z}$ modulo N. This decomposition is in general not unique (unless one specifies the canonical decomposition where $N_i \mid N_{i+1}, i = 1, 2, \ldots, r-1$) [8]. However the non-uniqueness does not play any role for our purposes.

Definition 2.1 A Lie algebra $L = (V, \mu)$ is called *G-graded* if the vector space V splits according to $V = \bigoplus_{j \in G} V_j$ and if the Lie product $\mu \colon V \times V \to V$ satisfies

$$\mu(V_j, V_k) \subset V_{j+k}. \tag{2.2}$$

for $j, k \in G$. (The Lie product is antisymmetric and satisfies the Jacobi identity $\mu(\mu(e, f), g) + \mu(\mu(f, g), e) + \mu(\mu(g, e), f) = 0$ for $e, f, g, \in V$.)

Definition 2.2 ([4]) We call a complex (resp., real) matrix $\gamma = (\gamma_{jk})$, $j, k \in G$, a *G-graded contraction* if γ is symmetric and satisfies the defining equations

$$\gamma_{jk}\gamma_{l,j+k} = \gamma_{jl}\gamma_{k,j+l} = \gamma_{kl}\gamma_{j,k+l}, \quad j, k, l \in G. \tag{2.3}$$

Definition 2.3 ([4]) Let $L = (V, \mu)$ be a complex (resp., real) G-graded Lie algebra and let γ be a complex (resp., real) G-graded contraction. Then with the obvious meaning,

$$\mu_\gamma(V_j, V_k) = \gamma_{jk}\mu(V_j, V_k), \quad j, k \in G, \tag{2.4}$$

defines a Lie product (the symmetry of γ ensures the antisymmetry of μ_γ and the defining equations guarantee the Jacobi identity for μ_γ).

The complex (resp., real) G-graded Lie algebra $L_\gamma = (V, \mu_\gamma)$ is called the G-graded contraction of L, in short

$$L \xrightarrow{\gamma} L_\gamma.$$

Remark 2.4 Note that this construction also works for an associative, commutative graded algebra (here the defining equations ensure the associativity) or a Lie superalgebra.

Definition 2.5

(i) Let γ' and γ'' be two G-graded contractions. (cf. [4]) Then the (elementwise) product $(j, k \in G)$

$$\gamma = \gamma' \cdot \gamma'' \text{ where } \gamma_{jk} = \gamma'_{jk}\gamma''_{jk} \tag{2.5}$$

is obviously a G-graded contraction.

(ii) Let γ be a G-graded contraction and γ' a G'-graded contraction. Then the tensor product $(j, k \in G\ j', k' \in G')$

$$\gamma \otimes \gamma' \text{ where } (\gamma \otimes \gamma')_{jj',kk'} = \gamma_{jk}\gamma'_{j'k'} \tag{2.6}$$

is obviously a $G \times G'$-graded contraction.

Definition 2.6 A G-graded contraction γ is called trivial if for all G-graded L we always have $L_\gamma \simeq L$ or we always have L_γ Abelian.

Example 2.7

(i) Trivial γ's are $\gamma_{jk} \equiv 0$, so that L_γ is Abelian, and the identity $\mathbf{1}$ where

$$\mathbf{1}_{jk} \equiv 1, \text{ so that } L_\gamma = L. \tag{2.7}$$

(ii) The trivial complex (resp., real) G-graded contraction da is defined [4, 6] by

$$(da)_{jk} = \frac{a_j a_k}{a_{j+k}} \quad j, k \in G \tag{2.8}$$

with arbitrary numbers $0 \neq a_j \in \mathbb{C}$ (resp., $\mathbb{R}$). And it corresponds to the change of basis $V_j \longrightarrow a_j V_j$ (which is compatible with the grading) so that $L_{da} \simeq L$. (The notation "*da*" reflects the fact that a general γ can be interpreted as a 2-cocycle and *da* as a 2-coboundary [6].)

(iii) In Section 3 (Lemma 3.4) the following real $\mathbb{Z}_N$-graded contraction, called "*db*", will occur which is a coboundary only in the complex, but not in the real case. Consider, for $G = \mathbb{Z}_N$ and $j, k = 0, 1, 2, \ldots, N-1$,

$$b_j = e^{ij\pi/N}. \tag{2.9}$$

Then

$$(db)_{jk} = \frac{b_j b_k}{b_{j+k}} = \begin{cases} 1 & 0 \leq j, k \leq j+k < N, \\ -1 & 0 \leq j+k < j, k < N. \end{cases} \tag{2.10}$$

Definition 2.8 Given a G-graded contraction γ:

(i) We call γ without zeroes, if $\gamma_{jk} \neq 0$ for all $j,k \in G$ and with zeroes, if $\gamma_{jk} = 0$ for at least one pair $(j,k) \in G \times G$. We define the *support* S of γ by

$$S(\gamma) = \{(j,k) \in G \times G \mid \gamma_{jk} \neq 0\}. \tag{2.11}$$

We denote by $\mathcal{S}(G)$ the set of all supports of a G-graded contraction.

(ii) We call γ a *projection* if $\gamma \cdot \gamma = \gamma$, which yields $\gamma_{jk} \in \{0,1\}$, so that (*cf.* (2.4))

$$\mu_\gamma(V_j, V_k) = \begin{cases} \mu(V_j, V_k) & \text{if } \gamma_{jk} = 1, \\ 0 & \text{if } \gamma_{jk} = 0. \end{cases}$$

We define the *projection* π *belonging to* γ by

$$\pi(\gamma)_{jk} = \begin{cases} 1 & \text{if } \gamma_{jk} \neq 0, \\ 0 & \text{if } \gamma_{jk} = 0. \end{cases} \tag{2.12}$$

Then γ and $\pi(\gamma)$ have the same support.

(iii) We say that an equation or an expression *survives* for some γ with zeroes if it is built from non-vanishing elements only.

Proposition 2.9 *The set of G-graded contractions with the same support S forms a group under (elementwise) multiplication.*

Proof Consider two G-graded contractions γ and γ' with $S(\gamma) = S(\gamma') = S$. Then $S(\gamma \cdot \gamma') = S$. The inverse of γ is $\tilde{\gamma}$ where $S(\tilde{\gamma}) = S$ and $\tilde{\gamma}_{jk} = \frac{1}{\gamma_{jk}}$ if $\gamma_{jk} \neq 0$, the unit being the projection π with $S(\pi) = S$. ■

Definition 2.10 Let γ be a real G-graded contraction.

(i) We call γ *positive* if $\gamma_{jk} > 0$ for all $j,k \in G$ and *non-negative* if $\gamma_{jk} \geq 0$ for all $j,k \in G$.

(ii) We define the non-negative G-graded contraction $|\gamma|$ by

$$|\gamma|_{jk} = |\gamma_{jk}| \tag{2.13}$$

and the real G-graded contraction $\operatorname{sgn}\gamma$ by

$$(\operatorname{sgn}\gamma)_{jk} = \begin{cases} 1 & \text{if } \gamma_{jk} > 0, \\ -1 & \text{if } \gamma_{jk} < 0, \\ 0 & \text{if } \gamma_{jk} = 0. \end{cases} \tag{2.14}$$

Then we have

$$\gamma = |\gamma| \cdot \operatorname{sgn}\gamma. \tag{2.15}$$

Definition 2.11 (*cf.* [13]) We call two elements γ_{jk} and $\gamma_{j'k'}(j, j', k, k' \in G)$ of a G-graded contraction γ *compatible* if their product appears in at least one of the defining equations (2.3) which is not trivial, (*i.e*,. not of the form $\gamma_{jk}\gamma_{j'k'} = \gamma_{jk}\gamma_{j'k'}$) otherwise *incompatible*.

Example 2.12

(i) For $G = \mathbb{Z}_N, N = 2M(M = 1, 2, \dots)$ the $\frac{M(M+1)}{2}$ elements

$$\{\gamma_{jk} \mid j \leq k,\ j, k \text{ odd},\ j, k = 0, 1, 2, \dots, N-1\}$$

are pairwise incompatible.

(ii) For $G = \mathbb{Z} \times G'$ where $N = 2, 3, \cdots$ and G' is a finite Abelian group, the elements

$$\{\gamma_{1j,1k} \mid j, k \in G'\}$$

are pairwise incompatible.

Remark 2.13 Let $\{\gamma_{s_i} \mid i = 1, 2, \dots, r, s_i \in G \times G\}$ be a set of r pairwise incompatible elements. Let $c_i \in \mathbb{C}$ (resp., $\mathbb{R}$) be arbitrary. If we choose

$$\gamma_{s_i} = c_i, ; \quad i = 1, 2, \dots, r,$$

and for all remaining elements $\gamma_s = 0$, then γ is a complex (resp., real) G-graded contraction.

Definition 2.14 (*cf.* [6]) Two complex (resp., real) G-graded contractions γ and γ' are called *equivalent*, written $\gamma \sim \gamma'$, if a coboundary da exists with $0 \neq a_j \in \mathbb{C}$ (resp., $\mathbb{R}$), $j \in G$, so that

$$\gamma = da \cdot \gamma'. \tag{2.16}$$

Conjecture 2.15 There is a second natural notion of equivalence, namely to say that γ is equivalent to γ' if and only if $L_\gamma \simeq L_{\gamma'}$ for all G-graded Lie algebras L. Now since da is trivial (*cf.* (2.8)) we have

$$\gamma \sim \gamma' \Longrightarrow L_\gamma \simeq L_{\gamma'}, \quad \text{for all } L.$$

If the converse holds, then both equivalence relations are the same. In fact, we believe this to be true [15].

Lemma 2.16 *We have*

(i)
$$da \sim \mathbf{1},$$

(ii)
$$\gamma_1 \sim \gamma_1',\ \gamma_2 \sim \gamma_2' \Longrightarrow \gamma_1 \cdot \gamma_2 \sim \gamma_1' \cdot \gamma_2'.$$

(iii) *For γ, γ' real,*

$$\gamma \sim \gamma' \quad \Longleftrightarrow \quad |\gamma| \sim |\gamma'| \textit{ and } \operatorname{sgn} \gamma \sim \operatorname{sgn} \gamma'.$$

Proof (i) Obvious. (ii) Use that $da_1 \cdot da_2 = da$ where $a_j = (a_1)_j (a_2)_j$. (iii) Assume $\gamma \sim \gamma'$ *i.e.*, $\gamma = da \cdot \gamma'$. Write $a_j = |a_j| \operatorname{sgn} a_j$. The converse follows from (ii) . ■

Definition 2.17 (*cf.* [6]) A complex (resp., real) G-graded contraction γ is called *continuous* if there exists a family $da(\varepsilon)$, $\varepsilon \in (0, 1]$, with $0 \neq a_j(\varepsilon) \in \mathbb{C}$ (resp., $\mathbb{R}$) for $j \in G$, such that

$$\gamma = \lim_{\varepsilon \to 0} da(\varepsilon),$$

otherwise it is called *discrete*.

Definition 2.18 Consider a complex (resp., real) G-graded contraction γ. A complex (resp., real) function f of elements of γ is called an *invariant* if f is constant on equivalence classes, *i.e.*,

$$f(da \cdot \gamma) = f(\gamma) \tag{2.17}$$

for all da with $0 \neq a_j \in \mathbb{C}$ (resp., $\mathbb{R}$), $j \in G$. Similarly, a property of γ is called invariant if it is constant on equivalence classes.

Example 2.19 Obvious invariants are (i) the support S, (ii) to be a coboundary or not, (iii) to be continuous or discrete .

3 G-Graded Contractions Without Zeroes

The simplest G-graded contractions without zeroes are the coboundaries da (2.8). They constitute precisely the equivalence class of the identity (2.7), (2.16). In [13] we proved that for positive (resp., complex) $\mathbb{Z}_N$-graded contractions, this is the only equivalence class. Here we extend our earlier results to include arbitrary (finite Abelian) grading groups G, and the real case (where there can be more than one equivalence class).

Let γ be a G-graded contraction without zeroes. Theorem 3.1 proves the existence of complex numbers $a_j \neq 0 (j \in G)$ such that $\gamma = da$. Hence all complex G-graded contractions without zeroes are coboundaries. For a real γ it is, in general, not possible to choose the a_j's real. Lemma 3.4 and Theorem 3.5 give a complete and detailed answer in this situation.

Each of the defining equations (2.3) combines four different elements, except for the case $l = 0$,

$$\gamma_{0j}\gamma_{jk} = \gamma_{0k}\gamma_{jk} = \gamma_{0,j+k}\gamma_{jk}, \tag{3.1}$$

and $j = l = 0$,

$$\gamma_{00}\gamma_{0k} = \gamma_{0k}^2. \tag{3.2}$$

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Equation (3.2) yields for all γ's without zeroes

$$\gamma_{0k} = \gamma_{00}, \quad k \in G, \tag{3.3}$$

which implies that (3.1) is trivially satisfied.

Theorem 3.1 *Given a complex G-graded contraction γ without zeroes, there exist complex numbers $0 \neq a_j \in \mathbb{C}(j \in G)$ such that $\gamma = da \sim \mathbf{1}$.*

Proof We present first the proof for $\mathbb{Z}_N$, then for $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ and finally for a general G.

Case 1: $G = \mathbb{Z}_N(N = 2, 3, \dots)$. Lemma A.1 shows that all elements of γ follow uniquely from the N elements $\gamma_{00}, \gamma_{11}, \gamma_{12} \cdots \gamma_{1,N-1}$. Hence, if we show that some da exists which agrees with γ on these N elements, we must have $\gamma = da$. The ansatz (2.8)

$$\gamma_{00} = a_0, \quad \gamma_{1j} = \frac{a_1 a_j}{a_{j+1}}, \quad j = 1, 2, \dots, N-1;$$

yields

$$a_0 = \gamma_{00}; \tag{3.4}$$

$$a_j \gamma_{11}\gamma_{12} \cdots \gamma_{1,j-1} = a_1^j, \quad j = 2, 3, \dots, N-1; \tag{3.5}$$

$$a_1^N = \gamma_{00}\gamma_{11}\gamma_{12} \cdots \gamma_{1,N-1} \tag{3.6}$$

which in turn satisfies the ansatz.

If we choose an arbitrary (complex) root of (3.6) for a_1 and then define a_0 by (3.4) and $a_2, a_3, \dots, a_{N-1}$ by (3.5), we get $\gamma = da$.

Due to (3.6), there is a 1-to-N correspondence between a complex $\mathbb{Z}_N$-graded contraction γ without zeroes and the set of complex numbers a_j, $j = 0, 1, 2, \dots, N-1$, which define the corresponding coboundary da. More precisely, if a_j is one solution, all solutions are of the form:

$$a_j' = e^{i2jl\pi/N} a_j, \quad l = 0, 1, 2, \dots, N-1. \tag{3.7}$$

Case 2: $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}(N_1, N_2 = 2, 3, \dots)$. Lemma A.2 shows that all elements of γ follow uniquely from the $N_1 \cdot N_2$ elements

$$\gamma_{00,00}, \gamma_{10,j0}, \gamma_{01,0k}, \gamma_{j0,0k}, \quad j = 1, 2, \dots, N_1 - 1, \ k = 1, 2, \cdots, N_2 - 1.$$

If we show that some da exists which agrees with γ on these $N_1 \cdot N_2$ elements, we must have $\gamma = da$. We find

$$a_{00} = \gamma_{00,00}, \tag{3.8}$$

and (as in Case 1 for $\mathbb{Z}_{N_1}$)

$$a_{j0}\prod_{j'=1}^{j-1}\gamma_{10,j'0}=a_{10}^{j},\qquad j=2,3,\dots,N_1-1, \tag{3.9}$$

where

$$a_{10}^{N_1}=\prod_{j'=0}^{N_1-1}\gamma_{10,j'0} \tag{3.10}$$

and (as in Case 1 for $\mathbb{Z}_{N_2}$)

$$a_{ok}\prod_{k'=1}^{k-1}\gamma_{01,0k'}=a_{01}^{k}\quad k=2,3,\dots,N_2-1, \tag{3.11}$$

where

$$a_{01}^{N_2}=\prod_{k'=0}^{N_2-1}\gamma_{01,0k'}. \tag{3.12}$$

Finally, a_{jk} follows uniquely from a_{j0} and a_{0k} according to

$$a_{jk}=\frac{a_{j0}a_{0k}}{\gamma_{j0,0k}}\quad j=1,2,\dots,N_1-1,\ k=1,2,\dots,N_2-1. \tag{3.13}$$

Altogether we get $\gamma=da$.

Due to (3.10) and (3.12) we get a 1-to-$N_1\cdot N_2$ correspondence between a complex $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$-graded γ without zeroes and the set of complex numbers a_{jk} which define the corresponding da.

Case 3: $G=\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\cdots\times\mathbb{Z}_{N_r}(N_i=2,3,\dots,i=1,2,\dots,r)$. We use the results of Lemma A.4, where we structure G as

$$G=\mathbb{Z}_{N_1}\times(\mathbb{Z}_{N_2}\times(\mathbb{Z}_{N_3}\times(\cdots\times(\mathbb{Z}_{N_{r-1}}\times\mathbb{Z}_{N_r})\cdots))).$$

We follow the same scheme as in Case 2. We find $a_{0\cdots0}=\gamma_{0\cdots0,0\cdots0}$. For all $\mathbb{Z}_{N_i}$-subgroups we get the corresponding results to (3.9) and (3.10) resp., (3.11) and (3.12), *e.g.*, for $i=1$

$$a_{j_10\cdots0}\prod_{j_1'=1}^{j_1-1}\gamma_{10\cdots0,j_1'0\cdots0}=a_{10\cdots0}^{j_1}\quad j_1=2,3,\dots,N_1-1;$$

where

$$a_{10\cdots0}^{N_1}=\prod_{j_1'=0}^{N_1-1}\gamma_{10\cdots0,j_1'0\cdots0}.$$

This allows us to determine ($j_i = 1, 2, \dots, N_i - 1$)

$$a_{j_1 0 \cdots 0},\ a_{0 j_2 0 \cdots 0},\ \dots,\ a_{0 \cdots 0 j_r}.$$

From these results we get *uniquely*, first

$$a_{0 \cdots 0 j_{r-1} j_r} = \frac{a_{0 \cdots 0 j_{r-1} 0} a_{0 \cdots 0 j_r}}{\gamma_{0 \cdots 0 j_{r-1} 0, 0 \cdots 0 j_r}},$$

then

$$a_{0 \cdots 0 j_{r-2} j_{r-1} j_r} = \frac{a_{0 \cdots 0 j_{r-2} 00} a_{0 \cdots 0 j_{r-1} j_r}}{\gamma_{0 \cdots 0 j_{r-2} 00, 0 \cdots 0 j_{r-1} j_r}}$$

and continuing in this way,

$$a_{j_1 j_2 j_3 \cdots j_r} = \frac{a_{j_1 0 \cdots 0} a_{0 j_2 j_3 \cdots j_r}}{\gamma_{j_1 0 \cdots 0, 0 j_2 j_3 \cdots j_r}}.$$

As above, we get $\gamma = da$. ■

Now we turn to the real case.

Lemma 3.2 *Given a real G-graded contraction γ without zeroes, we have $\gamma \sim \operatorname{sgn} \gamma$. In particular, $\gamma \sim \mathbf{1}$ if γ is positive.*

Proof We have (2.15) $\gamma = |\gamma| \cdot \operatorname{sgn} \gamma$. Since $|\gamma|$ is positive, the proof of Theorem 3.1 admits positive numbers $a_j > 0$, $j \in G$ so that $|\gamma| = da \sim \mathbf{1}$. (If $G = \mathbb{Z}_N$, (3.6) admits a positive root for a_1 and consequently positive values for all remaining a_j. The reasoning is similar for a general G.) Thus, $\gamma \sim \operatorname{sgn} \gamma$. ■

Lemma 3.3 *There are 2^N different G-graded contractions* $\operatorname{sgn} \gamma$ *without zeroes, where $N = |G|$.*

Proof We know that all elements of γ follow uniquely from the N elements of a natural basis as given in Lemmas A.1, A.2 and A.3. Theorem 3.1 shows that these N elements can be arbitrarily chosen. Now there are exactly 2^N different ways to distribute the signs "$\pm$" over these N elements. ■

How many different equivalence classes exist for these 2^N $\operatorname{sgn} \gamma$'s? Lemma 3.4 gives the answer for $G = \mathbb{Z}_N$, Theorem 3.5 for a general G.

Lemma 3.4 *For real $\mathbb{Z}_N$-graded contractions γ without zeroes, there is only one equivalence class if N is odd. If $N = 2M$ ($M = 1, 2, \dots$), there exist* two *equivalence classes which are separated by the sign of $\gamma_{00}\gamma_{MM}$. Representatives are* $\mathbf{1}$ *and db,* cf. (2.10).

Proof If N is odd, (3.6) admits a real root a_1 for γ. Therefore the proof of Theorem 3.1 yields ((3.4) and (3.5)) $\gamma = da \sim \mathbf{1}$, so that we get one equivalence class.

If $N = 2M$, (3.6) admits a real root a_1 for γ only if, *cf.* (A.4),

$$\operatorname{sgn}(\gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,2M-1}) = \operatorname{sgn}(\gamma_{00}\gamma_{MM}) = +1,$$

otherwise not. This means

$$\begin{aligned} \operatorname{sgn}(\gamma_{00}\gamma_{MM}) &= +1 \Rightarrow \gamma = da \sim \mathbf{1}, \\ \operatorname{sgn}(\gamma_{00}\gamma_{MM}) &= -1 \Rightarrow \gamma \not\sim \mathbf{1}. \end{aligned}$$

An example for the second case is the real $\mathbb{Z}_{2M}$-graded contraction db (*cf.* (2.10), Remark 3.6 and Example 3.7) since $(db)_{00}(db)_{MM} = -1$. In the case $\operatorname{sgn}(\gamma_{00}\gamma_{MM}) = -1$ we have $\operatorname{sgn}[(\gamma \cdot db)_{00}(\gamma \cdot db)_{MM}] = +1$ so that (see above) $\gamma \cdot db \sim \mathbf{1}$ which yields (since $db \cdot db = \mathbf{1}$) $\gamma \sim db$. Therefore we get two equivalence classes with representatives $\mathbf{1}$ and db which are separated by the sign of $\gamma_{00}\gamma_{MM}$. (In Section 4.1 we will see that $\operatorname{sgn}(\gamma_{00}\gamma_{MM})$ is a sign invariant of the second kind (Example 4.17).) ∎

The above result for $\mathbb{Z}_N$ makes the following theorem hardly surprising.

Theorem 3.5 *Let $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$ where r_e of the r numbers N_i, $N_i = 2, 3, \ldots,$ $i = 1, 2, \ldots, r$, are even and the others are odd, $0 \le r_e \le r$. Then there exist 2^{r_e} equivalence classes for real G-graded contractions γ without zeroes.*

Proof *Case 1:* $r = 1$. This follows from Lemma 3.4.

Case 2: $r = 2$. Due to (3.10) and (3.12) we have now exactly the same situation as in Lemma 3.4 for each individual $\mathbb{Z}_{N_i}$-subgroup. Note that (3.13) does not present any additional problem. We therefore get, as in Lemma 3.4, for a real γ without zeroes:

one equivalence class if N_1 and N_2 are odd with representative $\mathbf{1}$,

two equivalence classes if $N_1 = 2M_1$ ($M_1 = 1, 2, \ldots$) and N_2 is odd, separated by the sign of $\gamma_{00,00}\gamma_{M_10,M_10}$ with representatives (Definition 2.5(ii)) $\mathbf{1} \otimes \mathbf{1}$ and $db \otimes \mathbf{1}$,

four equivalence classes if $N_1 = 2M_1$ and $N_2 = 2M_2$ ($M_1, M_2 = 1, 2, \ldots$) separated by the signs of $\gamma_{00,00}\gamma_{M_10,M_10}$ and $\gamma_{00,00}\gamma_{0M_2,0M_2}$ with representatives $\mathbf{1} \otimes \mathbf{1}$, $db \otimes \mathbf{1}$, $\mathbf{1} \otimes db$, $db \otimes db$.

Case 3: $r \ge 3$ (Case 3 in the proof of Theorem 3.1). Exactly as in Case 2, each subgroup $\mathbb{Z}_{N_i}$ can be represented (only) by $\mathbf{1}$ if N_i is odd and by $\mathbf{1}$ or db if N_i is even. ∎

Remark 3.6 What is the result of a $\mathbb{Z}_{2M}$-graded contraction of a real Lie algebra L by db? Since the complex change of basis $V_j \longrightarrow b_j V_j$ (with the obvious meaning (2.9)) maps L onto L_{db}, L and L_{db} have the same complex extension.

Example 3.7 Consider the $\mathbb{Z}_2$-graded real compact Lie algebra $L = (V, \mu) = so(3)$ where $V = V_0 \oplus V_1$ with basis vectors $e_3 \in V_0, e_1, e_2 \in V_1$ and

$$\mu(e_1, e_2) = e_3 \quad \mu(e_2, e_3) = e_1 \quad \mu(e_3, e_1) = e_2.$$

The graded contraction of L by db ((2.9) and (2.10)) *i.e.*, $b_0 = 1, b_1 = i, (db)_{00} = (db)_{01} = +1, (db)_{11} = -1$ yields the real non-compact Lie algebra $L_{db} = (V, \mu_{db}) = so(2,1)$ where

$$\mu_{db}(e_1, e_2) = -e_3, \quad \mu_{db}(e_2, e_3) = e_1, \quad \mu_{db}(e_3, e_1) = e_2$$

i.e., L and L_{db} are two different real forms of the same simple complex Lie algebra A_1. The complex change of basis given by db, namely $V_0 \to V_0$ and $V_1 \to iV_1$, is known as Weyl's unitary trick.

Example 3.8 The decomposition of G as $G \simeq \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$ is in general not unique. Nevertheless the total number r_e of even numbers N_i is unique. We give two simple illustrations of this fact.

(i) Theorem 3.5 yields two equivalence classes for real $\mathbb{Z}_4$-graded contractions and four in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$. This result reflects the fact that $\mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) Theorem 3.4 yields two equivalence classes for $\mathbb{Z}_6$ which are separated by $\operatorname{sgn}(\gamma_{00}\gamma_{33})$. This result can be easily rewritten for $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$. Using the identification of the element 1 of $\mathbb{Z}_6$ with the element (1,1) of $\mathbb{Z}_2 \times \mathbb{Z}_3$ gives two equivalence classes which are separated by $\operatorname{sgn}(\gamma_{00,00}\gamma_{10,10})$, in agreement with Theorem 3.5.

Finally, we want to look at our results for real G-graded contractions $\gamma \sim \operatorname{sgn}\gamma$ without zeroes in a group theoretical way. Define

$$\operatorname{Sgn}(G) = \{\operatorname{sgn}\gamma \mid \gamma \text{ a real } G\text{-graded contraction without zeroes}\}. \tag{3.14}$$

$\operatorname{Sgn}(G)$ is a group with respect to elementwise multiplication (see also Proposition 2.9). The unit is **1** and each element agrees with its inverse. $\operatorname{Sgn}(G)$ has 2^N elements where $N = |G|$ (Lemma 3.3). Of particular interest is the subgroup

$$\operatorname{Sgn}_0(G) = \{da \mid a_j = \pm 1, \, j \in G\}, \tag{3.15}$$

since it implements the equivalence relation for $\operatorname{sgn}\gamma$ Lemma 2.16(iii)). It follows that the order of $\operatorname{Sgn}_0(G)$ is the number of elements in the equivalence class of any $\operatorname{sgn}\gamma$, and that the order of the quotient group is the number of equivalence classes.

Theorem 3.1 yields (see also Examples 2.7(iii))

$$\operatorname{Sgn}(\mathbb{Z}_N) = \operatorname{Sgn}_0(\mathbb{Z}_N) \text{ if } N \text{ is odd}, \tag{3.16}$$

$$\operatorname{Sgn}(\mathbb{Z}_N) = \operatorname{Sgn}_0(\mathbb{Z}_N) \cup db \cdot \operatorname{Sgn}_0(\mathbb{Z}_N) \text{ if } N = 2M, M = 1, 2 \dots. \tag{3.17}$$

Note that $\operatorname{Sgn}_0(\mathbb{Z}_N)$ has only 2^{N-1} elements for $N = 2M$ since in this case $a'_j = (-1)^j a_j$ yields $da' = da$. (This corresponds to $l = M, N = 2M$ in (3.7).) More generally,

$$\operatorname{Sgn}_0(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}) \text{ has } 2^{N-r_e} \text{ elements}, \tag{3.18}$$

where r_e of the r indices N_i are even (since 2^{r_e} different sets of a_j's belong to the same da).

4 Invariants

In this section we study invariants for G-graded contractions. We already know that the support is an invariant, *cf.* Examples 2.19(i). A second type of invariant arises from the G-dependent "higher-order identities" found by the author [13, Definition 4.1 and Lemma 4.4]. For real G-graded contractions there are also two types of sign invariants which we characterize completely (*cf.* Section 4.2).

In Theorem 7.1 we will show that our set of invariants is complete by proving that two inequivalent G-graded contractions must differ on one of these invariants.

4.1 Higher-Order Identities

Definition 4.1 Consider the set of all G-graded contractions γ for fixed G and two products $P_1(\gamma)$ and $P_2(\gamma)$ of the same number $r \geq 3$ of (not necessarily different) elements of γ. If

$$P_1(da) = P_2(da) \tag{4.1}$$

for all da where $0 \neq a_j \in \mathbb{C}$, $j \in G$, but if some γ exists with $P_1(\gamma) \neq P_2(\gamma)$ we call (4.1) a higher-order identity, for short, "$P_1 = P_2$"

Remark 4.2

(i) Since all G-graded contractions γ without zeroes are coboundaries da with $0 \neq a_j \in \mathbb{C}$ (Theorem 3.1), they automatically satisfy $P_1(\gamma) = P_2(\gamma)$ for all higher-order identities "$P_1 = P_2$".

(ii) We have ((2.8); $i = 1, 2$)

$$P_i(da) = \prod_{j \in G} a_j^{n_{ij}}, \quad n_{ij} \in \mathbb{Z}.$$

Since all a_j's can be chosen independently and arbitrarily, it follows that $P_1(da) = P_2(da)$ for all $da \Leftrightarrow n_{1j} = n_{2j}$ $j \in G$. Therefore, (4.1) can only hold if P_1 and P_2 have the same number of elements.

(iii) In Definition 4.1 the two cases $r = 1$ and $r = 2$ are omitted because of the following two reasons. If $r = 1$ (3.2) would yield the only candidate, namely

$$P_1(\gamma) = \gamma_{00}, \quad P_2(\gamma) = \gamma_{0k}, \quad 0 \neq k \in G.$$

We have $P_1(\gamma) = P_2(\gamma)$ for all γ's without zeroes (3.3), and this identity can only be *weakly* violated by γ's with zeroes (3.2). Therefore, and since this case is, in contrast to all others, independent of G, this identity will always be treated separately and not considered as a higher-order identity. If $r = 2$, one can easily verify that the ansatz

$$P_1(da) = \prod_{j \in G} a_j^{n_j} = P_2(da), \quad n_j \in \mathbb{Z}, \ \sum n_j = 2$$

only yields a trivial identity or a defining equation (2.3) which cannot be violated. Therefore higher-order identities with $r = 2$ do not exist.

We will see in Section 6 that it makes a crucial difference how a higher-order identity is violated.

Definition 4.3 A G-graded contraction γ *strongly violates* a higher-order identity "$P_1 = P_2$" if $0 \neq P_1(\gamma) \neq P_2(\gamma) \neq 0$ and *weakly violates* it if $0 = P_1(\gamma) \neq P_2(\gamma)$ or $0 = P_2(\gamma) \neq P_1(\gamma)$.

Note that each type of violation excludes the other. (The notation here is not ideal, since a strong violation does not imply a weak violation. However, it is appropriate in the sense that, as we shall see in Section 6, the consequences of a strong violation are stronger.)

The next lemma shows how invariants (Definition 2.18) arise from higher-order identities.

Lemma 4.4 *Let "$P_1 = P_2$" be a higher-order identity for G-graded contractions γ.*

(i) *The property $P_i(\gamma) = 0$, resp., $P_i(\gamma) \neq 0$, for $i = 1$ or $i = 2$ is invariant.*

(ii) *In the case $P_i(\gamma) \neq 0$ for $i = 1$ and 2, the complex (resp., real) number*

$$\frac{P_1(\gamma)}{P_2(\gamma)}$$

is an invariant.

Proof Since P_i is a product of matrix elements we have (Definition 2.5(i))

$$P_i(da \cdot \gamma) = P_i(da)P_i(\gamma), \quad i = 1, 2.$$

(i) follows immediately since $P_i(da) \neq 0$; (ii) follows immediately since (Definition 4.1) $P_1(da) = P_2(da)$. ■

Remark 4.5 (i) Higher-order identities can be combined to produce new ones. Consider the r invariants

$$\frac{P_1^{(l)}(\gamma)}{P_2^{(l)}(\gamma)}, \quad l = 1, 2, \dots r, \; r = 1, 2 \dots .$$

Then

$$\frac{P_1(\gamma)}{P_2(\gamma)} = \prod_{l=1}^{r} \Big(\frac{P_1^{(l)}(\gamma)}{P_2^{(l)}(\gamma)} \Big)^{n_l}, \quad n_l \in \mathbb{Z}.$$

is obviously again an invariant.

(ii) In [13] we began a study of the general structure of higher-order identities and listed all of them for $G = \mathbb{Z}_N$ with $N \leq 8$ (where "all" means that the remaining invariants follow from these as in (i)).

Fortunately, such a list for all G is not needed here, since Theorem 6.5 automatically produces all those higher-order identities which a G-graded contraction with given support can strongly violate. In contrast, the weak violations are completely determined by the support and they will not play a separate role in this paper (they do play a role in the characterization of continuous graded contractions [13, 15]).

Example 4.6 We give several examples for higher-order identities "$P_1 = P_2$" with $r = 3, 4, 5$ factors.

$r = 3$: The general structure is $P_1(\gamma) = \gamma_{j_1k_1}\gamma_{j_2k_2}\gamma_{j_3k_3}$ and $P_2(\gamma) = \gamma_{j_2k_1}\gamma_{j_3k_2}\gamma_{j_1k_3}$ [13], where

$$j_1 + k_1 = j_3 + k_2 \quad j_2 + k_2 = j_1 + k_3 \quad j_3 + k_3 = j_2 + k_1$$

and where all elements which occur are pairwise incompatible. Then "$P_1 = P_2$" is indeed a higher-order identity since

$$P_1(da) = \prod_{i=1}^{3} \frac{a_{j_i}a_{k_i}}{a_{j_i+k_i}} = P_2(da)$$

and since all elements which occur are pairwise incompatible so that P_1 and P_2 can take on arbitrary values (Remark 2.13) which leads to arbitrary strong and weak violations.

Another proof is the following. Using the defining equations 3 times yields

$$\gamma_{j_1+k_1,k_3}\gamma_{j_1k_1}\gamma_{j_2k_2}\gamma_{j_3k_3} = \gamma_{j_1+k_1,k_3}\gamma_{j_2k_1}\gamma_{j_3k_2}\gamma_{j_1k_3},$$

so we have for all γ without zeroes (but not necessarily for those with $\gamma_{j_1+k_1,k_3} = 0$) that $P_1(\gamma) = P_2(\gamma)$.

The element $\gamma_{j_1+k_1,k_3}$ works therefore as a "zipper-element" which overcomes the incompatibility of all other elements.

A simple counting argument shows that we need at least 6 different $j \in G$ to form such a higher-order identity (*e.g.,* 3 for the sums $j_i + k_i$ and a minimum of 3 for $j_i, k_i (i = 1, 2, 3)$). Therefore such a higher-order identity cannot exist for $N = |G| < 6$.

Examples for $N \geq 6$ are

$$\begin{aligned}
\mathbb{Z}_6\colon P_1(\gamma) &= \gamma_{11}\gamma_{33}\gamma_{55}; P_2(\gamma) = \gamma_{13}\gamma_{15}\gamma_{35}\\
\mathbb{Z}_N(N \geq 7)\colon P_1(\gamma) &= \gamma_{25}\gamma_{33}\gamma_{44}\ ;\ P_2(\gamma) = \gamma_{24}\gamma_{34}\gamma_{35}\\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\colon P_1(\gamma) &= \gamma_{100,100}\gamma_{101,111}\gamma_{110,111}; P_2(\gamma) = \gamma_{100,101}\gamma_{111,111}\gamma_{100,110}\\
\mathbb{Z}_2 \times \mathbb{Z}_4\colon P_1(\gamma) &= \gamma_{10,10}\gamma_{11,12}\gamma_{12,13}\ ;\ P_2(\gamma) = \gamma_{10,11}\gamma_{10,13}\gamma_{12,12}\\
\mathbb{Z}_N \times \mathbb{Z}_N(N \geq 3)\colon P_1(\gamma) &= \gamma_{10;N-1,0}\gamma_{0,N-1;12}\gamma_{01,21};\\
P_2(\gamma) &= \gamma_{01;0,N-1}\gamma_{N-1,0;21}\gamma_{10,12}.
\end{aligned}$$

$r = 4$: The general structure [13] is either of the cyclic type as for $r = 3$ or

$$P_1(\gamma) = \gamma_{j_1k_1}\gamma_{j_2k_2}\gamma_{j_3k_3}\gamma_{j_4k_4}, \quad P_2(\gamma) = \gamma_{j_1k_2}\gamma_{j_2k_1}\gamma_{j_3k_4}\gamma_{j_4k_3}$$

where

$$j_1 + k_1 = j_3 + k_4, \quad j_2 + k_2 = j_4 + k_3, \quad j_3 + k_3 = j_1 + k_2, \quad j_4 + k_4 = j_2 + k_1.$$

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Again all elements are pairwise incompatible so that arbitrary strong and weak violations are possible. Here we need the product of two "zipper-elements" *e.g.*, $\gamma_{j_1+k_1,k_2}\gamma_{j_4+k_4,k_3}$.

A simple counting argument shows that we need at least 8 different $j \in G$ to form a higher-order identity with $r = 4$. The first ones occur indeed for $N = |G| = 8$. All of them for $\mathbb{Z}_8$ are

$$\begin{aligned}
P_1(\gamma) &= \gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77} & P_2(\gamma) &= (\gamma_{15})^2(\gamma_{37})^2 \\
P_1(\gamma) &= \gamma_{11}\gamma_{33}(\gamma_{57})^2 & P_2(\gamma) &= (\gamma_{13})^2\gamma_{55}\gamma_{77} \\
P_1(\gamma) &= \gamma_{11}(\gamma_{35})^2\gamma_{77} & P_2(\gamma) &= (\gamma_{17})^2\gamma_{33}\gamma_{55} \\
P_1(\gamma) &= \gamma_{17}\gamma_{22}\gamma_{35}\gamma_{66} & P_2(\gamma) &= \gamma_{13}(\gamma_{26})^2\gamma_{57}.
\end{aligned}$$

Another example for $\mathbb{Z}_2 \times \mathbb{Z}_4$ is

$$P_1(\gamma) = \gamma_{10,10}\gamma_{11,11}\gamma_{12,12}\gamma_{13,13}, \quad P_2(\gamma) = (\gamma_{10,12})^2(\gamma_{11,13})^2.$$

$r = 5$: Here the first higher-order identities occur which are not built from pairwise incompatible elements, namely

$$P_1(\gamma) = \gamma_{j_1 j_2}\gamma_{j_1+j_2, j_3}\gamma_{j_4 j_7}\gamma_{j_5 j_8}\gamma_{j_6 j_9} \quad P_2(\gamma) = \gamma_{j_4 j_5}\gamma_{j_4+j_5, j_6}\gamma_{j_1 j_9}\gamma_{j_2 j_7}\gamma_{j_3 j_8},$$

where

$$j_1 + j_2 + j_3 = j_4 + j_5 + j_6$$

and

$$j_4 + j_7 = j_1 + j_9, \quad j_5 + j_8 = j_2 + j_7, \quad j_6 + j_9 = j_3 + j_8.$$

An example for $\mathbb{Z}_{29}$ is

$$P_1(\gamma) = \gamma_{13}\gamma_{48}\gamma_{2,14}\gamma_{20,26}\gamma_{19,15} \quad P_2(\gamma) = \gamma_{2,20}\gamma_{22,19}\gamma_{1,15}\gamma_{3,14}\gamma_{8,26}.$$

This brings the following two, and only these two, defining equations into play

$$\begin{gathered}
\gamma_{13}\gamma_{48} = \gamma_{18}\gamma_{39} = \gamma_{38}\gamma_{1,11} \\
\gamma_{2,20}\gamma_{22,19} = \gamma_{2,19}\gamma_{20,21} = \gamma_{19,20}\gamma_{2,10},
\end{gathered}$$

provided one sets all γ_{jk} which do not occur in the above 4 equations equal to zero. Then one can choose for all elements which occur in P_1 and P_2 arbitrary values, and for the remaining elements one chooses values so that both defining equations are satisfied. Then P_1 and P_2 can take on arbitrary values, and we can have arbitrary strong and weak violations.

Lemma 4.7 *Higher-order identities exist for G-graded contractions if and only if* $N = |G| \geq 6$.

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Proof (i) Let $N \leq 5$. Then we have at most 5 different $j \in G$ so that a higher-order identity cannot be constructed since we already need at least 6 different digits for the simplest one with 3 factors (Example 4.6 with $r = 3$).

(ii) Let $G \simeq \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$ where

$$N = \prod_{i=1}^{r} N_i \geq 6.$$

Then G must contain a subgroup which is isomorphic to one of the following groups

$$\mathbb{Z}_N (N \geq 6), \quad \mathbb{Z}_5 \times \mathbb{Z}_5, \quad \mathbb{Z}_4 \times \mathbb{Z}_4, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(Assume *e.g.*, that $\mathbb{Z}_5$ is the subgroup with the highest order contained in G. Since $|G| \geq 6$, G must contain as subgroup either $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{Z}_5 \times \mathbb{Z}_4 \simeq \mathbb{Z}_{20}$ or $\mathbb{Z}_5 \times \mathbb{Z}_3 \simeq \mathbb{Z}_{15}$ or $\mathbb{Z}_5 \times \mathbb{Z}_2 \simeq \mathbb{Z}_{10}$.) For all these groups higher-order identities do exist (Example 4.6 with $r = 3$) which can be trivially lifted to higher-order identities for G. ■

4.2 Sign Invariants

Here, we introduce sign invariants for *real* G-graded contractions. We characterize them and show that they split naturally into two kinds (Lemmas 4.8, 4.10, 4.11, Definition 4.12, Lemmas 4.13, 4.14, 4.15). We give examples of both kinds (Examples 4.17, 4.18). In the following sections we will see that sign invariants of the first kind play a crucial role for the existence of real γ's without zeroes (Theorems 5.12, 6.11), whereas sign invariants of the second kind define their equivalence class (Remark 4.16(i), Theorems 5.12, 6.11).

Lemma 4.8 *Consider a real G-graded contraction γ and a product*

$$P(\gamma) = \prod_{i=1}^{r} \gamma_{j_i k_i}, \qquad j_i, k_i \in G,$$

of r different, non-vanishing elements of γ. Then Sgn $P(\gamma)$ *is an invariant if and only if $P(da) > 0$ for all $0 \neq a_j \in \mathbb{R}$, $j \in G$.*

Proof The proof follows trivially from Definition 2.18 and the fact that $P(da \cdot \gamma) = P(da)P(\gamma)$. ■

Example 4.9 (i) Let $G = \mathbb{Z}_5$ and $P(\gamma) = \gamma_{13}\gamma_{22}\gamma_{33}\gamma_{44} \neq 0$. Due to the two defining equations $\gamma_{22}\gamma_{44} = \gamma_{24}\gamma_{12}$ and $\gamma_{12}\gamma_{33} = \gamma_{13}\gamma_{24}$ (which have to survive for all γ's with $P(\gamma) \neq 0$, no matter if γ has zeroes or not) we get

$$P(\gamma) = \gamma_{13}\gamma_{22}\gamma_{33}\gamma_{44} = \gamma_{12}^2\gamma_{33}^2 \tag{4.2}$$

which means (for $0 \neq a_j \in \mathbb{R}$) $P(da) > 0$. Therefore sgn $P(\gamma)$ is a sign invariant. Although we always have sgn $P(\gamma) = +1$, this still has some non-trivial consequences (Theorem 5.12).

(ii) Let $G = \mathbb{Z}_8$ and $P(\gamma) = \gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77} \neq 0$. We know that we have for all γ's without zeroes

$$P(\gamma) = \gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77} = (\gamma_{15})^2(\gamma_{37})^2 \tag{4.3}$$

since this is a higher-order identity (Example 4.6 with $r = 4$). This means $P(da) > 0$ (for $0 \neq a_j \in \mathbb{R}$), so that $\operatorname{sgn} P(\gamma)$ is a sign invariant (which is necessarily positive if γ has no zeroes). Since $P(\gamma)$ is built from pairwise non-compatible elements, we clearly can have, for γ's with zeroes (Remark 2.13), $\operatorname{sgn} P(\gamma) = -1$.

These two examples can be immediately generalized as follows.

Lemma 4.10 *Consider a real G-graded contraction γ and a product $P(\gamma)$ as in Lemma* 4.8*. Assume $P(\gamma)$ can be expressed for all γ's without zeroes as a product of squares of some elements of γ, then* $\operatorname{sgn} P(\gamma)$ *is a sign invariant (which is necessarily positive for γ's without zeroes).*

In practise, this lemma cannot, in general, be readily applied. We now develop a straightforward criterion. We begin by noting that when we have a basis $\{\gamma_{s_i} \mid s_i \in G \times G\,;\, i = 1, 2, \dots N = |G|\}$, we have for every element γ_{jk} a unique expression

$$\gamma_{jk} = \prod_{i=1}^{N} \gamma_{s_i}^{n_i}, \quad n_i \in \mathbb{Z},$$

((A.1) for $G = \mathbb{Z}_N$, Definition 5.1 (5.3)). This means that, in any basis, a product of squares of elements, when expressed in terms of the basis elements, always yields a product of squares. Thus we have

Lemma 4.11 *Let $P(\gamma)$ be as in Lemma* 4.8*, and let $\{\gamma_{s_i} \mid s_i \in G\times G; i = 1, 2, \dots N = |G|\}$ be a basis. For a γ without zeroes, we can write*

$$P(\gamma) = \prod_{i=1}^{N} \gamma_{s_i}^{m_i}, \quad m_i \in \mathbb{Z}. \tag{4.4}$$

If all powers m_i are even, then $\operatorname{sgn} P(\gamma)$ *is a sign invariant (which is necessarily positive for γ's without zeroes).*

Definition 4.12 A sign invariant which has the above form (4.4) with all m_i even, we call a *sign invariant of the first kind*. All other sign invariants (where some m_i are odd), we call *sign invariants of the second kind*.

In order to treat arbitrary sign invariants, we now give a useful general criterion.

Lemma 4.13 *Consider a real G-graded contraction γ and a product*

$$P(\gamma) = \prod_{i=1}^{r} \gamma_{j_i k_i}, \, j_i, k_i \in G$$

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of r different, non-vanishing elements of γ. Sgn $P(\gamma)$ is an invariant if and only if r is even and we have for all da with $0 \neq a_j \in \mathbb{R}$, $(j \in G)$

$$P(da) = \prod_{j \in G} a_j^{n_j}, \quad n_j \in \mathbb{Z} \tag{4.5}$$

where all powers n_j are even.

Proof We have

$$P(da) = \prod_{i=1}^{r} \frac{a_{j_i} a_{k_i}}{a_{j_i+k_i}} = \prod_{j \in G} a_j^{n_j}, \quad n_j \in \mathbb{Z}.$$

For a sign invariant we need (Lemma 4.8) $P(da) > 0$ for all $0 \neq a_j \in \mathbb{R}$. This requires all powers $n_j \in \mathbb{Z}$ to be even. Since $r = \sum_{j \in G} n_j$, this implies furthermore that r is even. ∎

Now we are in a position to characterize all sign invariants. We start with $G = \mathbb{Z}_N$.

Lemma 4.14 *Consider a real $\mathbb{Z}_N$-graded contraction γ. If N is odd, all sign invariants are of the first kind. If $N = 2M$, $M = 1, 2, \ldots$, we have for all sign invariants of the second kind for all γ's without zeroes*

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00}\gamma_{11}\gamma_{12}\ldots\gamma_{1,N-1}) = \operatorname{sgn}(\gamma_{00}\gamma_{MM}). \tag{4.6}$$

Proof If γ is without zeroes, we can express any product $P(\gamma)$ by the elements of the natural basis (A.1) to bring it into the form $(j = 0, 1, 2, \ldots N-1)$

$$P(\gamma) = \gamma_{00}^{m_0}\gamma_{11}^{m_1}\gamma_{12}^{m_2}\cdots\gamma_{1,N-1}^{m_{N-1}}, \quad m_j \in \mathbb{Z}. \tag{4.7}$$

This yields for an arbitrary da with $0 \neq a_j \in \mathbb{R}$ (2.8)

$$P(da) = a_0^{m_0 - m_{N-1}} a_1^{2m_1 + m_2 + m_3 + \cdots + m_{N-1}} a_2^{m_2 - m_1} a_3^{m_3 - m_2} \ldots a_{N-1}^{m_{N-1} - m_{N-2}}.$$

According to Lemma 4.13, sgn $P(\gamma)$ is an invariant if and only if all powers in this expression are even, *i.e.*, if

$$m_0 - m_{N-1}, \quad m_2 - m_1, \quad m_3 - m_2 \quad , \ldots, \quad m_{N-1} - m_{N-2}$$

and

$$2m_1 + m_2 + m_3 + \cdots + m_{N-1}$$

are all even. This set of relations has exactly two types of solutions, namely:

(a) Assume m_1 is even. Then the relations above force all other m_j to be even, too, so that we get a sign invariant of the first kind (Definition 4.12).

(b) Assume m_1 is odd. Then the relations above force all other m_j to be odd, too, plus $(2m_1 + m_2 + m_3 + \cdots + m_{N-1})$ to be even which means $N \cdot \text{odd} = \text{even}$, so that N has to be even. In this case we have

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00}^{m_0}\gamma_{11}^{m_1}\gamma_{12}^{m_2}\ldots\gamma_{1,N-1}^{m_{N-1}})$$

where all m_j are odd so that (A.4)

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00}\gamma_{11}\gamma_{12}\dots\gamma_{1,N-1}) = \operatorname{sgn}(\gamma_{00}\gamma_{MM})$$

where $N = 2M$. ■

Lemma 4.15 *Consider a real G-graded contraction G where $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$ ($r = 2, 3, \dots$). Assume r_e of the r numbers N_i are even, the others are odd. If $r_e = 0$, all sign invariants are of the first kind. If $r_e > 0$, there are also sign invariants* $\operatorname{sgn} P(\gamma)$ *of the second kind. They behave for all γ's without zeroes either like a sign invariant of the second kind for one individual subgroup $\mathbb{Z}_{N_i}$ with $N_i = 2M_i$,* i.e., *we have by Lemma* 4.14, $\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{0\cdots0,0\cdots0}\gamma_{0\cdots0M_i0\cdots0,0\cdots0M_i0\cdots0})$ *or like products of these sign invariants for several different such subgroups,* i.e., *we have*

$$\operatorname{sgn} P(\gamma) = \prod_i \operatorname{sgn}(\gamma_{0\cdots0,0\cdots0}\gamma_{0\cdots0M_i0\cdots0,0\cdots0M_i0\cdots0})$$

where the product is taken over $2, 3\dots$ *or all r_e different subgroups $\mathbb{Z}_{N_i}$ with $N_i = 2M_i$.*

Proof *Case 1:* $r = 2$. We follow the proof for $G = \mathbb{Z}_N$ (Lemma 4.14). By expressing $P(\gamma)$ by elements of a natural basis (*cf.* proof of Theorem 3.1, Case 2) we get

$$(4.8)\quad P(\gamma) = \gamma_{00,00}^{m_{00}} \Big(\prod_{j=1}^{N_1-1} \gamma_{10,j0}^{m_{j0}}\Big)\Big(\prod_{k=1}^{N_2-1} \gamma_{01,0k}^{m_{0k}}\Big) \times \Big(\prod_{j=1}^{N_1-1}\prod_{k=1}^{N_2-1} \gamma_{j0,0k}^{m_{jk}}\Big) \quad m_{00}, m_{j0}, m_{0k}, m_{jk} \in \mathbb{Z}.$$

This yields for some arbitrary da

$$(4.9)\quad P(da) = a_{00}^{m_{00}-m_{N_1-1,0}-m_{0,N_2-1}} \cdot a_{10}^{2m_{10}+m_{20}+m_{30}+\cdots+m_{N_1-1,0}+\sum_{k=1}^{N_2-1} m_{1k}} \cdot \Big(\prod_{j=2}^{N_1-1} a_{j0}^{m_{j0}-m_{j-1,0}+\sum_{k=1}^{N_2-1} m_{jk}}\Big) \cdot a_{01}^{2m_{01}+m_{02}+m_{03}+\cdots+m_{0,N_2-1}+\sum_{j=1}^{N_1-1} m_{j1}} \cdot \Big(\prod_{k=2}^{N_2-1} a_{0k}^{m_{0k}-m_{0,k-1}+\sum_{j=1}^{N_1-1} m_{jk}}\Big) \cdot \Big(\prod_{j=1}^{N_1-1}\prod_{k=1}^{N_2-1} a_{jk}^{-m_{jk}}\Big).$$

Sgn $P(\gamma)$ is an invariant if and only if all powers in this expression are even (Lemma 4.13). Obviously, this forces all m_{jk} to be even for $j = 1, 2, \dots, N_1 - 1$, $k = 1, 2, \dots, N_2 - 1$. Therefore the remaining requirements for $m_{10}, m_{20}, \cdots m_{N_1-1,0}$ are then identical to those for the sign invariants for the subgroup $\mathbb{Z}_{N_1}$ alone (which we studied in Lemma 4.14) and independent of those for $m_{01}, m_{02}, \cdots m_{0,N_2-1}$ which in turn are identical to those for sign invariants of the subgroup $\mathbb{Z}_{N_2}$ alone. The requirement for m_{00} which belongs to both subgroups then follows accordingly.

Altogether we get exactly the following sign invariants.

If all powers in (4.8) are even, we have a sign invariant of the first kind (Definition 4.12).

If $N_1 = 2M_1$ and N_2 is odd, we have for all sign invariants $\operatorname{sgn} P(\gamma)$ of the second kind for all γ's without zeroes,

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00,00}\gamma_{M_1 0,M_1 0}) = \operatorname{sgn}(\gamma_{00,00}\gamma_{10,10}\gamma_{10,20}\cdots\gamma_{10;N_1-1,0}).$$

If $N_1 = 2M_1$ and $N_2 = 2M_2$, we have for all sign invariants of the second kind for all γ's without zeroes one of the following three possibilities:

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00,00}\gamma_{M_1 0,M_1 0}) = \operatorname{sgn}(\gamma_{00,00}\gamma_{10,10}\gamma_{10,20}\cdots\gamma_{10;N_1-1,0})$$

or

$$\operatorname{sgn} P(\gamma) = \operatorname{sgn}(\gamma_{00,00}\gamma_{0M_2,0M_2}) = \operatorname{sgn}(\gamma_{00,00}\gamma_{01,01}\gamma_{01,02}\cdots\gamma_{01;0,N_2-1)}$$

or

$$\begin{aligned}\operatorname{sgn} P(\gamma) &= \operatorname{sgn}(\gamma_{M_1 0,M_1 0}\gamma_{0M_2,0M_2})\\ &= \operatorname{sgn}(\gamma_{10,10}\gamma_{10,20}\cdots\gamma_{10;N_1-1,0}\gamma_{01,01}\gamma_{01,02}\cdots\gamma_{01;0,N_2-1}).\end{aligned}$$

Case 2: $r \geq 3$. We use a natural basis for G (Lemma A.4) and proceed as in Case 1. With the same reasoning we get exactly the corresponding results. ■

Remark 4.16 (i) Note that the two equivalence classes of real $\mathbb{Z}_{2M}$-graded contractions without zeroes ($M = 1, 2, \dots$) are separated by the sign invariant $\operatorname{sgn}(\gamma_{00}\gamma_{MM})$ of the second kind (proof of Lemma 3.4).

Similarly, the equivalence classes of real G-graded contractions without zeroes, where $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$, are separated by the sign invariants of the second kind of all those subgroups $\mathbb{Z}_{N_i}$ where $N_i = 2M_i$ (proof of Theorem 3.5).

(ii) Consider a real G-graded contraction γ with two different sign invariants $\operatorname{sgn} P_1(\gamma)$ and $\operatorname{sgn} P_2(\gamma)$. Then the product $P_1(\gamma)P_2(\gamma) \neq 0$ is obviously again a sign invariant (after all squares of elements, which are necessarily positive, have been removed, *i.e.,* if we take all exponents modulo 2).

In particular, the product of two sign invariants of the first kind is again one of the first kind. In contrast, the product of two sign invariants of the second kind for $\mathbb{Z}_{2M}$ is one of the first kind. In the case of a general $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$ with $r_e \geq 2$ (Lemma 4.15) we must make the following distinction. If both sign invariants of the second kind belong to the same subgroups $\mathbb{Z}_{N_i}$ where $N_i = 2M_i$, *i.e.,* have odd powers at exactly the same places when expressed in a natural basis as in (4.8), their product is a sign invariant of the first kind (since now all powers have become even). If they differ on one such subgroup, their product is a sign invariant of the second kind (since some odd powers remain).

(iii) Let "$P_1 = P_2$" be a higher-order identity for G-graded contractions. Then $\operatorname{sgn}[P_1(\gamma)P_2(\gamma)]$ is trivially a sign invariant (if all exponents are taken modulo 2), since ((4.1), Lemma 4.8) $P_1(da) = P_2(da)$. Since $P_1(\gamma) = P_2(\gamma)$ for all real γ's without zeroes (Remark 4.2(i)), this sign invariant is obviously of the first kind.

Example 4.17 (Sign invariants of the second kind) Lemma 4.14 tells us that for real $\mathbb{Z}_{2M}$-graded γ's without zeroes all sign invariants of the second kind agree. Therefore we can restrict ourselves to one, *e.g.,* $\operatorname{sgn} P(\gamma)$ where $P(\gamma) = \gamma_{00}\gamma_{MM}$. But for $\mathbb{Z}_{2M}$-graded γ's with zeroes this is different. Some sign invariants of the second kind may vanish for γ, while some others are positive and still others are negative (see the examples below). Therefore we have to consider all others as well.

All sign invariants $\operatorname{sgn} P(\gamma)$ of the second kind for $G = \mathbb{Z}_{2M}$ with only two factors are, besides $\gamma_{00}\gamma_{MM}$,

$$P(\gamma) = \gamma_{jj}\gamma_{M+j,M+j}\ 0 < j < M; \quad P(\gamma) = \gamma_{jM}\gamma_{M,M+j}\ 0 < j < M.$$

We only give an explicit proof for the first one. It is an invariant (Lemma 4.13) since

$$P(da) = (da)_{jj}(da)_{M+j,M+j} = \frac{a_j^2 a_{M+j}^2}{a_{2j}^2}.$$

It is an invariant of the second kind since we have for γ's without zeroes (A.1)

$$\begin{aligned} P(\gamma) &= \frac{\gamma_{1j}\gamma_{1,j+1}\cdots\gamma_{1,2j-1}}{\gamma_{11}\gamma_{12}\cdots\gamma_{1,j-1}} \cdot \frac{\gamma_{1,M+j}\gamma_{1,M+j+1}\cdots\gamma_{1,2j-1}}{\gamma_{11}\gamma_{12}\cdots\gamma_{1,M+j-1}} \\ &= \frac{\gamma_{1j}\gamma_{1,j+1}\cdots\gamma_{1,2j-1}}{\gamma_{11}\gamma_{12}\cdots\gamma_{1,j-1}} \cdot \frac{\gamma_{1,M+j}\gamma_{1,M+j+1}\cdots\gamma_{00}}{\gamma_{1,2j}\gamma_{1,2j+1}\cdots\gamma_{1,M+j-1}}, \end{aligned}$$

since $0 < j \leq 2j-1 < M+j-1 < M+j < 2M$, so that all elements of the natural basis occur with an odd power (namely one).

For such an invariant we can have both signs for $\mathbb{Z}_{2M}$-graded contractions with and without zeroes. Take *e.g.,* $G = \mathbb{Z}_4$ and $P(\gamma) = \gamma_{11}\gamma_{33} \neq 0$ and define

$$\gamma_{11} = +1, \quad \gamma_{33} = \pm 1$$

plus either $\gamma_{jk} = 0$ otherwise ($j, k \in \mathbb{Z}_4$) or $\gamma_{00} = \gamma_{0k} = \gamma_{12} = +1, \gamma_{13} = \gamma_{22}\gamma_{23} = \pm 1$.

Furthermore for γ's with zeroes we can have distinct signs for two different sign invariants of the second kind. Take *e.g.,* $G = \mathbb{Z}_8$ and $P_1(\gamma) = \gamma_{11}\gamma_{55}, P_2(\gamma) = \gamma_{33}\gamma_{77}$ and define (Remark 2.13) $\gamma_{11} = \gamma_{33} = \gamma_{55} = +1, \gamma_{77} = -1$ and $\gamma_{jk} = 0$ otherwise. Then $\operatorname{sgn} P_1(\gamma) = +1$ while $\operatorname{sgn} P_2(\gamma) = -1$.

From the structure of the sign invariants of the second kind with two factors we can guess how to construct those with more factors. We only give the following three examples with 4, resp., 6 factors.

$$P(\gamma) = \begin{cases} \gamma_{jk}\gamma_{j,M+k}\gamma_{M+j,k}\gamma_{M+j,M+k} & 0 < j < k < M, \\ \gamma_{jk}\gamma_{j,j+k}\gamma_{M+j,k}\gamma_{M+j,M+j+k} & 0 < j, k < M, \\ \gamma_{jk}\gamma_{jl}\gamma_{kl}\gamma_{M+j,M+k}\gamma_{M+j,M+l}\gamma_{M+k,M+l} & 0 < j < k < l < M. \end{cases}$$

Example 4.18 (Sign invariants of the first kind) (i) Consider the higher-order identity "$P_1 = P_2$" for $G = \mathbb{Z}_8$ where (Example 4.6 with $r = 4$) $P_1(\gamma) = \gamma_{17}\gamma_{22}\gamma_{35}\gamma_{66}$, $P_2(\gamma) = \gamma_{13}(\gamma_{26})^2\gamma_{57}$. Then (Remark 4.16(ii), (iii); Example 4.17)

$$\operatorname{sgn}[P_1(\gamma)P_2(\gamma)] = \operatorname{sgn}(\gamma_{13}\gamma_{17}\gamma_{35}\gamma_{57}) \cdot \operatorname{sgn}(\gamma_{22}\gamma_{66})$$

is a sign invariant of the first kind which is the product of two sign invariants of the second kind.

Similarly, the two higher-order identities "$P_1 = P_2$" for $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ (Examples 4.6 with $r = 3$, resp., $r = 4$) where $P_1(\gamma) = \gamma_{10,10}\gamma_{11,12}\gamma_{12,13}$, $P_2(\gamma) = \gamma_{10,11}\gamma_{10,13}\gamma_{12,12}$, resp., $P_1(\gamma) = \gamma_{10,10}\gamma_{11,11}\gamma_{12,12}\gamma_{13,13}$, $P_2(\gamma) = \gamma_{10,12}^2\gamma_{11,13}^2$, yield the sign invariants of the first kind, $\operatorname{sgn}(\gamma_{10,10}\gamma_{10,11}\gamma_{10,13}\gamma_{11,12}\gamma_{12,12}\gamma_{12,13})$, resp., $\operatorname{sgn}(\gamma_{10,10}\gamma_{11,11}\gamma_{12,12}\gamma_{13,13})$.

(ii) Both sign invariants given in Example 4.9 are of the first kind. The first one is always positive, no matter if γ has zeroes or not. The second one illustrates part (ii) and (iii) of Remark 4.16, since we have for $G = \mathbb{Z}_8$ (Example 4.17)

$$\operatorname{sgn}(\gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77}) = \operatorname{sgn}(\gamma_{11}\gamma_{55})\operatorname{sgn}(\gamma_{33}\gamma_{77}).$$

Note that in the case of "contradicting" signs for these two sign invariants of the second kind, *e.g.*, $\operatorname{sgn}(\gamma_{11}\gamma_{55}) = +1$, $\operatorname{sgn}(\gamma_{33}\gamma_{77}) = -1$, (Examples 4.17) we automatically get by multiplication a negative sign invariant of the first kind, namely $\operatorname{sgn}(\gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77}) = -1$.

5 Pseudobasis

In [13] we introduced the concept of a basis in order to unveil the structure of non-negative $\mathbb{Z}_N$-graded contractions. Since we encounter new features here, we introduce the notion of a pseudobasis. In Example 5.3(i) we present natural bases for all G-graded contractions γ.

The property of individual elements γ_{jk} which dominates this section is independence. Although this notion is defined in an unusual way we will see in Lemma 5.5 that our definition is not in disagreement with normal usage. Even though the defining equations are not linear, we can make use of several ideas and methods from linear algebra.

Algorithm A constructs all pseudobases. Algorithm B constructs all graded contractions without zeroes which agree on a given pseudobasis. Finally, Theorem 5.12 proves that real values for all elements of a pseudobasis yield a real γ without zeroes if and only if all sign invariants of the first kind which can be expressed in terms of the pseudobasis, are positive.

Definition 5.1 A set of matrix elements $\{\gamma_{s_i} \mid s_i \in G \times G, i = 1, 2, \ldots, r\}$ is called independent if, for every choice $\gamma_{s_i} = c_i$ with $0 \neq c_i \in \mathbb{C}$, there exists a complex G-graded contraction γ without zeroes whose elements γ_{s_i} have the assigned values (*cf.* [13, Definition IV.1]).

A maximal set of independent elements is called a *pseudobasis*. If the resulting complex G-graded contraction is unique, the pseudobasis is called a *basis*.

If a relation of the form

$$\gamma_{jk}^{n} = \prod_{i=1}^{r} \gamma_{s_i}^{ni}, \quad n = \prod_{i=1}^{r} n_i \in \mathbb{N},\ n_i \in \mathbb{Z}, \tag{5.1}$$

holds, we call the matrix elements $\gamma_{jk}(j,k \in G)$ *dependent* on the set $\{\gamma_{s_i} \mid i = 1,2,\dots,r\}$ (since $\{\gamma_{s_i} \mid i = 1,2,\dots,r\} \cup \{\gamma_{jk}\}$ is certainly not independent, even if $\{\gamma_{s_i}\}$ is).

Remark 5.2 Note that all dependence relations between elements must have the form of (5.1) since the combination of several defining equations which all have the form $\gamma_{\cdot}\gamma_{\cdot} = \gamma_{\cdot}\gamma_{\cdot}$ can only lead to expressions of the form $\gamma_{\cdot}\gamma_{\cdot}\cdots\gamma_{\cdot} = \gamma_{\cdot}\gamma_{\cdot}\cdots\gamma_{\cdot}$, with the same number of elements on both sides. Numerical factors cannot occur.

Example 5.3 (i) Natural basis for $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$

The proof of Theorem 3.1 (Case 3) shows that for arbitrary non-vanishing complex values of the $N = |G|$ elements listed in Lemma A.4, a unique complex G-graded contraction without zeroes exists (namely the specific *da* constructed there), thus verifying that these N elements constitute a basis, which we call natural.

This means that the N elements (Lemma A.1) $\{\gamma_{00}, \gamma_{11}, \gamma_{12}, \dots, \gamma_{1,N-1}\}$ constitute a basis for $\mathbb{Z}_N$-graded contractions, the $N_1 \cdot N_2$ elements (Lemma A.2)

$$\{\gamma_{00,00}; \gamma_{10,j0}; \gamma_{01,0k}; \gamma_{j0,0k} \mid j = 1,2,\dots,N_1-1; k = 1,2,\dots,N_2-1\}$$

one for $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$-graded contractions, *etc.*

(ii) For $\mathbb{Z}_5$-graded contractions γ, $\{\gamma_{00}, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{44}\}$ is a pseudobasis which is not a basis. The dependent elements follow from the defining equations as

$$\gamma_{13} = \frac{\gamma_{11}\gamma_{22}}{\gamma_{12}} \quad \gamma_{14} = \frac{\gamma_{12}^2\gamma_{33}}{\gamma_{00}\gamma_{22}} \quad \gamma_{23} = \frac{\gamma_{12}\gamma_{33}}{\gamma_{00}}$$

$$\gamma_{24} = \frac{\gamma_{12}^2\gamma_{33}}{\gamma_{11}\gamma_{22}} \quad \gamma_{34} = \frac{\gamma_{12}\gamma_{33}}{\gamma_{22}}, \quad \text{where } \gamma_{12}^3 = \frac{\gamma_{11}\gamma_{22}^2\gamma_{44}}{\gamma_{33}}.$$

Conversely, these six equations imply all defining equations. It is therefore clear that for arbitrary non-vanishing complex values of the elements of the pseudobasis, three complex $\mathbb{Z}_5$-graded contractions without zeroes exist.

Expressing all dependent elements in terms of the pseudobasis yields

$$\gamma_{12}^3 = \frac{\gamma_{11}\gamma_{22}^2\gamma_{44}}{\gamma_{33}}, \qquad \gamma_{13}^3 = \frac{\gamma_{11}^2\gamma_{22}\gamma_{33}}{\gamma_{44}},$$

$$\gamma_{14}^3 = \frac{\gamma_{11}^2\gamma_{22}\gamma_{33}\gamma_{44}^2}{\gamma_{00}^3}, \qquad \gamma_{23}^3 = \frac{\gamma_{11}\gamma_{22}^2\gamma_{33}^2\gamma_{44}}{\gamma_{00}^3},$$

$$\gamma_{24}^3 = \frac{\gamma_{22}\gamma_{33}\gamma_{44}^2}{\gamma_{11}}, \qquad \gamma_{34}^3 = \frac{\gamma_{11}\gamma_{33}^2\gamma_{44}}{\gamma_{22}}.$$

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Note that we cannot read off directly from these equations how many different γ's agree on the pseudobasis in question since, as we see above, the different roots cannot be chosen independently of each other.

Warning 5.4 If for a set of matrix elements $\{\gamma_{s_i} \mid i = 1, 2, \dots, r\}$ and arbitrary $0 \neq c_i \in \mathbb{C}$, a complex γ with zeroes exists where $\gamma_{s_i} = c_i$, the set $\{\gamma_{s_i}\}$ need not be independent since a complex γ without zeroes need not exist (Remark 5.6, Example 5.7).

Lemma 5.5 *Given a set of r matrix elements $\{\gamma_{s_i} \mid s_i \in G \times G, i = 1, 2, \dots, r\}$, the following statements are equivalent.*

(i) *The $\{\gamma_{s_i}\}$ are independent.*

(ii) *There exists no non-trivial dependence relation*

$$\prod_{i=1}^{r} \gamma_{s_i}^{n_i} = 1, n_i \in \mathbb{Z}.$$

"Non-trivial" means some $n_i \neq 0$.

(iii) *The ansatz*

$$\gamma_{j_i k_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}}, \quad i = 1, 2, \dots, r,$$

where $\gamma_{s_i} = \gamma_{j_i k_i}$ ($j_i, k_i \in G$) determines r different a_j's (not necessarily uniquely).

Proof (i) $\Rightarrow$ (ii) is trivial (Definition 5.1).

(ii) $\Rightarrow$ (iii) (Example 5.9 illustrates the following procedure for a maximal set of independent elements, *i.e.,* for a pseudobasis.) The general idea is straightforward. Solve one of the equations in the ansatz for some a_j, and replace this a_j in all remaining equations by the resulting expression. Continue this procedure. At each step it must be possible to solve for some remaining a_j since otherwise γ_{s_i} would be expressed in terms of the other γ_{s_k} which would contradict (ii). The only difficulty is that this procedure can yield expressions for some integer power of the a_j, and we must be careful to handle the resulting non-uniqueness of the a_j in a self-consistent way.

We therefore use the following procedure. Namely, when solving for some a_j, we solve for $a_j^{\nu_j}$ where $\nu_j \in \mathbb{N}$ is the least possible power. Since we can multiply and divide the given equations, ν_j is the generator of the ideal generated by the powers of a_j which occur in the given equations. It follows that all remaining equations can only contain a_j in the form $a_j^{s\nu_j}, s \in \mathbb{Z}$. Hence we can just substitute our expression for $a_j^{\nu_j}$ in all remaining equations. Finally, we solve for the a_j backwards. Namely, the last $a_{j_r}^{\nu_{j_r}}$ will be expressed in terms of the $(|G| - r)$ remaining a_k's plus the $\{\gamma_{s_i} \mid i = 1, 2, \dots, r\}$, and we can choose an arbitrary root. Continuing in this way produces the desired result.

(iii) $\Rightarrow$ (i) Set all $(|G| - r)$ remaining a_k's equal to 1. Then, for an arbitrary choice $0 \neq \gamma_{s_i} \in \mathbb{C} (i = 1, 2, \dots, r)$, the resulting G-graded contraction $\gamma = da$ without zeroes takes on the given values on the γ_{s_i}. ∎

Remark 5.6 Let "$P_1 = P_2$" be a higher-order identity for G-graded contractions. Then we have (Remark 4.2(i)) $P_1(\gamma) = P_2(\gamma)$ for all G-graded contractions γ without zeroes. This means that each higher-order identity represents a dependence relation which exists automatically for all γ's without zeroes, but not necessarily for those with zeroes.

Example 5.7 Consider the $\mathbb{Z}_6$-graded contraction γ where $\gamma_{11}, \gamma_{13}, \gamma_{15}, \gamma_{33}, \gamma_{35}, \gamma_{55}$ are nonzero and $\gamma_{jk} = 0$ otherwise. Although these six non-vanishing elements can take on arbitrary values (Examples 4.6 with $r = 3$), only five of them are independent due to the higher-order identity "$P_1 = P_2$" where $P_1(\gamma) = \gamma_{11}\gamma_{33}\gamma_{55}$ and $P_2(\gamma) = \gamma_{13}\gamma_{15}\gamma_{35}$.

In [13] we dealt mostly with non-negative graded contractions. In this case there is no difference between a basis and a pseudobasis since for positive values of a pseudobasis there is a unique positive solution. In the following we collect all relevant results on pseudobases (Lemma 5.8 and Algorithms A and B).

Lemma 5.8 *A pseudobasis for G-graded contractions has $N = |G|$ elements.*

Proof Let $\{\gamma_{s_i} \mid s_i \in G \times G; i = 1, 2, \dots, M\}$ be a pseudobasis [13, Corollary IV.1]. Since the set $\{\gamma_{s_i}\}$ is independent, we can determine (Lemma 5.5) from the ansatz

$$\gamma_{s_i} = \gamma_{j_i k_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}} \quad j_i, k_i \in G, i = 1, 2, \dots, M;$$

M different a_j's (resp., some natural power of these a_j's), so that $M \le N$.

Since $\{\gamma_{s_i}\}$ is a maximal set of independent elements, all other elements must be dependent, *i.e.,* follow (not necessarily uniquely) from this set. Therefore we must be able to determine all a_j's from the ansatz above which implies $M \ge N$.

Altogether we have $M = N$. (An alternate proof uses a replacement principle identical in spirit to the one used for different sets of independent vectors spanning a vector space.) ■

The following example shows that a pseudobasis does not define all a_j's uniquely, a phenomenon we already encountered in part (ii) $\Longrightarrow$ (iii) of the proof of Lemma 5.5.

Example 5.9 Consider for $G = \mathbb{Z}_5$ the pseudobasis $\{\gamma_{00}, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{44}\}$ of Example 5.3(ii). The ansatz

$$\gamma_{00} = a_0 \quad \gamma_{11} = \frac{a_1^2}{a_2} \quad \gamma_{22} = \frac{a_2^2}{a_4} \quad \gamma_{33} = \frac{a_3^2}{a_1} \quad \gamma_{44} = \frac{a_4^2}{a_3}$$

yields

$$a_0 = \gamma_{00} \quad a_1 = \frac{a_3^2}{\gamma_{33}} \quad a_2 = \frac{a_3^4}{\gamma_{11}\gamma_{33}^2} \quad a_4 = \frac{a_3^8}{\gamma_{11}^2\gamma_{22}\gamma_{33}^4}$$

where $a_3^{15} = \gamma_{11}^4\gamma_{22}^2\gamma_{33}^8\gamma_{44}$, so that we get $3 \cdot 5$ solutions. This corresponds to three different γ's which agree on this pseudobasis (due to the 1-to-5 correspondence between one γ and its a_j's (proof of Theorem 3.1, (3.7)). They differ, *e.g.,* on the dependent element γ_{12} which can take on either root of

$$\gamma_{12}^3 = \frac{\gamma_{11}\gamma_{22}^2\gamma_{44}}{\gamma_{33}}.$$

5.1 Algorithms

All pseudobases for G-graded contractions are constructed from the following algorithm [13, Remark IV.1].

Algorithm A Start with the set of all matrix elements γ_{jk}, $j, k \in G$, and the set of all defining equations (2.3). After replacing $\gamma_{0k}(0 \neq k \in G)$ by γ_{00} (3.3) and taking into account $\gamma_{jk} = \gamma_{kj}$, there are L_N elements

$$\{\gamma_{t_r} \mid t_r \in G \times G, \quad r = 1, 2, \dots, L_N\}$$

where $L_N = \frac{N(N-1)}{2} + 1$, $N = |G|$.

Use any one of the defining equations. If it has become a trivial identity, go on to another one. Otherwise choose any one element in it and express it by the others. Now this element has become dependent and gets replaced by this expression in all remaining equations. Consider another equation and continue with the same procedure.

At the end, all defining equations have been considered, so that no more dependence relations exist. This means that the remaining elements are independent (Lemma 5.5) and constitute a pseudobasis $\{\gamma_{s_i} \mid i = 1, 2, \dots, N\}$.

Since we are free to go through the defining equations in any order and since we are free to choose any element in an equation under consideration as the independent one, we must get all pseudobases in this way.

Note that a set of r independent elements $\{\gamma_{s_i} \mid i = 1, 2, \dots, r < N\}$ can always be completed to a pseudobasis. (Just do not pick these r elements as dependent ones in this algorithm.)

Assume $\{\gamma_{s_i} \mid i = 1, 2, \dots, N = |G|\}$ is a pseudobasis. Algorithm B below produces all complex γ's without zeroes which take on fixed, but arbitrarily chosen values, *e.g.,*

$$\gamma_{s_i} = c_i \neq 0, \quad c_i \in \mathbb{C}, \ i = 1, 2, \dots, N,$$

on this pseudobasis.

Preliminary remarks What information does Algorithm A provide us with? Let γ_{t_r} denote the element which is chosen as the dependent one at the r-th step, $r = 1, 2, \dots, L_N - N$. The expression of dependence we get for γ_{t_r} can always be brought into the form

$$\gamma_{t_r}^{n_r} = \prod_{l=r+1}^{L_N - N} \gamma_{t_l}^{m_{lr}} \prod_{i=1}^{N} \gamma_{s_i}^{n_{ir}} \tag{5.2}$$

where

$$n_r = \sum_{l=r+1}^{L_N-N} m_{lr} + \sum_{i=1}^{N} n_{ir} \in \mathbb{N}, \quad m_{lr}, n_{ir} \in \mathbb{Z}.$$

But the powers n_r, m_{lr}, n_{ir} are not unique since the elements on the right-hand side of (5.2) are not independent. The only exception is the "last" dependent element γ_{t_r} with $r = L_N - N$.

The collection of dependence relations (5.2) allows us to express each dependent element $\gamma_t, t \in G \times G$ by the pseudobasis alone. We get

$$\gamma_t^n = \prod_{i=1}^{N} \gamma_{s_i}^{n_i} \tag{5.3}$$

where

$$n = \sum_{i=1}^{N} n_i \in \mathbb{N}, \quad n_i \in \mathbb{Z}.$$

(In the case of a basis, we have $n = 1$ for all γ_t.) In contrast to (5.2), the powers n and n_i in (5.3) are unique for every γ_t (up to trivial natural multiples $n' = mn, n'_i = mn_i, 0 \neq m \in \mathbb{N}$, since a non-trivial relation between these independent elements cannot hold. But (5.3) does not contain enough information to allow us to read off all possible γ's that agree on this pseudobasis, since the different roots in (5.3) cannot be chosen independently for different elements (Example 5.3(ii)). We solve this problem by running Algorithm A backwards in a more refined way, as follows.

Algorithm B Assume we know the values of the pseudobasis $\{\gamma_{s_i} \mid i = 1, 2, \dots, N\}$. Assume further we have already assigned values to the dependent elements γ_{t_l} for $l = r+1, r+2, \dots, L_N - N$. Which values are allowed for $\gamma_{t_r} (r \in \{1, 2, \dots, L_N - N\})$? We look at all valid relations between this "new" dependent element and all those elements we already know, *i.e.,* at all valid relations of the type

$$\gamma_{t_r}^{n_r} = \prod_{l=r+1}^{L_N-N} \gamma_{t_l}^{m_{lr}} \prod_{i=1}^{N} \gamma_{s_i}^{n_{ir}} \tag{5.4}$$

where

$$n_r = \sum_{l=r+1}^{L_N-N} m_{lr} + \sum_{i=1}^{N} n_{ir} \in \mathbb{N}, \quad m_{lr}, n_{ir} \in \mathbb{Z}.$$

Let μ_r be the minimal power of γ_{t_r} for which such a relation exists, *i.e.,*

$$\mu_r = \min n_r \geq 1, \quad \mu_r \in \mathbb{N}. \tag{5.5}$$

Then we must have $\frac{n_r}{\mu_r} \in \mathbb{N}$ for all powers n_r which occur in (5.4), since otherwise μ_r would not be the smallest power possible.

Can we get contradictory results? The answer is "no" as can be seen in the following way. Assume first that two different relations exist for the same power n_r *i.e.*,

$$\gamma_{t_r}^{n_r} = \prod_{l=r+1}^{L_N-N} \gamma_{t_l}^{m_{lr}} \prod_{i=1}^{N} \gamma_{s_i}^{n_{ir}} = \prod_{l=r+1}^{L_N-N} \gamma_{t_l}^{m'_{lr}} \prod_{i=1}^{N} \gamma_{s_i}^{n'_{ir}}$$

where we do not have $m_{lr} = m'_{lr}$, for all l.

Therefore this identity represents a valid relation of the type of (5.4), but for one of the dependent elements already determined in an earlier step. This means that this relation has already been taken into account, so that the identity above is a trivial one.

Assume next that a relation exists for $\gamma_{t_r}^{\nu\mu_r}$, where $\nu \in \mathbb{N}\backslash\{1\}$. In the same way as above we argue that it has to agree with the relation for $\gamma_{t_r}^{\mu_r}$ taken to the ν-th power.

Therefore the knowledge of the values of the pseudobasis plus the values of γ_{t_l} for all $l = r+1, r+2, \dots, L_N - N$ provides us with a unique value for $\gamma_{t_r}^{\mu_r}$, so that we are free to choose any of the μ_r possible roots for γ_{t_r}.

Continuing in this way we construct one possible γ out of the total number of

$$\prod_{r=1}^{L_N-N} \mu_r \tag{5.6}$$

different γ's which agree on this pseudobasis.

A pseudobasis is a basis if and only if $\mu_r = 1$ for all $r = 1, 2, \dots, L_N - N$, which implies $n = 1$ in (5.3) for all dependent elements.

Alternative Method An alternative way to find all γ's which agree on a given pseudobasis $\{\gamma_{s_i} = \gamma_{j_i k_i} \mid i = 1, 2, \dots, N = |G|\}$ is based on Lemma 5.5 and is illustrated by Example 5.9. According to part (ii) $\Longrightarrow$ (iii) of the proof of Lemma 5.5 the ansatz

$$\gamma_{j_i k_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i+k_i}}, \quad i = 1, 2, \dots, N$$

for a maximal set of independent elements yields unique expressions for $a_j^{\nu_j}$ ($j \in G$, $\nu_j \in \mathbb{N}$), where we are free to take any of the ν_j roots. Therefore we get $\prod_{j\in G} \nu_j$ different solution sets for the a_j, which corresponds (due to the one-to-N correspondence between a γ and its a_j's) to $\frac{1}{N}\prod_{j\in G} \nu_j$ different γ's.

Remark 5.10 Algorithm B obviously constitutes an explicit proof that the set $\{\gamma_{s_i}\}$ which is produced by Algorithm A is indeed a pseudobasis (without referring to Lemma 5.5).

Finally we turn our attention to real G-graded contractions without zeroes. If we choose real values for a given basis, this uniquely defines a real contraction. For a pseudobasis, the situation is much more involved. Lemma 3.3 yields all admissible sign distributions over the elements of a pseudobasis. It follows that if we choose real values for a given pseudobasis, there may be no real γ taking on these values. Furthermore, if real γ do exist, they could belong to different equivalence classes. The following example illustrates these two features.

Example 5.11 Consider $G = \mathbb{Z}_6$ and the pseudobasis $\{\gamma_{00}, \gamma_{11}, \gamma_{22}, \gamma_{34}, \gamma_{35}, \gamma_{45}\}$. The two defining equations $\gamma_{12}\gamma_{35} = \gamma_{13}\gamma_{45}$ and $\gamma_{12}\gamma_{13} = \gamma_{11}\gamma_{22}$ yield

$$\gamma_{13}^2 = \frac{\gamma_{11}\gamma_{22}\gamma_{35}}{\gamma_{45}}.$$

Therefore a real γ_{13} exists if and only if $\operatorname{sgn}(\gamma_{11}\gamma_{22}\gamma_{35}\gamma_{45}) = +1$. (Note that this is a sign invariant of the first kind.) In this case a real γ exists since all elements of the natural basis of $\mathbb{Z}_6$ follow uniquely from the elements of the pseudobasis plus γ_{13}, namely

$$\gamma_{12} = \gamma_{13}\frac{\gamma_{45}}{\gamma_{35}}, \quad \gamma_{14} = \frac{\gamma_{11}\gamma_{34}}{\gamma_{35}}, \quad \gamma_{15} = \gamma_{13}\frac{\gamma_{45}}{\gamma_{00}}.$$

But the equivalence class of γ (Lemma 3.4) is not defined, since we cannot form any sign invariant of the second kind from the elements of the pseudobasis. In fact, the equivalence class of γ is given, *e.g.,* by the sign invariant of the second kind $\operatorname{sgn}(\gamma_{13}\gamma_{34})$ (*cf.* Lemma 4.14, Example 4.17), which depends on the root we take for γ_{13}. (For a general G, a pseudobasis will, in general, yield an incomplete determination of the equivalence class, depending on which sign invariants of the second kind can be constructed from the pseudobasis.)

Theorem 5.12 *Given a pseudobasis $\{\gamma_{s_i} \mid i = 1, 2 \dots, N = |G|\}$ for G-graded contractions and arbitrary real number $0 \neq c_i \in \mathbb{R}$:*

(i) *A real G-graded contraction γ without zeroes where $\gamma_{s_i} = c_i$ exists if and only if all sign invariants of the first kind which can be formed by the elements of the pseudobasis are positive.*

(ii) *Assume a real γ as in* (i) *exists. Then there exists such a γ in each equivalence class which is compatible with the values of all sign invariants of the second kind which can be formed by the elements of the pseudobasis. In particular, if all these sign invariants are positive, there exists such a γ with $\gamma \sim \mathbf{1}$.*

Proof (i) Condition (i) is clearly necessary (Lemma 4.11). To prove sufficiency, we will use Algorithm B to show that when the condition in (i) holds, a real γ with $\gamma_{s_i} = c_i$ exists. Since a positive $|\gamma|$ with $|\gamma|_{s_i} = |c_i|$ always exists, by Lemma 3.2 we only have to care about $\operatorname{sgn}\gamma$.

For each dependent element γ_t, Algorithm B yields a $\gamma_t^{\mu_t}$ with minimal power μ_t (5.4), (5.5). Whenever μ_t is odd, $\operatorname{sgn}\gamma_t$ follows uniquely from $\{\operatorname{sgn}\gamma_{s_i}\}$ plus the signs of all dependent elements which have been determined prior to γ_t. Therefore all these γ_t can be replaced by $\{\gamma_{s_i}\}$ and by those of the dependent elements which have already been determined with an even power.

Let $\gamma_{t_k}, k = 1, 2, \dots, K$ be the K dependent elements where the algorithm yields $\gamma_{t_k}^{\mu_k}$ with μ_k even. Then we have, if the algorithm produces first γ_{t_1}, then γ_{t_2}, *etc.,*

$$\gamma_{t_k}^{\mu_k} = \prod_{l=1}^{k-1} \gamma_{t_l}^{m_{lk}} \prod_{i=1}^{N} \gamma_{s_i}^{n_{ik}}, \quad m_{lk}, n_{ik} \in \mathbb{Z}. \tag{5.7}$$

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Case 1: If $K = 0$, the algorithm yields a unique real γ, which means that our pseudobasis behaves, with respect to real γ's, like a basis.

Case 2: If $K \geq 1$, it follows from (5.7) that γ_{t_k} can be chosen real if and only if

$$\text{(5.8)} \qquad \operatorname{sgn} \gamma_{t_k}^2 = \operatorname{sgn}\Big[\prod_{l=1}^{k-1} \gamma_{t_l}^{\delta_{lk}} \prod_{i=1}^{N} \gamma_{s_i}^{\varepsilon_{ik}}\Big] = +1, \quad k = 1, 2, \dots, K,$$

where

$$\delta_{lk} = \begin{cases} 1 & \text{if } m_{lk} \text{ is odd,} \\ 0 & \text{if } m_{lk} \text{ id even,} \end{cases} \quad \text{and} \quad \varepsilon_{ik} = \begin{cases} 1 & \text{if } n_{ik} \text{ is odd,} \\ 0 & \text{if } n_{ik} \text{ is even.} \end{cases}$$

We now show that there exist real γ_{t_k} satisfying (5.8).

Consider first (5.8) for $k = 1$. Then only the γ_{s_i} occur, and this equation is automatically satisfied, since otherwise it would be a negative sign invariant of the first kind which can be formed from the elements of the pseudobasis. Consider next (5.8) for $k = 2$. If $\delta_{12} = 1$, we solve for $\operatorname{sgn} \gamma_{t_1}$ and substitute this result in all remaining equations. If $\delta_{12} = 0$, then only the γ_{s_i} occur, and as above, this equation is satisfied by assumption. We continue in this way: (5.8) for $\operatorname{sgn} \gamma_{t_k}^2$ $(k = 2, 3, \dots, K)$ can either be solved for some $\operatorname{sgn} \gamma_{t_l}$, $l = 1, 2, \dots, k-1$ or it contains only the γ_{s_i} and is therefore satisfied.

At the end, we have solved for maximally $K-1$ of the $\operatorname{sgn} \gamma_{t_k}$. Thus $\lambda = 1, 2, \dots, K$ of them are free and can take on both values.

Taking Case 1 and Case 2 together we get therefore 2^λ, $\lambda = 0, 1, 2, \dots, K$ different real γ's which agree on our pseudobasis.

(ii) Lemma 4.15 exhibits clearly the structure of the sign invariants of the second kind. There are 2^{r_e} different equivalence classes resulting from the r_e obviously independent sign invariants of the second kind, one for each of the r_e subgroups $\mathbb{Z}_{N_l}$ of G where N_l is even (Example 6.15). Furthermore, it is easy to see from this structure that from any collection of sign invariants of the second kind one can extract a set of independent sign invariants which contain all the information contained in the original set.

Now, in the present situation it can happen that some of the sign invariants of the second kind can be expressed in terms of our given pseudobasis as

$$\text{(5.9)} \qquad \operatorname{sgn} P(\gamma) = \operatorname{sgn} \prod_{i=1}^{N} \gamma_{s_i}^{\delta_i}; \delta_i \in \{0, 1\}.$$

Hence the values $\{\operatorname{sgn} \gamma_{s_i} = \operatorname{sgn} c_i\}$ partially determine the equivalence class. We will now show that the construction in part (i) yields a γ in every equivalence class which is compatible with these given values.

We start by defining the number ϱ of independent sign invariants of the second kind not given by the pseudobasis. Namely, we define ϱ to be the minimal number of sign invariants $\operatorname{sgn} P_l(\gamma)$, $l = 1, 2, \dots, \varrho$ with the property that every sign invariant of the second kind can be expressed in terms of these $\varrho P_l(\gamma)$ plus the sign invariants

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of the form of (5.9). This means that there are 2^{ϱ} equivalence classes compatible with $\gamma_{s_i} = c_i$.

Case 1: $\varrho = 0$. In this case all sign invariants of the second kind can be expressed as in Eq. (5.9). Then the $\{\operatorname{sgn} c_i\}$ define a unique equivalence class, and hence all γ constructed in part (i) must lie in this equivalence class. This is necessarily the case when $r_e = 0$. It also happens when $K = 0$.

Case 2: $\varrho = 1$. (as in Example 5.11). Then there is one sign invariant sgn $P(\gamma)$ of the second kind which is not of the form of (5.9), and there are precisely two equivalence classes which are compatible with $\gamma_{s_i} = c_i$. Now $P(\gamma)$ must contain at least one element which, in the construction in part (i), can take on both signs, since otherwise $P(\gamma)$ would be expressed in terms of the pseudobasis. But then our construction yields solutions γ with both signs for $P(\gamma)$.

Case 3: $\varrho = 2$. Now, two independent sign invariants sgn $P_1(\gamma)$ and sgn $P_2(\gamma)$ of the second kind are not of the form of (5.9). In this case there must be elements γ_{t_l} contained in $P_l(l = 1, 2)$ which can not only take on both signs, but can do this independently, since otherwise we would have $\varrho = 1$.

Case 4: $\varrho \geq 3$. Just use the obvious extension of the argument for $\varrho = 2$. ■

6 *G*-Graded Contractions with Zeroes

In this section we study G-graded contractions γ with zeroes. The support $S(\gamma)$ and the strong violations of higher-order identities play a deciding role. For real γ's, there are also the sign invariants, and negative sign invariants of the first kind are now possible.

In Definition 6.3 we introduce, for a given support $S \in \mathcal{S}(G)$, two integers $N'(S) \leq N''(S)$. $N'(S) < N = |G|$ is the maximal number of independent elements in S, whereas N'' is the maximal number of "quasi-independent" elements which seem to be independent if we only consider all *surviving* defining equations. Theorem 6.5 proves that the difference $Q = N'' - N' \geq 0$ is the number of higher-order identities which can be violated strongly in an arbitrary way by γ when γ is either complex or non-negative. (For sgn γ the strong violation of a higher-order identity simply means the existence of a negative sign invariant of the first kind (Remark 4.16(iii)).

In the real case, N'' also determines the number of different sgn γ, which is $2^{N''}$ (Lemma 6.10) in contrast to 2^N when γ has no zeroes (Lemma 3.3).

If there are no strong violations, we have in the complex (resp., non-negative) case (Theorem 6.7) $\gamma \sim \pi(\gamma)$ which corresponds to $\gamma \sim \mathbf{1}$ for a complex (resp., positive) γ without zeroes (Theorem 3.1 (resp., Lemma 3.2)). In the real case, if in addition there are no negative sign invariants of the first kind, we have $\gamma = \Gamma \cdot \pi(\gamma)$ where Γ is a real G-graded contraction *without* zeroes (Theorem 6.11).

Finally, we introduce the notion of independent sign invariants (Definition 6.13, Lemma 6.16). The $J(S)$ independent sign invariants which exist for a given support S (Lemma 6.14) can be constructed from the Q higher-order identities and the J' independent sign invariants for the N' independent elements alone so that $J = Q + J'$ (Lemma 6.17, Algorithm 6.18).

Remark 6.1 For a γ with zeroes, (3.2) yields, instead of (3.3),

$$\gamma_{0k} = \gamma_{00} \quad \text{or} \quad \gamma_{0k} = 0, \quad 0 \neq k \in G. \tag{6.1}$$

In the case $\gamma_{0j} = \gamma_{0k} = \gamma_{0,j+k}$ (3.1) is trivially satisfied, otherwise it yields $\gamma_{jk} = 0$.

Remark 6.2 All possible supports $S \in \mathcal{S}(G)$ can be determined easily as follows. Take an arbitrary subset S_0 of elements of γ, and declare them to be non-zero. The defining equations then determine uniquely the smallest possible support S containing S_0. (If both elements on one side of a defining equation are non-zero, then both elements on the other side must also be non-zero.) There are, for example, 5 different supports for $G = \mathbb{Z}_2$, 15 for $\mathbb{Z}_3$, 47 for $\mathbb{Z}_4$, and 41 for $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 6.3 Let $S \in \mathcal{S}(G)$. We say that elements in S are *quasi-independent* if there exists no non-trivial dependence relation coming from the surviving defining equations. We define $N''(S)$ to be the maximal number of quasi-independent elements in S, and $N'(S)$ to be the maximal number of independent elements in S.

Remark 6.4 To determine N'', use all surviving defining equations to remove as many dependent elements as possible. Then N'' is the number of elements which are left. Note that $N'' < N = |G|$, $N'' = N$, and $N'' > N$ are all possible (Examples 5.7 and 6.6). Although these N'' elements can obviously take on arbitrary (non-vanishing) values, they are in general not independent (Warning 5.4) because there can be surviving higher-order identities (Remark 5.6). Hence $N' \leq N''$. To determine N' we use Lemma 5.5 (see the proof of Theorem 6.5 below). We must have $N' < N$ (otherwise there could not be any zeroes).

The uniqueness of N' and N'' follows, *e.g.,* from a replacement principle (alternate proof of Lemma 5.8).

Theorem 6.5 *Let $S \in \mathcal{S}(G)$. There exist $Q = N''(S) - N'(S) \geq 0$ surviving higher-order identities "$P_1^{(l)} = P_2^{(l)}$" for $l = 1, 2, \dots, Q$ with the following property: for any non-zero complex (resp., positive) numbers α_l, $l = 1, 2, \dots, Q$, there exists a complex (resp., non-negative) γ with support S such that*

$$P_1^{(l)}(\gamma) = \alpha_l P_2^{(l)}(\gamma), \quad l = 1, 2, \dots, Q.$$

Furthermore, all strong violations of γ are direct consequences of these Q ones.

Proof Let S be given and therefore N'' and N' (Definition 6.3). Consider a complex (resp., non-negative) γ with support S and let $\{\gamma_{j_i k_i} \mid j_i, k_i \in G, i = 1, 2, \dots, N''\} \subset S$ be N'' quasi-independent elements. By renumbering if necessary, we let $\{\gamma_{j_i k_i} \mid i = 1, 2, \dots, N' \leq N''\}$ be a subset of N' independent elements. This means that the ansatz (Lemma 5.5)

$$\gamma_{j_i k_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}}, \quad i = 1, 2, \dots, N' \tag{6.2}$$

allows us to solve for N' of the a_j's.

If $Q = 0$ there is nothing to prove, so we assume $N' < N''$. Consider any $\gamma_{j_{N'+l}k_{N'+l}}$ and the ansatz

$$\gamma_{j_{N'+l}k_{N'+l}} = \frac{a_{j_{N'+l}}a_{k_{N'+l}}}{a_{j_{N'+l}+k_{N'+l}}}, \quad 0 < l \leq Q = N'' - N'. \tag{6.3}$$

We replace the a_j's on the right-hand side, whenever possible, by the expressions obtained from (6.2), if necessary by taking the smallest appropriate power of our ansatz. Then the a_j's must completely drop out since otherwise we could solve for another one so that our set would contain more than N' independent elements. Instead we are left with a dependence relation for $\gamma_{j_{N'+l}k_{N'+l}}$ (resp., some smallest possible power of it) in terms of $\{\gamma_{j_ik_i} \mid i = 1, 2, \ldots, N'\}$. By assumption, this dependence relation does not follow from the surviving defining equations and consequently does not have to be satisfied by γ. Therefore, it can be transformed into the standard form of a higher-order identity "$P_1^{(l)} = P_2^{(l)}$" which will of course be satisfied by our ansatz *da*.

Since we can choose the values of all $\gamma_{j_ik_i}$, $i = 1, 2, \ldots, N'$ plus the values of all $\gamma_{j_{N'+l}k_{N'+l}}$, $l = 1, 2, \ldots, Q$ arbitrarily, we can define a complex (resp., non-negative) γ for which

$$P_1^{(l)}(\gamma) = \alpha_l P_2^{(l)}(\gamma), \quad l = 1, 2, \ldots, Q, \tag{6.4}$$

whatever the chosen complex (resp., positive) numbers $\alpha_l \neq 0$ are.

These Q higher-order identities can of course be combined to produce "new" ones (Remark 4.5(i)). That γ cannot violate strongly any higher-order identity which is not a consequence of (6.4) can be seen in the following way. Assume γ as defined in (6.4) violates strongly another higher-order identity "$P_1 = P_2$", *i.e.,*

$$\frac{P_1(\gamma)}{P_2(\gamma)} = \alpha \neq 0, \quad 1 \neq \alpha \in \mathbb{C} \quad (\text{resp., } \mathbb{R}^+).$$

First we express all elements in this relation by our N'' elements $\{\gamma_{j_ik_i} \mid i = 1, 2, \ldots, N''\}$. Then we replace all elements $\{\gamma_{j_ik_i} \mid i = N' + 1, \ldots, N''\}$ by their expressions stemming from (6.3)). This leaves us with a dependence relation for our independent elements $\{\gamma_{j_ik_i} \mid i = 1, 2, \ldots, N'\}$ alone, which has to be a trivial identity. Therefore all numerical factors have to drop out, *i.e.,* there must exist $n \in \mathbb{N}$ and $n_l \in \mathbb{Z}$ so that $\alpha^n = \prod_{l=1}^{Q} \alpha_l^{n_l}$. This means

$$\Big[\frac{P_1(\gamma)}{P_2(\gamma)}\Big]^n = \prod_{l=1}^{Q} \Big[\frac{P_1^{(l)}(\gamma)}{P_2^{(l)}(\gamma)}\Big]^{n_l}, \tag{6.5}$$

so that $\big[\frac{P_1(\gamma)}{P_2(\gamma)}\big]^n$ is a direct consequence of the Q higher-order identities above.

If we have $n = 1$ in (6.5), the proof is complete. Therefore we assume $n > 1$ and $\alpha_l = 1, l = 1, 2, \ldots, Q$ in the following, so that (6.5) reads $P_1(\gamma) = \sqrt[n]{1} P_2(\gamma)$. This is obviously a dependence relation between some non-vanishing elements of γ. Since we know that N' of these elements are independent while the rest depend on them either through surviving defining equations or through the ansatz *da* (6.3), we must have $\sqrt[n]{1} = 1$ (Remark 5.2). ■

Example 6.6 Consider all $\mathbb{Z}_8$-graded contractions γ, with $\gamma_{jk} \neq 0$ if j and k are odd and $\gamma_{jk} = 0$, otherwise. Since all 10 non-vanishing elements are pairwise incompatible ,there are no surviving defining equations so that $N'' = 10$. Since γ has zeroes we know that $N' \leq 7$. The ansatz

$$\gamma_{17} = \frac{a_1 a_7}{a_0} \quad \gamma_{33} = \frac{a_3^2}{a_6} \quad \gamma_{35} = \frac{a_3 a_5}{a_0}$$
$$\gamma_{37} = \frac{a_3 a_7}{a_2} \quad \gamma_{55} = \frac{a_5^2}{a_2} \quad \gamma_{57} = \frac{a_5 a_7}{a_4} \quad \gamma_{77} = \frac{a_7^2}{a_6}$$

allows us to solve for 7 of the 8 a_j's (as functions of a_3), namely

$$a_6 = \frac{a_3^2}{\gamma_{33}} \quad a_5^4 = a_3^4 \frac{\gamma_{55}^2 \gamma_{77}}{\gamma_{33}\gamma_{37}^2} \quad a_7 = \frac{a_5^2}{a_3}\frac{\gamma_{37}}{\gamma_{55}} \quad a_1 = \frac{a_3 a_5}{a_7}\frac{\gamma_{17}}{\gamma_{35}}$$
$$a_2 = \frac{a_3 a_7}{\gamma_{37}} \quad a_4 = \frac{a_5 a_7}{\gamma_{57}} \quad a_0 = \frac{a_1 a_7}{\gamma_{17}}.$$

Therefore, $N' = 7$.

For the remaining three elements the corresponding ansatz yields the following dependence relations

$$\gamma_{11} = \frac{a_1^2}{a_2} = \frac{\gamma_{17}^2\gamma_{33}\gamma_{55}}{\gamma_{35}^2\gamma_{77}} \quad \gamma_{13} = \frac{a_1 a_3}{a_4} = \frac{\gamma_{17}\gamma_{33}\gamma_{57}}{\gamma_{35}\gamma_{77}} \quad \gamma_{15} = \frac{a_1 a_5}{a_6} = \frac{\gamma_{17}\gamma_{33}\gamma_{55}}{\gamma_{35}\gamma_{37}}.$$

Therefore, the following $Q = N'' - N' = 3$ surviving higher-order identities "$P_1^{(l)} = P_2^{(l)}$", $l = 1, 2, 3$ can be arbitrarily violated or satisfied by γ.

$$P_1^{(1)}(\gamma) = \gamma_{11}\gamma_{35}^2\gamma_{77} \qquad P_2^{(1)}(\gamma) = \gamma_{17}^2\gamma_{33}\gamma_{55}$$
$$P_1^{(2)}(\gamma) = \gamma_{13}\gamma_{35}\gamma_{77} \qquad P_2^{(2)}(\gamma) = \gamma_{17}\gamma_{33}\gamma_{57}$$
$$P_1^{(3)}(\gamma) = \gamma_{15}\gamma_{35}\gamma_{37} \qquad P_2^{(3)}(\gamma) = \gamma_{17}\gamma_{33}\gamma_{55}.$$

Now we want to characterize those γ's which satisfy $\gamma \sim \pi(\gamma)$ (Theorem 6.7 in the complex case and Theorem 6.11 in the real case).

Theorem 6.7 *Given a complex (resp., non-negative) G-graded contraction γ with zeroes, we have $\gamma \sim \pi(\gamma)$ if and only if γ does not violate strongly any surviving higher-order identity.*

Proof The condition is clearly necessary since $\gamma \sim \pi(\gamma)$ means $\gamma = da \cdot \pi(\gamma)$ and $\pi(\gamma)$ can only have weak violations and da none at all. To show sufficiency we assume that γ has no strong violations. We will prove the existence of a complex (resp., positive) G-graded contraction Γ without zeroes such that

$$\gamma = \Gamma \cdot \pi(\gamma) \tag{6.6}$$

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which means $\Gamma_{jk} = \gamma_{jk}$ if $\gamma_{jk} \neq 0$, $j, k \in G$. Since $\Gamma = da \sim \mathbf{1}$ (Theorem 3.1, resp., Lemma 3.2), (6.6) trivially implies $\gamma \sim \pi(\gamma)$.

By following the proof of Theorem 6.5 we determine first the numbers N'' and N' and we then choose a maximal set of $N' < N$ independent elements $\{\gamma_{s_i} \mid s_i \in G \times G, i = 1, 2, \ldots, N'\}$. (The $Q = N'' - N'$ higher-order identities (6.4) are all satisfied by assumption.)

By definition of independence, a complex G-graded contraction $\tilde{\Gamma}$ without zeroes exists which agrees with γ on these N' independent elements. If all non-vanishing dependent elements of γ follow uniquely from these independent ones, $\tilde{\Gamma}$ has to agree with γ there, too. Hence we can take $\Gamma = \tilde{\Gamma}$ in the complex case.

Otherwise we complete these N' independent elements $\{\gamma_{s_i} \mid i = 1, 2, \ldots, N' < N\}$ to a pseudobasis $\{\gamma_{s_i} \mid i = 1, 2, \ldots, N\}$. We define

$$\Gamma_{s_i} = \gamma_{s_i}, \quad i = 1, 2, \ldots, N' \quad \text{and} \quad \Gamma_{s_i} = c_i, \quad i = N' + 1, \ldots, N$$

for some arbitrary $0 \neq c_i \in \mathbb{C}$ (resp., $c_i > 0$).

Now we only have to convince ourselves that for those dependent elements which do not vanish for γ, we can choose for Γ the same values γ has. This is certainly true for Γ positive since in this case a pseudobasis behaves like a basis. In the complex case we look at Algorithm A which produces all possible Γ's. We organize this algorithm in such a way that we determine first all dependent elements Γ_t with $\gamma_t \neq 0$. For such a Γ_t we have for *all* valid relations in (5.4) obviously $n_{ir} = 0, i = N' + 1, \ldots, N$.

All these relations hold by assumption for γ, as well, since they constitute either surviving defining equations (or consequences thereof) or surviving higher-order identities (which are all those dependence relations which arise through elements which vanish for γ). Thus, we have at each step of the algorithm the identical choices for γ as for Γ, so that we can choose $\Gamma_t = \gamma_t$, $\gamma_t \neq 0$. This yields Γ. ∎

Now we turn to the real case.

Lemma 6.8 *Given a real G-graded contraction γ with zeroes. We have $\gamma \sim \operatorname{sgn} \gamma$ if and only if $|\gamma|$ does not violate strongly any surviving higher-order identity.*

Proof Theorem 6.7 yields that $|\gamma| \sim \pi(\gamma)$ if and only if $|\gamma|$ has no strong violations. In this case we have (2.15) $\gamma = |\gamma| \cdot \operatorname{sgn} \gamma \sim \pi(\gamma) \cdot \operatorname{sgn} \gamma = \operatorname{sgn} \gamma$. ∎

For real G-graded contractions γ without zeroes there are 2^N different $\operatorname{sgn} \gamma$ by Lemma 3.3. In contrast, if zeroes occur, we get $2^{N''}$ different $\operatorname{sgn} \gamma$ (see Lemma 6.10 below). To prove this result we need part (i) of the following lemma. (Both parts will be used in Lemma 6.17.)

Lemma 6.9 *Let $S \in \mathcal{S}(G)$.*

(i) *We can choose $N''(S)$ quasi-independent elements $\{\gamma_{s_i} \mid i = 1, 2, \ldots, N''\}$ in such a way that the signs of all dependent elements of $\operatorname{sgn} \gamma$ follow uniquely from $\{\operatorname{sgn} \gamma_{s_i}\}$.*

(ii) *We can choose from these N'' elements $N'(S)$ independent elements in such a way that each of the remaining $Q = N'' - N'$ elements occurs in its individual higher-order identity, as produced in Theorem* 6.5*, with an odd power.*

Proof (i) Assume $\{\gamma_{s_i} \mid i = 1, 2, \dots, N''\}$ is a maximal set of quasi-independent elements. If $\{\gamma_{s_i}\}$ does not have the desired property, we use an algorithm identical in spirit to Algorithm A to determine the dependent elements of γ one by one from $\{\gamma_{s_i} \mid i = 1, 2, \dots, N''\}$. The only difference is that we only use all valid relations which stem from the surviving defining equations alone (*i.e.,* the $Q(S)$ higher-order identities are ignored).

Assume γ_t is the first dependent element of γ in S for which our algorithm does not yield γ_t itself, but γ_t^μ, *i.e.,*

$$\gamma_t^\mu = \prod_{i=1}^{N''} \gamma_{s_i}^{n_i} \quad \mu = \sum_{i=1}^{N''} n_i \in \mathbb{N}, \quad n_i \in \mathbb{Z}, \tag{6.7}$$

with a minimal power $\mu \geq 2$ ((5.4), (5.5)). (Since by assumption all dependent elements determined prior to γ_t follow uniquely from $\{\gamma_{s_i}\}$, they were replaced in (6.7) by $\{\gamma_{s_i}\}$.)

Since

$$\operatorname{sgn} \gamma_t^\mu = \begin{cases} \operatorname{sgn} \gamma_t & \text{if } \mu \text{ odd,} \\ +1 & \text{if } \mu \text{ even,} \end{cases}$$

$\operatorname{sgn} \gamma_t$ follows from $\{\operatorname{sgn} \gamma_{s_i}\}$ uniquely only if μ is odd. If μ is even, we are free to choose both signs for γ_t. But now the right-hand side of (6.7) obviously constitutes a sign invariant of the first kind which must be positive, *i.e.,*

$$\operatorname{sgn} \prod_{i=1}^{N''} \gamma_{s_i}^{\delta_i} = +1, \qquad \text{where } \delta_i = \begin{cases} 0 & \text{if } n_i \text{ even,} \\ 1 & \text{if } n_i \text{ odd.} \end{cases} \tag{6.8}$$

Therefore we cannot choose $\operatorname{sgn} \gamma_{s_i}$ arbitrarily for all i. But we can get around this complication. Since not all powers n_i which occur in (6.7) can be even (otherwise we could divide all occurring powers μ and n_i by 2 so that μ would not be minimal), we can solve (6.8) for some γ_{s_k} with n_k odd. This means that $\operatorname{sgn} \gamma_{s_k}$ follows uniquely from $\{\operatorname{sgn} \gamma_{s_i} \mid i \neq k\}$, and consequently that the signs of all dependent elements considered prior to γ_t also follow uniquely from $\{\operatorname{sgn} \gamma_{s_i} \mid i \neq k\}$. Therefore we can consider from now on instead of $\{\gamma_{s_i}\}$, the set $\{\gamma_{s_i} \mid i \neq k\} \cup \{\gamma_t\}$. Continuing in this way we see that we can always get the desired property.

(ii) Let (after renumbering if necessary) $\{\gamma_{s_i} \mid i = 1, 2, \dots, N' \leq N''\}$ be N' independent elements. Theorem 6.5 yields, for the remaining Q elements

$$\{\gamma_{s_{N'+l}} \mid l = 1, 2, \dots, Q = N'' - N'\},$$

individual higher-order identities "$P_1^{(l)}(\gamma) = P_2^{(l)}(\gamma)$", ($l = 1, 2, \dots, Q$) which look like

$$\text{“}\gamma_{s_{N'+l}}^{n_l} = \prod_{i=1}^{N'} \gamma_{s_i}^{n_{li}}\text{”}, \quad n_l = \sum_{i=1}^{N'} n_{li} \in \mathbb{N}, \quad n_{li} \in \mathbb{Z}. \tag{6.9}$$

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Consider (6.9) for $l = 1$. If n_1 is odd, we leave this equation alone. If n_1 is even, we solve this equation for one of the γ_{s_i} with n_{1i} odd. (Such a γ_{s_i} must exist since otherwise we could divide all powers in this equation by 2.) Then $\gamma_{s_{N'+1}}$ and this specific γ_{s_i} change places, *i.e.*, $\gamma_{s_{N'+1}}$ becomes one of the N' independent elements instead of this γ_{s_i}. In all other equations this γ_{s_i} gets replaced as well. This may require taking some natural power of these equations. But since this power will always be odd, all odd powers n_l $(l \neq 1)$ in (6.9) will remain odd.

We continue this procedure with the next of our now possibly modified equations. At the end, the new N' independent elements have the desired property. ∎

Lemma 6.10 *Let $S \in \mathcal{S}(G)$. There are $2^{N''}$ different* $\operatorname{sgn} \gamma$ *(where $N''(S)$ is defined in Definition* 6.3*).*

Proof Select $\{\gamma_{s_i} \mid i = 1, 2, \dots, N''\}$ quasi-independent elements as in Lemma 6.9. Then we can choose $\operatorname{sgn} \gamma_{s_i}, i = 1, 2, \dots, N''$ arbitrarily, which yields $2^{N''}$ different $\operatorname{sgn} \gamma$. ∎

Now we can extend Theorem 6.7 to the real case.

Theorem 6.11 *Let γ be a real G-graded contraction with zeroes.*

(i) *A real G-graded contraction Γ without zeroes satisfying $\gamma = \Gamma \cdot \pi(\gamma)$ exists if and only if γ does not violate strongly any surviving higher-order identity, and γ has no negative sign invariants of the first kind.*

(ii) *Assume Γ as in* (i) *exists. We have $\gamma \sim \pi(\gamma)$ if and only if all sign invariants of the second kind are positive for γ.*

Proof (i) The conditions in (i) are clearly necessary (Remark 4.2(i), Lemma 4.11, Definition 4.12). Therefore we assume in the following that γ has at most weak violations which means (Lemma 6.8) $\gamma \sim \operatorname{sgn} \gamma$ so that we only have to care about the signs. Furthermore we assume that all sign invariants of the first kind are positive for γ.

We start as in the proof of Theorem 6.7 by choosing N' independent elements $\{\gamma_{s_i} \mid i = 1, 2, \dots, N'\}$. But here we can make use of Lemma 6.9(i) and choose these elements in such a way that all dependent elements which survive for γ follow uniquely from $\{\gamma_{s_i}\}$.

Now we complete $\{\gamma_{s_i} \mid i = 1, 2, \dots, N'\}$ to a pseudobasis $\{\Gamma_{s_i} \mid i = 1, 2, \dots, N\}$ where $\Gamma_{s_i} = \gamma_{s_i}, i = 1, 2, \dots, N'$. Let $\Gamma_{s_i} = c_i, i = N'+1, \dots, N$, for some arbitrary $0 \neq c_i \in \mathbb{R}$. Due to our special choice of $\{\gamma_{s_i} \mid i = 1, 2, \dots, N'\}$ we obviously have $\Gamma_{jk} = \gamma_{jk}$ if $\gamma_{jk} \neq 0$. When can the remaining dependent elements of Γ be chosen to be real, too? According to Theorem 5.12 a real Γ without zeroes exists as long as the sign distribution $\{\operatorname{sgn} c_i\}$ does not create negative sign invariants of the first kind.

Consider therefore the (finite) set of all sign invariants of the first kind which have the form

$$\operatorname{sgn}\Big[\prod_{l=N'+1}^{N} \Gamma_{s_l}^{\delta_l} \prod_{i=1}^{N'} \gamma_{s_i}^{\epsilon_i}\Big], \delta_l, \epsilon_i \in \{0, 1\}.$$

If this set is empty, we can obviously choose $\{\operatorname{sgn} c_l\}$ arbitrarily. Otherwise we show that it is possible to choose $\{\operatorname{sgn} c_l \mid l = N' + 1, \dots, N\}$ in such a way that for all these invariants we have

$$\operatorname{sgn}\Big[\prod_{l=N'+1}^{N} c_l^{\delta_l} \prod_{i=1}^{N'} \gamma_{s_i}^{\epsilon_i}\Big] = +1. \tag{6.10}$$

If no equation exists with $\delta_{N'+1} = 1$, we can obviously choose either sign for $c_{N'+1}$. Otherwise we take one equation with $\delta_{N'+1} = 1$, solve it for $\operatorname{sgn} c_{N'+1}$ and substitute this result in all remaining equations. Then we repeat this procedure with all these remaining equations for $\delta_{N'+2}$, then for $\delta_{N'+3}$ and finally for δ_N. After $(N - N')$ steps we have determined $\{\operatorname{sgn} c_l\}$. All remaining equations in (6.10) must now look like $\operatorname{sgn} \prod_{i=1}^{N'} \gamma_{s_i}^{\epsilon_i} = +1$, and they are satisfied by assumption.

(ii) Now we show that we can choose $\Gamma \sim \mathbf{1}$ (which means $\gamma \sim \pi(\gamma)$) if all sign invariants of the second kind are positive for γ. To see this, we add to (6.10) all sign invariants of the second kind which can be formed by our pseudobasis $\{\Gamma_{s_i} \mid i = 1, 2, \dots, N\}$ as well and demand that they be positive, too. The solution will be obtained in the same way as above. ■

To characterize the equivalence classes for $\operatorname{sgn} \gamma$ we need the notion of independent sign invariants.

Remark 6.12 Let $S \in \mathcal{S}(G)$ and choose N'' quasi-independent elements $\{\gamma_{s_i} \mid i = 1, 2, \dots, N''\}$ as in Lemma 6.9. Since all the signs of all non-vanishing elements of γ follow uniquely from these $\{\operatorname{sgn} \gamma_{s_i}\}$, any surviving sign invariant can be uniquely expressed as $\operatorname{sgn} P(\gamma)$ where $P(\gamma) = \prod_{i=1}^{N''} \gamma_{s_i}^{\delta_i}$, $\delta_i \in \{0, 1\}$. Such a sign invariant is non-trivial if it can take on both signs for some γ with support S, which means that not all $\delta_i = 0$.

Definition 6.13 Given $S \in \mathcal{S}(G)$ and a family of non-trivial surviving sign invariants expressed as in Remark 6.12, we say that they are independent if none of them is a product of the remaining ones (where we take all exponents modulo 2 (Remark 4.16(ii)).

Lemma 6.14 *Let $S \in \mathcal{S}(G)$. Any maximal set of surviving independent sign invariants has the same number $J(S)$ of elements. All surviving sign invariants are products of the $J(S)$ elements of such a set.*

Proof Consider the set of all surviving sign invariants expressed as in Remark 6.12. We remove from this set as many sign invariants as possible that are products of the remaining ones, until we are left with $J(S)$ independent ones. All sign invariants are then obviously products of these $J(S)$ ones. The uniqueness of $J(S)$ follows from a straightforward replacement argument. ■

Example 6.15 Consider all G-graded contractions without zeroes where $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$. According to Lemma 4.15 there are exactly r_e independent sign invariants of the second kind (while all sign invariants of the first kind must have trivially the value 1).

The following lemma shows that, as expected, independent sign invariants can indeed take on independent values.

Lemma 6.16 *Assume given $S \in \mathcal{S}(G)$, N'' quasi-independent elements γ_{s_i} as in Lemma* 6.9*, and L sign invariants* sgn $P_l(\gamma)$*, where*

$$P_l(\gamma) = \prod_{i=1}^{N''} \gamma_{s_i}^{\delta_{li}}, \delta_{li} \in \{0, 1\}, \quad l = 1, 2, \dots, L.$$

Then for arbitrary $\alpha_l = \pm 1$, a real γ with support S satisfying sgn $P_l(\gamma) = \alpha_l$*, exists if and only if these L sign invariants are independent.*

Proof The condition is clearly necessary. To show sufficiency, we first recall that for arbitrary $\{\text{sgn } \gamma_{s_i} \mid i = 1, 2, \dots, N''\}$ a real γ with support S exists (Lemma 6.10). Now we show that it is possible to choose $\{\text{sgn } \gamma_{s_i}\}$ in such a way that

$$\text{(6.11)} \qquad \text{sgn } P_l(\gamma) = \text{sgn} \prod_{i=1}^{N''} \gamma_{s_i}^{\delta_{li}} = \alpha_l; l = 1, 2, \dots, L.$$

If no equation exists with $\delta_{l1} = 1$, we can obviously choose either sign for sgn γ_{s_1}. Otherwise we select one equation with $\delta_{l1} = 1$, solve it for sgn γ_{s_1} and substitute this result into all remaining equations. Note that this substitution yields the same result as the multiplication of all remaining equations where γ_{s_1} occurs, *i.e.*, where $\delta_{l1} = 1$, by the chosen equation. (Assume *e.g.*,

$$\text{sgn } P_1(\gamma) = \text{sgn}\Big(\gamma_{s_1} \prod_{i=2}^{N''} \gamma_{s_i}^{\delta_{1i}}\Big) = \alpha_1, \quad \text{sgn } P_2(\gamma) = \text{sgn}\Big(\gamma_{s_1} \prod_{i=2}^{N''} \gamma_{s_i}^{\delta_{2i}}\Big) = \alpha_2.$$

Then the first equation yields sgn $\gamma_{s_1} = \alpha_1$ sgn $\prod_{i=2}^{N''} \gamma_{s_i}^{\delta_{1i}}$, which turns the second one into $\text{sgn}(P_1(\gamma)P_2(\gamma)) = \text{sgn} \prod_{i=2}^{N''} \gamma_{s_i}^{(\delta_{1i}+\delta_{2i}) \bmod 2} = \alpha_1\alpha_2$.)

Then we repeat this procedure with all remaining equations for δ_{l2} and so on. The only problem we can run into is that we encounter an equation where all γ_{s_i}'s have dropped out, so that it says that a certain product of α_l's should be equal to 1. But this means that the corresponding product of the P_l's is equal to 1, *i.e.*, that one of these P_l's is a product of the remaining ones, so that our L sign invariants are not independent. ∎

Finally we determine the number $J(S)$ (Lemma 6.17) and we give an algorithm which constructs a maximal set of independent sign invariants for any given set of independent elements $\{\gamma_{s_i} \mid i = 1, 2, \dots, r \leq N = |G|\}$ (Algorithm 6.18).

Lemma 6.17 *Let $S \in \mathcal{S}(G)$. As in Lemma 6.9, choose $N''(S)$ quasi-independent and $N'(S)$ independent elements. Let $J'(S)$ be the maximal number of independent sign invariants which can be constructed from these N' elements alone. Then $J(S) = Q(S) + J'(S)$ where $Q = N'' - N'$.*

Proof First we choose our elements $\{\gamma_{s_i} \mid i = 1, 2, \ldots, N'\}$ and $\{\gamma_{s_i} \mid i = N'+1, \ldots, N''\}$ according to Lemma 6.9. The Q higher-order identities "$P_1^{(l)} = P_2^{(l)}$," $l = 1, 2, \ldots, Q = N'' - N'$ which get produced by the proof of Theorem 6.5, automatically yield Q sign invariants $\operatorname{sgn}[P_1^{(l)}(\gamma)P_2^{(l)}(\gamma)]$, $l = 1, 2, \ldots, Q$ of the first kind (Remark 4.16(iii)). Since each of these sign invariants contains exactly one and always a different element from the set $\{\gamma_{s_i} \mid i = N' + 1, \ldots, N''\}$, they are clearly independent from each other and from the J' ones for the N' elements alone.

Using Remark 6.12 we write all sign invariants in terms of $\{\gamma_{s_i} \mid i = 1, 2, \ldots, N''\}$. Assume there exists a sign invariant which contains some of the elements from the set $\{\gamma_{s_i} \mid i = N'+1, \ldots, N''\}$. By multiplying this sign invariant by the corresponding sign invariants for the individual elements which occur, we obviously get a sign invariant for the N' elements alone. Therefore it cannot be independent from the Q ones above, plus the J' ones.

Thus, the maximal number J of independent sign invariants is $J = Q + J'$. ∎

To determine J' and to construct the corresponding J' independent sign invariants we present the following algorithm which works for any set of independent elements $\{\gamma_{s_i}\}$.

Algorithm 6.18 Given r independent elements

$$\{\gamma_{s_i} = \gamma_{j_i k_i} \mid j_i, k_i \in G, i = 1, 2, \ldots, r\}, 1 \leq r \leq N = |G|$$

of a G-graded contraction γ, we construct a maximal set of independent sign invariants which can be formed by these r elements, *i.e.,* which have the form $\operatorname{sgn} P(\gamma)$ where $P(\gamma) = \prod_{i=1}^{r} \gamma_{s_i}^{\delta_i}, \delta_i \in \{0, 1\}$. According to Lemma 4.13, $\operatorname{sgn} P(\gamma)$ is a sign invariant if and only if we have for all da, $P(da) = \prod_{j=1}^{N} a_j^{n_j}$ $(a_j \in \mathbb{R})$ where all $n_j \in \mathbb{Z}$ are even. We will see that the sign invariants for $\{\gamma_{s_i}\}$ are directly related to the ambiguities in the solutions of the ansatz

$$\gamma_{s_i} = \gamma_{j_i k_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}}, \quad i = 1, 2, \ldots, r, a_j = \pm 1, \tag{6.12}$$

where all $\gamma_{s_i} = +1$. (In order to recognize the sign invariants later on, we do not replace the γ_{s_i}'s by their common value 1 in the following.)

According to Lemma 5.5 we can solve (6.12) for some power of exactly r of the Na_j's as functions of the remaining $(N - r)a_j$'s plus the γ_{s_i}. Since here we have $a_j = \pm 1$, we only care if the power of a_j is even or odd (*i.e.,* we consider the powers modulo 2).

Assume $\tilde{r}(1 \leq \tilde{r} \leq r)$ of the a_j's can be determined from (6.12) with an odd power, and the remaining $(r - \tilde{r})$ ones not. This means that (6.12) yields

$$2^{N-\tilde{r}} \tag{6.13}$$

different solutions for the a_j's where $a_j = \pm 1$. We will see that the maximal number of independent sign invariants turns out to be $(r - \tilde{r})$.

We start by solving (6.12) as long as possible for such a_j's which occur with an *odd* power and we immediately replace the expression we get in all remaining equations. By renumbering if necessary, we can assume that this is possible by solving the first $\tilde{r}$ equations in (6.12) in the natural order.

Assume we have solved the equation for γ_{s_1} for a_{l_1} with an odd power. Replacing a_{l_1} by this expression in all remaining equations is equivalent to the following procedure. If the equation for γ_{s_i} $(i > 1)$ contains a_{l_1} with an *even* power, we leave it alone. If it contains a_{l_1} with an *odd* power, we multiply this equation by the one for γ_{s_1} *i.e.*, we replace

$$\gamma_{s_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}}$$

by the equation

$$\gamma_{s_i} \gamma_{s_1} = \frac{a_{j_i} a_{k_i}}{a_{j_i + k_i}} \frac{a_{j_1} a_{k_1}}{a_{j_1 + k_1}}$$

where a_{l_1} now occurs with an even power. (Note that we do not lose any information in this way).

Next we solve the equation for γ_{s_2} (resp., $\gamma_{s_2}\gamma_{s_1}$) for a_{l_2} with an odd power. Again we multiply all remaining equations for γ_{s_i} (resp., $\gamma_{s_i}\gamma_{s_1}$), $i > 2$; by the one for γ_{s_2} (resp., $\gamma_{s_2}\gamma_{s_1}$) if and only if they contain a_{l_2} with an odd power.

We continue in this way until the remaining $(r - \tilde{r})$ equations only contain a_j's with an even power. Therefore these equations must have picked up some powers of $\gamma_{s_1}, \gamma_{s_2}, \dots, \gamma_{s_{\tilde{r}}}$ along the way. They look like

$$\gamma_{s_i} \prod_{k=1}^{\tilde{r}} \gamma_{s_k}^{m_{ik}} = \prod_{j \in G} a_j^{n_{ij}}, \tag{6.14}$$

$i = \tilde{r} + 1, \dots, r$, $m_{ik} = 0, 1, 2, \dots,$ n_{ij} even. By Lemma 4.13 these expressions constitute $(r - \tilde{r})$ sign invariants $\operatorname{sgn} P_i(\gamma)$ where

$$P_i(\gamma) = \gamma_{s_i} \prod_{k=1}^{\tilde{r}} \gamma_{s_k}^{m_{ik} mod 2}, \quad i = \tilde{r} + 1, \dots, r, \tag{6.15}$$

which are obviously independent since each of them contains one element all others do not.

Now we show that this set is maximal. Assume that $\operatorname{sgn} P(\gamma)$ is a sign invariant where

$$P(\gamma) = \prod_{i=1}^{r} \gamma_{s_i}^{\delta_i}, \quad \delta_i \in \{0, 1\}.$$

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Following the same strategy as in the proof of Lemma 6.17, we multiply $P(\gamma)$ by $P_i(\gamma)$ whenever $\delta_i = 1$ for some $i \in \{\tilde{r}+1, \dots, r\}$. In this way we get a sign invariant for the first $\tilde{r}$ elements alone, *i.e.*, some sgn $P'(\gamma)$ where

$$P'(\gamma) = \prod_{i=1}^{\tilde{r}} \gamma_{s_i}^{\delta_i'}, \quad \delta_i' \in \{0, 1\}. \tag{6.16}$$

We will show that this must be a trivial invariant, *i.e.*, $\delta_i' = 0$ for all i. This means that sgn $P(\gamma)$ is a product of some of the sgn $P_i(\gamma)$'s, so that it is not independent.

Assume $\delta_l' = 1$ in (6.16) for some $l \in \{1, 2, \dots, \tilde{r}\}$. This would lead to twice the number of solutions of (6.12) than was stated in (6.13), as can be seen in the following way. If $\delta_l' = 1$, it is obviously the same to solve either $\gamma_{s_i} = \frac{a_{j_i} a_{k_i}}{a_{j_i+k_i}}$ for all $i = 1, 2, \dots, \tilde{r}$ or to replace the equation for γ_{s_l} by the following new one

$$\prod_{i=1}^{\tilde{r}} \gamma_{s_i}^{\delta_i'} = \prod_{i=1}^{\tilde{r}} \Big(\frac{a_{j_i} a_{k_i}}{a_{j_i+k_i}} \Big)^{\delta_i'}.$$

But since this is a sign invariant, all a_j's enter as squares only, so that we cannot solve this equation for any one of them. Therefore we would only get $(\tilde{r} - 1)$ solutions instead of $\tilde{r}$ solutions, which is clearly a contradiction.

7 The Classification

In Theorem 7.1 we prove that our set of invariants (support, higher-order identities and sign invariants) is complete by showing that two inequivalent G-graded contractions must differ on at least one of these invariants. This result immediately yields a straightforward classification of G-graded contractions.

In the complex (resp., non-negative) case, we already know that for each support S exactly $Q(S)$ higher-order identities can be independently and arbitrarily violated (Theorem 6.5), which therefore immediately gives all equivalence classes.

In the real case, we already know that there exist exactly $J(S)$ independent surviving sign invariants for each support S which can independently take on both values (Lemmas 6.16, 6.17). This immediately gives all equivalence classes for sgn γ.

Theorem 7.1 *Let γ and γ' be inequivalent G-graded contractions. Then either they have different supports, or (if their supports agree) they differ on some surviving higher-order identity, or, in the real case, on a surviving sign invariant.*

Proof Assume that γ and γ' agree on all invariants. We will prove that then $\gamma \sim \gamma'$. Since $S(\gamma) = S(\gamma')$, the equation

$$\tilde{\gamma}_{jk} = \frac{\gamma_{jk}}{\gamma'_{jk}}$$

defines a G-graded contraction with the same support S (Proposition 2.9). We will show that $\tilde{\gamma} \sim \pi(\tilde{\gamma})$, which trivially implies that $\gamma \sim \gamma'$.

If "$P_1 = P_2$" is any surviving higher-order identity, then P_1/P_2 is an invariant (Lemma 4.4) and by assumption we have

$$\frac{P_1(\gamma)}{P_2(\gamma)} = \frac{P_1(\gamma')}{P_2(\gamma')} \neq 0,$$

which implies $P_1(\tilde{\gamma}) = P_2(\tilde{\gamma})$. Similarly, in the real case, if sgn $P(\gamma)$ is any surviving sign-invariant, we have sgn $P(\gamma) =$ sgn $P(\gamma')$ and hence sgn $P(\tilde{\gamma}) = +1$. That $\tilde{\gamma} \sim \pi(\tilde{\gamma})$ now follows from Theorem 6.7 in the complex and non-negative cases and from Theorem 6.11 in the real case. ■

7.1 The Classification of G-Graded Contractions

(i) *The complex case.* For a given G, one first determines all possible supports $\mathcal{S}(G)$. There is a straightforward algorithm for this (Remark 6.2). Let $S \in \mathcal{S}(G)$. In Section 6 (Definition 6.3, Remark 6.4, Theorem 6.5) we have shown that there is a straightforward procedure for determining the two natural numbers $N'(S) \leq N''(S)$ and the $Q(S) = N''(S) - N'(S) \geq 0$ surviving higher-order identities "$P_1^{(l)} = P_2^{(l)}$", $l = 1, 2, \dots, Q$, with the property that for arbitrary complex $\alpha_l \neq 0$, there exists a γ with $P_1^{(l)}(\gamma) = \alpha_l P_2^{(l)}(\gamma), l = 1, 2, \dots, Q$. Furthermore, all strong violations of higher-order identities follow from these Q identities. Thus for each support S there is a $Q(S)$-parameter family of inequivalent γ's.

Note that by Lemma 4.7 there are no higher-order identities if and only if $|G| \leq 5$. Hence in this case, the number of equivalence classes is just the number of supports (5 for $\mathbb{Z}_2$, 15 for $\mathbb{Z}_3$, 47 for $\mathbb{Z}_4$, and 41 for $\mathbb{Z}_2 \times \mathbb{Z}_2$, for example (Remark 6.2)).

(ii) *The non-negative case.* This is identical to the complex case, except that the parameters α_l are positive.

(iii) *The real case.* Since $\gamma \sim \gamma'$ if and only if $|\gamma| \sim |\gamma'|$ and sgn $\gamma \sim$ sgn γ' (Lemma 2.16(iii)), we only have to add the classification of sgn γ to the non-negative case. Since for a given support S there are only finitely many sgn γ, there can only be a finite number of equivalence classes. In fact, because of Theorem 7.1 and Lemma 6.16, the $2^{N''(S)}$ possible sgn γ (Lemma 6.10) split into $2^{J(S)}$ equivalence classes where $J(S)$ is the maximal number of independent surviving sign invariants (Lemma 6.17, see also Remark 7.3).

The following examples illustrate the classification of sgn γ.

Example 7.2

(i) Consider all G-graded contractions sgn γ without zeroes. Then we have (Example 6.15) $N = N' = N''$, $Q = 0$, $J = J' = r_e$, so that we get 2^{r_e} equivalence classes, in agreement with Theorem 3.5.

(ii) We give the number of equivalence classes of sgn γ for small $|G|$. For $\mathbb{Z}_3$ there is only one equivalence class for each support, since sign invariants do not exist. For $\mathbb{Z}_2$, resp., $\mathbb{Z}_4$, we get two equivalence classes for those supports where sgn$(\gamma_{00}\gamma_{11})$, resp., sgn$(\gamma_{00}\gamma_{22})$, or sgn$(\gamma_{11}\gamma_{33})$ survives, otherwise one. For $\mathbb{Z}_2 \times \mathbb{Z}_2$ there are 1, 2 or 4 equivalence classes since 0, 1, or 2 independent sign invariants like sgn$(\gamma_{00,00}\gamma_{01,01})$ or sgn$(\gamma_{00,00}\gamma_{10,10})$ or sgn$(\gamma_{00,00}\gamma_{11,11})$ can survive for a specific support.

(iii) Consider all $\mathbb{Z}_8$-graded contractions $\operatorname{sgn}\gamma$ with support

$$\begin{cases} \gamma_{jk} \neq 0 & \text{if } j, k \text{ odd}, \\ \gamma_{jk} = 0 & \text{otherwise}. \end{cases}$$

The relevant numers are (Example 6.6) $N = 8, N'' = 10, N' = 7, Q = 3; r_e = 1$. For the N' independent elements $\{\gamma_{17}, \gamma_{33}, \gamma_{35}, \gamma_{37}, \gamma_{55}, \gamma_{57}, \gamma_{77}\}$ which we have chosen in Example 6.6 and which satisfy our assumptions in Lemma 6.17, there exists exactly one sign invariant, namely $\operatorname{sgn}(\gamma_{33}\gamma_{77})$ (which is of the second kind (Examples 4.17)). Therefore $J' = 1$ which means that the $2^{N''} = 2^{10}$ different $\operatorname{sgn}\gamma$ split into $2^{Q+J'} = 2^4$ equivalence classes.

A maximal set of $J = 4$ independent sign invariants which separate them is

$$\operatorname{sgn}(\gamma_{33}\gamma_{77}), \quad \operatorname{sgn}(\gamma_{11}\gamma_{33}\gamma_{55}\gamma_{77}) = \operatorname{sgn}(\gamma_{11}\gamma_{55}) \operatorname{sgn}(\gamma_{33}\gamma_{77})$$
$$\operatorname{sgn}(\gamma_{13}\gamma_{17}\gamma_{33}\gamma_{35}\gamma_{57}\gamma_{77}) = \operatorname{sgn}(\gamma_{33}\gamma_{77}) \operatorname{sgn}(\gamma_{13}\gamma_{17}\gamma_{35}\gamma_{57}),$$
$$\operatorname{sgn}(\gamma_{15}\gamma_{17}\gamma_{33}\gamma_{35}\gamma_{37}\gamma_{55}).$$

(Example 6.6 for the 3 sign invariants of the first kind which stem from the Q higher-order identities)

Finally we determine the equivalence classes of $\operatorname{sgn}\gamma$ in an alternate way by using a counting argument based on the group structure.

Remark 7.3 There are $2^{N''}$ different $\operatorname{sgn}\gamma$ with support S (Lemma 6.10). The equivalence class of any such $\operatorname{sgn}\gamma$ is given by $\operatorname{Sgn}_0(G) \cdot \operatorname{sgn}\gamma$ where $\operatorname{Sgn}_0(G) = \{da \mid a_j = \pm 1, j \in G\}$ has 2^{N-r_e} elements ((3.15), (3.18)). The number of elements in this equivalence class is just the number of elements in the quotient group of $\operatorname{Sgn}_0(G)$ by its stabilizer subgroup of all da satisfying

$$da \cdot \operatorname{sgn}\gamma = \operatorname{sgn}\gamma, \tag{7.1}$$

i.e.,

$$(da)_s = +1, \quad s \in S, a_j = \pm 1. \tag{7.2}$$

Since the order of $\operatorname{Sgn}_0(G)$ is a power of 2, the order of its stabilizer subgroup must also be some power of 2. Therefore (7.2) must have $2^{N_{ST}}$ solutions for the da's for some integer N_{ST}. Hence the number of elements in any equivalence class is

$$2^{N-r_e-N_{ST}} \tag{7.3}$$

and the number of equivalence classes is

$$2^{N''-(N-r_e-N_{ST})}. \tag{7.4}$$

Now we calculate N_{ST}. We use the notation of Lemma 6.9. Since all dependent elements follow uniquely from our N'' elements, it follows that $(da)_{s_i} = +1$, $i = 1, 2, \dots, N''$, $a_j = \pm 1$ implies (7.2). Furthermore, because each of the Q higher-order identities is a dependence relation which must hold for any da, it suffices to have (6.9)

$$(da)_{s_i} = +1, \quad i = 1, 2, \dots, N', a_j = \pm 1. \tag{7.5}$$

Now (7.5) corresponds directly to (6.12) which we solved in Lemma 6.17 if we set $r = N'$. Therefore (7.5) and hence (7.2) admit exactly $2^{N-(N'-J')}$ different solutions for the a_j's where $a_j = \pm 1$ and where J' is the maximal number of independent sign invariants for our N' independent elements.

Since, for a fixed da, the equation

$$(da)_{jk} = \frac{a_j a_k}{a_{j+k}}$$

has 2^{r_e} different solutions (3.18), we have $2^{N-(N'-J')-r_e}$ different solutions for the da's so that we finally get $N_{ST} = N - N' + J' - r_e$. It then follows from (7.3) that the number of elements in any equivalence class is $2^{N-J'}$. From (7.4) the number of equivalence classes for sgn γ follows as 2^J where $J = Q + J'$.

A Natural Basis for G-Graded Contractions

In this appendix, we show that all elements of a G-graded contraction γ without zeroes follow uniquely from $N = |G|$ specific elements. We treat first the case $\mathbb{Z}_N$ in Lemma A.1, then $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ in Lemma A.2, and $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ in Lemma A.3. This provides the necessary preparation for the general case in Lemma A.4.

In fact, these N elements constitute a basis (Example 5.3) which we call a "natural" basis for G-graded contractions, since it uses, in a straightforward way, the structure of G and, above all, because it is easy to work with (see the proofs of Theorem 3.1 and Lemmas 4.14, 4.15). To prove the basis property we have to show that for arbitrary complex, non-vanishing values of these N elements, a complex γ without zeroes exists which takes on these values. This could be done directly via the defining equations, but a much simpler argument is given in Theorem 3.1 which constructs γ as a specific coboundary da.

Lemma A.5 is a technical lemma for real $\mathbb{Z}_N$-graded contractions.

Lemma A.1 *All elements of a $\mathbb{Z}_N$-graded contraction γ without zeroes follow uniquely from the N elements $\{\gamma_{00}, \gamma_{11}, \gamma_{12}, \dots, \gamma_{1,N-1}\}$, $N = 2, 3, \dots$.*

Proof Consider a $\mathbb{Z}_N$-graded contraction γ without zeroes. We have $\gamma_{0j} = \gamma_{00}$ for $j, k = 0, 1, 2, \dots, N-1$, *cf.* (3.3) . Multiplying the defining equations (2.3) $\gamma_{1l}\gamma_{l+1,k} = \gamma_{lk}\gamma_{1,k+l}$ for $l = 1, 2, \dots, j-1$ where $j \le k$, $j \ne 0, 1$ yields

$$\gamma_{11}\gamma_{12} \cdots \gamma_{1,j-1}\gamma_{2k}\gamma_{3k} \cdots \gamma_{jk} = \gamma_{1k}\gamma_{2k} \cdots \gamma_{j-1,k}\gamma_{1,k+1}\gamma_{1,k+2} \cdots \gamma_{1,j+k-1}.$$

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Since γ is without zeroes we get

(A.1)
$$\gamma_{jk} = \frac{\gamma_{1k}\gamma_{1,k+1}\cdots\gamma_{1,j+k-2}\gamma_{1,j+k-1}}{\gamma_{11}\gamma_{12}\cdots\gamma_{1,j-1}}.$$

Therefore all elements of γ follow uniquely from the N elements

$$\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1,N-1}, \gamma_{1N} = \gamma_{10} = \gamma_{00}.$$

(One way to show that these N elements constitute a basis is to prove that γ_{jk}, when defined by (A.1), satisfies all defining equations.) ∎

The generalization of this lemma from $\mathbb{Z}_N$ to an arbitrary G is not entirely straightforward. The obvious guess for $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ would be to try the "tensor product" of the bases for $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$. This, however, does not work. We therefore proceed in several steps.

Lemma A.2 *All elements of a $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$-graded contraction γ without zeroes follow uniquely from the $N_1 \cdot N_2$ elements $(N_1, N_2 = 2, 3, \ldots)$*

$$\{\gamma_{00,00}, \gamma_{10,j0}, \gamma_{01,0k}, \gamma_{j0,0k} \mid j = 1, 2, \ldots, N_1 - 1,\ k = 1, 2, \ldots, N_2 - 1\}.$$

Proof Consider a $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$-graded contraction γ without zeroes. We have $(j, j' \in \mathbb{Z}_{N_1}, k, k' \in \mathbb{Z}_{N_2}$, see(3.3)

(A.2)
$$\gamma_{00,jk} = \gamma_{00,00}.$$

The defining equations (2.3) yield

$$\gamma_{j0,0k}\gamma_{jk,j'k'} = \gamma_{j0,j'k'}\gamma_{0k;j+j',k'} = \gamma_{0k,j'k'}\gamma_{j0;j',k+k'};$$

especially for $k' = 0$,

$$\gamma_{j0,0k}\gamma_{jk,j'0} = \gamma_{j0,j'0}\gamma_{0k;j+j',0},$$

and for $j' = 0$,

$$\gamma_{j0,0k}\gamma_{jk,0k'} = \gamma_{0k,0k'}\gamma_{j0;0,k+k'}.$$

Altogether we get (since γ is symmetric)

$$\gamma_{jk,j'k'} = \frac{\gamma_{j0,j'k'}\gamma_{0k;j+j',k'}}{\gamma_{j0,0k}} = \frac{1}{\gamma_{j0,0k}} \cdot \frac{\gamma_{j0,j'0}\gamma_{0k';j+j',0}}{\gamma_{j'0,0k'}} \cdot \frac{\gamma_{0k,0k'}\gamma_{j+j',0;0,k+k'}}{\gamma_{j+j',0;0k'}},$$

so that

(A.3)
$$\gamma_{jk,j'k'} = \frac{\gamma_{j0,j'0}\gamma_{0k,0k'}\gamma_{j+j',0;0,k+k'}}{\gamma_{j0,0k}\gamma_{j'0,0k'}}.$$

From the $\mathbb{Z}_{N_1}$ (resp., $\mathbb{Z}_{N_2}$)-subgroup structure we know that (A.1) all elements $\gamma_{j0,j'0}$ (resp., $\gamma_{0k,0k'}$) follow uniquely from the elements $\gamma_{10,j0}$ (resp., $\gamma_{01,0k}$). Altogether we see that all elements of γ follow uniquely from the elements $\gamma_{00,00}, \gamma_{10,j0}, \gamma_{01,0k}, \gamma_{j0,0k}$. ∎

Lemma A.3 *For $N_1, N_2, N_3 = 2, 3, \dots$, all elements of a $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$-graded contraction γ without zeroes follow uniquely from the $N_1 \cdot N_2 \cdot N_3$ elements*

$$\gamma_{000,000}, \qquad \gamma_{100,j00},\ j \neq 0, \qquad \gamma_{010,0k0},\ k \neq 0, \qquad \gamma_{001,00l},\ l \neq 0,$$
$$\gamma_{j00,0kl},\ j \neq 0,\ (k,l) \neq (0,0), \qquad \gamma_{0k0,00l},\ k,l \neq 0,$$

where $(j,k,l) \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$.

Proof We start by splitting off one of the three subgroups, *e.g.*, by writing $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} = \mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3})$. (Splitting off one of the other two subgroups leads to a different, but equally natural basis.) Now we proceed nearly verbatim as in the proof of Lemma A.2, the only difference being that $\mathbb{Z}_{N_2}$ is replaced by $\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. This means that k and k' are replaced by (k,l) and (k',l'), where $k, k' \in \mathbb{Z}_{N_2}$ and $l, l' \in \mathbb{Z}_{N_3}$. Then (A.2), (A.3) yield that all elements $\gamma_{jkl,j'k'l'}$ follow uniquely from the elements $\gamma_{000,000}, \gamma_{j00,j'00}, \gamma_{0kl,0k'l'}, \gamma_{j00,0kl}$. From the $\mathbb{Z}_{N_1}$-subgroup structure we know (Lemma A.1) that all elements $\gamma_{j00,j'00}$ follow uniquely from the elements $\gamma_{100,j00} (j \in \mathbb{Z}_{N_1})$. From the $\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$-subgroup structure we know (Lemma A.2) that all elements $\gamma_{0kl,0k'l'}$ follow uniquely from the elements $\gamma_{010,0k0}, \gamma_{001,00l}, \gamma_{0k0,00l}$.

Altogether we see that all elements of γ follow uniquely from the elements $\gamma_{000,000}$, $\gamma_{100,j00}, \gamma_{010,0k0}, \gamma_{001,00l}, \gamma_{j00,0kl}, \gamma_{0k0,00l}$. ∎

Lemma A.4 *For $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r}$, $(N_i = 2, 3, \dots)$, all elements of a G-graded contraction γ without zeroes follow uniquely from the $N_1 \cdot N_2 \cdots N_r$ elements*

$$\gamma_{0\cdots0,0\cdots0}, \quad \gamma_{10\cdots0,j_10\cdots0}\,(j_1 \neq 0), \quad \gamma_{010\cdots0,0j_20\cdots0}\,(j_2 \neq 0), \quad \dots \quad \gamma_{0\cdots01,0\cdots0j_r}\,(j_r \neq 0),$$
$$\gamma_{j_10\cdots0,0j_2j_3\cdots j_r} \quad (j_1 \neq 0,\ (j_2, j_3, \cdots j_r) \neq (0,0,\cdots 0)),$$
$$\gamma_{0j_20\cdots0,00j_3\cdots j_r} \quad (j_2 \neq 0,\ (j_3, \dots, j_r) \neq (0, \dots, 0)),$$
$$\dots \quad \gamma_{0\cdots0j_{r-1}0,0\cdots0j_r} \quad (j_{r-1}, j_r \neq 0),$$

where $j_i \in \mathbb{Z}_{N_i}$.

Proof We start by splitting off individual subgroups in the following way

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_r} = \mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_3} \times (\cdots \times (\mathbb{Z}_{N_{r-1}} \times \mathbb{Z}_{N_r}) \cdots))).$$

Then we use the proof of Lemma A.3 repeatedly. (Splitting the subgroups in a different order leads to a different, but equally natural, basis for G). ∎

For real $\mathbb{Z}_N$-graded contractions we need the following calculation (Lemma 4.14).

Lemma A.5 *We have for real $\mathbb{Z}_N$-graded contractions γ without zeroes where $N = 2M, (M = 1, 2, \dots)$*

$$\operatorname{sgn}(\gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,N-1}) = \operatorname{sgn}(\gamma_{00}\gamma_{MM}). \tag{A.4}$$

Proof Multiplying the defining equations (2.3) $\gamma_{1l}\gamma_{l+1,2M-l-1} = \gamma_{1,2M-l-1}\gamma_{l,2M-l}$ for $l = 1, 2, \dots, M-1$ yields

$$\begin{aligned}\gamma_{11}\gamma_{12}\cdots\gamma_{1,M-1}\gamma_{2,2M-2}\gamma_{3,2M-3}\cdots\gamma_{M-1,M+1}\gamma_{MM}\\ = \gamma_{1,2M-2}\gamma_{1,2M-3}\cdots\gamma_{1M}\gamma_{1,2M-1}\gamma_{2,2M-2}\gamma_{3,2M-3}\cdots\gamma_{M-1,M+1}.\end{aligned}$$

Since γ is without zeroes, we get

$$\gamma_{11}\gamma_{12}\cdots\gamma_{1,M-1}\gamma_{MM} = \gamma_{1,2M-1}\gamma_{1,2M-2}\cdots\gamma_{1,M+1}\gamma_{1M}.$$

Therefore, $\gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,N-1} = \gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,M-1}\gamma_{1M}\cdots\gamma_{1,2M-1}$ can be rewritten as $\gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,N-1} = \gamma_{00}\gamma_{MM}\gamma_{11}^2\gamma_{12}^2\cdots\gamma_{1,M-1}^2$. Thus, we have for a real γ without zeroes, $\operatorname{sgn}(\gamma_{00}\gamma_{11}\gamma_{12}\cdots\gamma_{1,N-1}) = \operatorname{sgn}(\gamma_{00}\gamma_{MM})$. ■

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