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# RAMSEY NUMBERS FOR TREES

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**Abstract**

For $n \geq 5$, let $T_n'$ denote the unique tree on $n$ vertices with $\Delta(T_n') = n-2$, and let $T_n^* = (V, E)$ be the tree on $n$ vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In this paper, we evaluate the Ramsey numbers $r(G_m, T_n')$ and $r(G_m, T_n^*)$, where $G_m$ is a connected graph of order $m$. As examples, for $n \geq 8$ we have $r(T_n', T_n^*) = r(T_n^*, T_n^*) = 2n-5$, for $n > m \geq 7$ we have $r(K_{1,m-1}, T_n^*) = m+n-3$ or $m+n-4$ according to whether $m-1 \mid n-3$ or $m-1 \nmid n-3$, and for $m \geq 7$ and $n \geq (m-3)^2+2$ we have $r(T_m^*, T_n^*) = m+n-3$ or $m+n-4$ according to whether $m-1 \mid n-3$ or $m-1 \nmid n-3$.

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## 1. Introduction

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in $G$ and let $\Delta(G)$ be the maximal degree of $G$. For a forbidden graph $L$, let $\mathrm{ex}(p; L)$ denote the maximal number of edges in a graph of order $p$ not containing $L$ as a subgraph. The corresponding Turán problem is to evaluate $\mathrm{ex}(p; L)$.

Let $\mathbb{N}$ be the set of positive integers, and let $p, n \in \mathbb{N}$ with $p \geq n \geq 3$. For a given tree $T_n$ on $n$ vertices, it is difficult to determine the value of $\mathrm{ex}(p; T_n)$. The famous Erdős–Sós conjecture asserts that $\mathrm{ex}(p; T_n) \leq (n-2)p/2$ for every tree $T_n$ on $n$ vertices. For progress on the Erdős–Sós conjecture, see [4, 8, 9, 11]. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Let $P_n$ be the path on $n$ vertices. In [5], Faudree and Schelp showed that

$$\mathrm{ex}(p; P_n) = k\binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}. \tag{1.1}$$

In the special case $r = 0$, (1.1) is due to Erdős and Gallai [3]. Let $K_{1,n-1}$ denote the unique tree on $n$ vertices with $\Delta(K_{1,n-1}) = n-1$, and for $n \geq 4$ let $T_n'$ denote the unique

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tree on $n$ vertices with $\Delta(T'_n) = n - 2$. In [10], the author and Lin-Lin Wang obtained exact values for $\mathrm{ex}(p; K_{1,n-1})$ and $\mathrm{ex}(p; T'_n)$; see Lemmas 2.4 and 2.5.

For $n \geq 5$ let $T^*_n = (V, E)$ be the tree on $n$ vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In [10], we also determined the value of $\mathrm{ex}(p; T^*_n)$; see Lemmas 2.6–2.8.

As usual, $\overline{G}$ denotes the complement of a graph $G$. Let $G_1$ and $G_2$ be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer $n$ such that, for every graph $G$ with $n$ vertices, either $G$ contains a copy of $G_1$ or else $\overline{G}$ contains a copy of $G_2$.

Let $n \in \mathbb{N}$ with $n \geq 6$. If the Erdős–Sós conjecture is true, it is known that $r(T_n, T_n) \leq 2n - 2$ (see [8]). Let $m, n \in \mathbb{N}$. In 1973, Burr and Roberts [2] showed that, for $m, n \geq 3$,

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases}$$

In 1995, Guo and Volkmann [6] proved that, for $n \geq m \geq 5$,

$$r(T'_m, T'_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-5 & \text{if } m = n \equiv 0 \pmod 2, \\ m+n-4 & \text{otherwise} \end{cases}$$

and, for $n > m \geq 4$,

$$r(K_{1,m-1}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1), \\ m+n-4 & \text{if } 2 \nmid m(n-1). \end{cases}$$

Let $m, n \in \mathbb{N}$ with $n \geq m \geq 6$. In this paper, we evaluate the Ramsey number $r(T_m, T^*_n)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}$. As examples, for $n \geq 8$,

$$r(P_n, T^*_n) = r(T^*_n, T^*_n) = 2n - 5;$$

for $n > m \geq 7$,

$$r(K_{1,m-1}, T^*_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3; \end{cases}$$

and, for $m \geq 7$ and $n \geq (m-3)^2 + 2$,

$$r(P_m, T^*_n) = r(T'_m, T^*_n) = r(T^*_m, T^*_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3. \end{cases}$$

In addition to the above notation, throughout the paper we also use the following notation: $\lfloor x \rfloor$ is the greatest integer not exceeding $x$, $K_n$ is the complete graph on $n$ vertices, $K_{m,n}$ is the complete bipartite graph with $m$ and $n$ vertices in the bipartition, $d_G(v)$ is the degree of the vertex $v$ in a given graph $G$, and $d(u, v)$ is the distance between the two vertices $u$ and $v$ in a graph.

## 2. Basic lemmas

LEMMA 2.1. *Let $G_1$ and $G_2$ be two graphs. Suppose that $p \in \mathbb{N}$, $p \geq \max\{|V(G_1)|, |V(G_2)|\}$ and $\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

PROOF. Let $G$ be a graph of order $p$. If $e(G) \leq \mathrm{ex}(p; G_1)$ and $e(\overline{G}) \leq \mathrm{ex}(p; G_2)$, then

$$\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}.$$

This contradicts the assumption. Hence, either $e(G) > \mathrm{ex}(p; G_1)$ or $e(\overline{G}) > \mathrm{ex}(p; G_2)$. Therefore, $G$ contains a copy of $G_1$ or $\overline{G}$ contains a copy of $G_2$. This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved. □

LEMMA 2.2. *Let $k, p \in \mathbb{N}$ with $p \geq k + 1$. Then there exists a $k$-regular graph of order $p$ if and only if $2 \mid kp$.*

This is a known result; see, for example, [10, Corollary 2.1].

LEMMA 2.3. *Let $G_1$ and $G_2$ be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then the following results hold.*

(i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.

(ii) *Suppose that $G_1$ is a connected graph of order $m$ and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.*

(iii) *Suppose that $G_1$ is a connected graph of order $m$ and $d_2 > m$. Then $r(G_1, G_2) \geq d_1 + d_2$ if one of the following conditions holds:*

  (1) $2 \mid d_1 + d_2 - m$;

  (2) $d_1 \neq m - 1$;

  (3) *$G_2$ has two vertices $u$ and $v$ such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$.*

PROOF. We first consider (i). If $2 \mid (d_1 - 1)(d_2 - 1)$, then $2 \mid (d_1 - 1)(d_1 + d_2 - 1)$. Since $d_1 - 1 \geq 1$, by Lemma 2.2 we may construct a $(d_1 - 1)$-regular graph $G$ of order $d_1 + d_2 - 1$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 1$, $G$ does not contain $G_1$ as a subgraph and $\overline{G}$ does not contain $G_2$ as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2$. Now we assume that $2 \nmid (d_1 - 1)(d_2 - 1)$. Then $2 \mid d_1$, $2 \mid d_2$ and so $2 \mid d_1 + d_2 - 2$. By Lemma 2.2, we may construct a $(d_1 - 1)$-regular graph $G$ of order $d_1 + d_2 - 2$. Since $\Delta(G) = d_1 - 1$ and $\Delta(\overline{G}) = d_2 - 2$, $G$ does not contain $G_1$ as a subgraph and $\overline{G}$ does not contain $G_2$ as a subgraph. Hence $r(G_1, G_2) \geq 1 + |V(G)| = d_1 + d_2 - 1$. This proves (i).

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Next we consider (ii). Suppose that $G_1$ is a connected graph of order $m$ and $d_1 < d_2 \le m$. Since $K_{d_2-1} \cup K_{d_2-1}$ does not contain any copies of $G_1$, and its complement $K_{d_2-1,d_2-1}$ does not contain any copies of $G_2$, we see that

$$r(G_1, G_2) \ge 1 + 2(d_2 - 1) = 2d_2 - 1 \ge d_1 + d_2.$$

This proves (ii).

Finally, we consider (iii). Suppose that $G_1$ is a connected graph of order $m$ and $d_2 > m$. By Lemma 2.2, we may construct a graph

$$G = \begin{cases} K_{m-1} \cup H_1 & \text{if } 2 \mid d_1 + d_2 - m, \\ K_{m-2} \cup H_2 & \text{if } 2 \nmid d_1 + d_2 - m, \end{cases}$$

where $H_1$ is a $(d_1 - 1)$-regular graph of order $d_1 + d_2 - m$ and $H_2$ is a $(d_1 - 1)$-regular graph of order $d_1 + d_2 - m + 1$. It is easily seen that $G$ does not contain any copies of $G_1$ and

$$\Delta(\overline{G}) = \begin{cases} d_2 - 1 & \text{if } 2 \mid d_1 + d_2 - m \text{ or } d_1 \ne m - 1, \\ d_2 & \text{if } 2 \nmid d_1 + d_2 - m \text{ and } d_1 = m - 1. \end{cases}$$

If $2 \mid d_1 + d_2 - m$ or $d_1 \ne m - 1$, then $\overline{G}$ does not contain any copies of $G_2$ and so $r(G_1, G_2) \ge 1 + |V(G)| = d_1 + d_2$. Now assume that $2 \nmid d_1 + d_2 - m$ and $d_1 = m - 1$. For $v_0 \in V(H_2)$, $d_{\overline{G}}(v_0) = d_2 - 1$. Suppose that $v_1, \ldots, v_{m-2} \in V(G)$ and $v_1, \ldots, v_{m-2}$ induce a copy of $K_{m-2}$. Then $\{v_1, \ldots, v_{m-2}\}$ is an independent set in $\overline{G}$ and $d_{\overline{G}}(v_i) = d_2$ for $i = 1, 2, \ldots, m - 2$. If $G_2$ has two vertices $u$ and $v$ such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$, we see that $G$ does not contain any copies of $G_2$ and so $r(G_1, G_2) \ge 1 + |V(G)| = d_1 + d_2$. This proves (iii) and the lemma is proved. □

Lemma 2.4 [10, Theorem 2.1]. *Let $p, n \in \mathbb{N}$ with $p \ge n - 1 \ge 1$. Then* $\mathrm{ex}(p; K_{1,n-1}) = \lfloor (n - 2)p/2 \rfloor$.

Lemma 2.5 [10, Theorem 3.1]. *Let $p, n \in \mathbb{N}$ with $p \ge n \ge 5$. Let $r \in \{0, 1, \ldots, n - 2\}$ be given by $p \equiv r \pmod{n - 1}$. Then*

$$\mathrm{ex}(p; T_n') = \begin{cases} \left\lfloor \dfrac{(n-2)(p-1) - r - 1}{2} \right\rfloor & \text{if } n \ge 7 \text{ and } 2 \le r \le n - 4, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 [10, Theorems 4.1–4.3]. *Let $p, n \in \mathbb{N}$ with $p \ge n \ge 6$, and let $p = k(n - 1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n - 5, n - 4, n - 3, n - 2\}$. Then*

$$\mathrm{ex}(p; T_n^*) = \begin{cases} \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n - 5, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

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Lemma 2.7 [10, Theorem 4.4]. *Let* $p, n \in \mathbb{N}$, $p \ge n \ge 11$, $r \in \{2, 3, \ldots, n-6\}$ *and* $p \equiv r \pmod{n-1}$. *Let* $t \in \{0, 1, \ldots, r+1\}$ *be given by* $n - 3 \equiv t \pmod{r+2}$. *Then*

$$\mathrm{ex}(p; T_n^*) = \begin{cases} \left\lfloor \dfrac{(n-2)(p-1) - 2r - t - 3}{2} \right\rfloor & \text{if } r \ge 4 \text{ and } 2 \le t \le r-1, \\ \dfrac{(n-2)(p-1) - t(r+2-t) - r - 1}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.8 [10, Theorem 4.5]. *Let* $p, n \in \mathbb{N}$ *with* $6 \le n \le 10$ *and* $p \ge n$, *and let* $r \in \{0, 1, \ldots, n-2\}$ *be given by* $p \equiv r \pmod{n-1}$.

(i) *If* $n = 6, 7$, *then* $\mathrm{ex}(p; T_n^*) = ((n-2)p - r(n-1-r))/2$.

(ii) *If* $n = 8, 9$, *then*

$$\mathrm{ex}(p; T_n^*) = \begin{cases} \dfrac{(n-2)p - r(n-1-r)}{2} & \text{if } r \ne n-5, \\ \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } r = n-5. \end{cases}$$

(iii) *If* $n = 10$, *then*

$$\mathrm{ex}(p; T_n^*) = \begin{cases} 4p - \dfrac{r(9-r)}{2} & \text{if } r \ne 4, 5, \\ 4p - 7 & \text{if } r = 5, \\ 4p - 9 & \text{if } r = 4. \end{cases}$$

Lemma 2.9. *Let* $p, m \in \mathbb{N}$ *with* $p \ge m \ge 5$, *and* $T_m \in \{P_m, K_{1,m-1}, T_m', T_m^*\}$. *Then* $\mathrm{ex}(p; T_m) \le (m-2)p/2$. *Moreover, if* $m - 1 \nmid p$ *and* $T_m \in \{P_m, T_m', T_m^*\}$, *then* $\mathrm{ex}(p; T_m) \le (m-2)(p-1)/2$.

Proof. This is immediate from (1.1) and Lemmas 2.4–2.8. □

Lemma 2.10. *Let* $m, n \in \mathbb{N}$ *with* $m, n \ge 5$. *Let* $G_m$ *be a connected graph on* $m$ *vertices. If* $m + n - 5 = (m-1)x + (m-2)y$ *for some nonnegative integers* $x$ *and* $y$, *then* $r(G_m, T_n) \ge m + n - 4$ *for* $T_n \in \{K_{1,n-1}, T_n', T_n^*\}$.

Proof. Let $G = xK_{m-1} \cup yK_{m-2}$. Then $|V(G)| = m + n - 5$, $\Delta(G) \le m - 1$ and $\Delta(\overline{G}) \le n - 3$. Clearly, $G$ does not contain $G_m$ as a subgraph, and $\overline{G}$ does not contain $T_n$ as a subgraph. So the result is true. □

Lemma 2.11 [7, Theorem 8.3, pp. 11–12]. *Let* $a, b, n \in \mathbb{N}$. *If* $a$ *is coprime to* $b$ *and* $n \ge (a-1)(b-1)$, *then there are two nonnegative integers* $x$ *and* $y$ *such that* $n = ax + by$.

Conjecture 2.12. Let $p, n \in \mathbb{N}$, $p \ge n \ge 5$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Let $T_n \ne K_{1,n-1}, T_n'$ be a tree on $n$ vertices. Then $\mathrm{ex}(p; T_n) \le \mathrm{ex}(p; T_n^*)$. Hence:

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(i) if $r \in \{0, 1, n-4, n-3, n-2\}$, then

$$\mathrm{ex}(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2};$$

(ii) if $2 \le r \le n-5$, then

$$\mathrm{ex}(p; T_n) \le \frac{(n-2)(p-1) - r - 1}{2}.$$

We note that

$$\mathrm{ex}(p; T_n) \ge e(kK_{n-1} \cup K_r) = \frac{(n-2)p - r(n-1-r)}{2} = \mathrm{ex}(p; P_n).$$

Definition 2.13. For $n \ge 5$ let $T_n$ be a tree on $n$ vertices. View $T_n$ as a bipartite graph with $s_1$ and $s_2$ vertices in the bipartition. Define $\alpha_2(T_n) = \max\{s_1, s_2\}$.

Conjecture 2.14. Let $p, n \in \mathbb{N}$ with $p \ge n \ge 5$. Let $T_n^{(1)}$ and $T_n^{(2)}$ be two trees on $n$ vertices. If $\alpha_2(T_n^{(1)}) < \alpha_2(T_n^{(2)})$, then $\mathrm{ex}(p; T_n^{(1)}) \le \mathrm{ex}(p; T_n^{(2)})$.

## 3. The Ramsey number $r(G_n, T_n^*)$

Lemma 3.1. *Let $n \in \mathbb{N}$, $n \ge 6$, and let $G_n$ be a connected graph on $n$ vertices such that $\mathrm{ex}(2n-5; G_n) < n^2 - 5n + 4$. Then $r(G_n, T_n^*) = 2n-5$.*

Proof. As $2K_{n-3}$ does not contain any copies of $G_n$ and $\overline{2K_{n-3}} = K_{n-3,n-3}$ does not contain any copies of $T_n^*$, we see that $r(G_n, T_n^*) > 2(n-3)$. By Lemma 2.6,

$$\mathrm{ex}(2n-5; T_n^*) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11.$$

Thus,

$$\begin{aligned} \mathrm{ex}(2n-5; G_n) + \mathrm{ex}(2n-5; T_n^*) &< n^2 - 5n + 4 + n^2 - 6n + 11 \\ &= 2n^2 - 11n + 15 = \binom{2n-5}{2}. \end{aligned}$$

Appealing to Lemma 2.1, we obtain $r(G_n, T_n^*) \le 2n-5$. So $r(G_n, T_n^*) = 2n-5$ as asserted. □

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n \ge 8$. Then*

$$r(P_n, T_n^*) = r(T_n', T_n^*) = r(T_n^*, T_n^*) = 2n-5.$$

Proof. By Lemma 2.6,

$$\mathrm{ex}(2n-5; T_n^*) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

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By Lemma 2.5,

$$\mathrm{ex}(2n-5;T_n') = \Big\lfloor \frac{(n-2)(2n-6)-(n-4)-1}{2} \Big\rfloor = \Big\lfloor n^2 - \frac{11}{2}n + \frac{15}{2} \Big\rfloor \le n^2 - \frac{11}{2}n + \frac{15}{2} < n^2 - 5n + 4.$$

By (1.1),

$$\mathrm{ex}(2n-5;P_n) = \binom{n-1}{2} + \binom{n-4}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

Thus, applying Lemma 3.1, we deduce the result. □

Conjecture 3.3. Let $n \in \mathbb{N}$, $n \ge 8$, and let $T_n \ne K_{1,n-1}$ be a tree on $n$ vertices. Then $r(T_n, T_n^*) = 2n - 5$.

Remark 3.4. Let $n \in \mathbb{N}$ with $n \ge 4$. From [6, Theorem 3.1(ii)] we know that $r(K_{1,n-1}, T_n^*) = 2n - 3$.

## 4. The Ramsey number $r(G_m, T_n^*)$ for $m < n$

Theorem 4.1. *Let $m, n \in \mathbb{N}$, $n > m \ge 5$ and $m - 1 \mid n - 3$. Let $G_m$ be a connected graph of order $m$ such that*

$$\mathrm{ex}(m+n-3;G_m) \le \frac{(m-2)(m+n-3)}{2} \quad \text{or} \quad G_m \in \{P_m, K_{1,m-1}, T_m', T_m^*\}.$$

*Then $r(G_m, T_n^*) = m + n - 3$.*

Proof. By Lemma 2.9 we may assume that $\mathrm{ex}(m+n-3;G_m) \le (m-2)(m+n-3)/2$. Suppose that $n - 3 = k(m-1)$. Clearly $(k+1)K_{m-1}$ does not contain $G_m$ as a subgraph and $\overline{(k+1)K_{m-1}}$ does not contain $T_n^*$ as a subgraph. Thus,

$$r(G_m, T_n^*) > (k+1)(m-1) = m + n - 4.$$

Since $1 \le m - 4 \le n - 6$, using Lemma 2.9 we see that

$$\mathrm{ex}(m+n-3;T_n^*) \le \frac{(n-2)(m+n-4)}{2}.$$

Thus,

$$\begin{aligned}\mathrm{ex}(m+n-3;G_m) + \mathrm{ex}(m+n-3;T_n^*) &\le \frac{(m-2)(m+n-3)}{2} + \frac{(n-2)(m+n-4)}{2} \\ &< \frac{(m-2+n-2)(m+n-3)}{2} = \binom{m+n-3}{2}.\end{aligned}$$

Hence, by Lemma 2.1, $r(G_m, T_n^*) \le m + n - 3$, and the result follows. □

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Lemma 4.2. *Let $m, n \in \mathbb{N}$, $n > m \geq 7$ and $m - 1 \nmid n - 3$. Let $G_m$ be a connected graph of order m such that*

$$\mathrm{ex}(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2} \quad or \quad G_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}.$$

*Then $r(G_m, T^*_n) \leq m + n - 4$.*

Proof. By Lemma 2.9, we may assume that

$$\mathrm{ex}(m+n-4; G_m) \leq (m-2)(m+n-4)/2.$$

As $m + n - 4 = n - 1 + m - 3$ and $m - 1 \nmid n - 3$, we see that $2 \leq m - 3 \leq n - 4$ and $m - 3 \neq n - 5$. Thus, applying Lemmas 2.6–2.8,

$$\mathrm{ex}(m+n-4; T^*_n) < \frac{(n-3)(m+n-4)}{2}.$$

Hence,

$$\mathrm{ex}(m+n-4; G_m) + \mathrm{ex}(m+n-4; T^*_n) < \frac{(m-2)(m+n-4)}{2} + \frac{(n-3)(m+n-4)}{2} = \binom{m+n-4}{2}.$$

Applying Lemma 2.1, we obtain the result. □

Theorem 4.3. *Let $m, n \in \mathbb{N}$, $n > m \geq 7$ and $m - 1 \nmid n - 3$. Let $G_m$ be a connected graph of order m such that*

$$\mathrm{ex}(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2} \quad or \quad G_m \in \{P_m, T'_m, T^*_m\}.$$

*If $m + n - 5 = (m-1)x + (m-2)y$ for some $x, y \in \{0, 1, 2, \ldots\}$, then $r(G_m, T^*_n) = m + n - 4$.*

Proof. By Lemma 4.2, $r(G_m, T^*_n) \leq m + n - 4$, and by Lemma 2.10, $r(G_m, T^*_n) \geq m + n - 4$. Thus the result follows. □

Theorem 4.4. *Suppose that $m, n \in \mathbb{N}$, $n > m \geq 7$, $n = k(m-1) + b = q(m-2) + a$, $k, q \in \mathbb{N}$, $a \in \{0, 1, \ldots, m-3\}$ and $b \in \{0, 1, \ldots, m-2\} - \{3\}$. Let $G_m$ be a connected graph of order m such that*

$$\mathrm{ex}(m+n-4; G_m) \leq \frac{(m-2)(m+n-4)}{2} \quad or \quad G_m \in \{P_m, T'_m, T^*_m\}.$$

*If one of the conditions:*

(i) $b \in \{1, 2, 4\}$;

(ii) $b = 0$ *and* $k \geq 3$;

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(iii) $n \geq (m-3)^2 + 2$;
(iv) $n \geq m^2 - 1 - b(m-2)$; *or*
(v) $a \geq 3$ *and* $n \geq (a-4)(m-1) + 4$

*holds, then* $r(G_m, T_n^*) = m + n - 4$.

PROOF. For $b \in \{1, 2, 4\}$,

$$m + n - 5 = \begin{cases} (k-2)(m-1) + 3(m-2) & \text{if } b = 1, \\ (k-1)(m-1) + 2(m-2) & \text{if } b = 2, \\ (k+1)(m-1) & \text{if } b = 4. \end{cases}$$

For $b = 0$ and $k \geq 3$, $m + n - 5 = (k-3)(m-1) + 4(m-2)$. For $n \geq (m-3)^2 + 2$, $m + n - 5 \geq (m-2)(m-3)$ and so $m + n - 5 = (m-1)x + (m-2)y$ for some $x, y \in \{0, 1, 2, \ldots\}$ by Lemma 2.11. For $n \geq m^2 - 1 - b(m-2)$, $k \geq m + 1 - b$ and

$$m + n - 5 = (k + b - m - 1)(m-1) + (m + 3 - b)(m-2).$$

For $a \geq 3$ and $n \geq (a-4)(m-1) + 4$, $q \geq a - 4$ and

$$m + n - 5 = (a-3)(m-1) + (q + 4 - a)(m-2).$$

Combining all the above with Theorem 4.3, we obtain the result. □

THEOREM 4.5. *Suppose that* $m, n \in \mathbb{N}$, $n > m \geq 7$ *and* $m - 1 \nmid n - 3$. *Then*

$$r(K_{1,m-1}, T_n^*) = m + n - 4,$$
$$r(T_m', T_n^*) = m + n - 4 \text{ or } m + n - 5,$$
$$m + n - 6 \leq r(T_m^*, T_n^*) \leq m + n - 4.$$

PROOF. From Lemma 4.2,

$$r(T_m, T_n^*) \leq m + n - 4 \quad \text{for } T_m \in \{K_{1,m-1}, T_m', T_m^*\}.$$

By Lemma 2.3,

$$r(K_{1,m-1}, T_n^*) \geq m - 1 + n - 3, \quad r(T_m', T_n^*) \geq m - 2 + n - 3 \quad (n > m + 1)$$

and

$$r(T_m^*, T_n^*) \geq m - 3 + n - 3.$$

By Theorem 4.4, $r(T_m', T_n^*) = m + n - 4$ for $n = 1, m + 3$. Thus the theorem is proved. □

THEOREM 4.6. *Suppose that* $m, n \in \mathbb{N}$, $n > m \geq 7$, $n = k(m-1) + b$, $k \in \mathbb{N}$, $b \in \{0, 1, \ldots, m-2\}$, $b \neq 3$ *and* $(m-b)/2 \leq k \leq m + 2 - b$. *Let* $G_m$ *be a connected graph of order* $m$ *such that* $\mathrm{ex}(m + n - 4; G_m) \leq \frac{1}{2}(m-2)(m+n-4)$ *or* $G_m \in \{P_m, T_m^*\}$. *Then* $r(G_m, T_n^*) = m + n - 4$ *or* $m + n - 5$.

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PROOF. By Lemma 4.2 we only need to show that $r(G_m, T_n^*) > m + n - 6$. Set

$$G = (2k + b - m)K_{m-2} \cup (m + 2 - b - k)K_{m-3}.$$

Then

$$|V(G)| = (2k + b - m)(m - 2) + (m + 2 - b - k)(m - 3) = m + n - 6.$$

We also have $\Delta(G) \le m - 2$ and $\Delta(\overline{G}) \le m + n - 6 - (m - 3) = n - 3$. Now it is clear that $G_m$ is not a subgraph of $G$ and that $T_n^*$ is not a subgraph of $\overline{G}$. So $r(G_m, T_n^*) > |V(G)|$, which completes the proof. □

REMARK 4.7. If $p \ge m \ge 6$ and $T_m$ is a tree on $m$ vertices with a vertex adjacent to at least $\lfloor (m - 1)/2 \rfloor$ vertices of degree one, in [9] Sidorenko proved that $\mathrm{ex}(p; T_m) \le (m - 2)p/2$. Thus, $G_m$ can be replaced by $T_m$ in Lemma 4.2 and Theorems 4.1, 4.3, 4.4 and 4.6.

## 5. The Ramsey number $r(G_m, T_n')$ for $m < n$

THEOREM 5.1. *Let $m, n \in \mathbb{N}$, $n > m \ge 6$ and $m - 1 \mid n - 3$. Suppose that $G_m$ is a connected graph of order $m$ satisfying*

$$\mathrm{ex}(m + n - 3; G_m) \le \frac{(m - 2)(m + n - 3) + m + n - 4}{2} \quad \text{or} \quad G_m \in \{T_m^*, P_m\}.$$

*Then $r(G_m, T_n') = m + n - 3$.*

PROOF. By Lemma 2.9 we may assume that

$$\mathrm{ex}(m + n - 3; G_m) \le (m - 2)(m + n - 3)/2 + (m + n - 4)/2.$$

Suppose that $n - 3 = k(m - 1)$ and $G = (k + 1)K_{m-1}$. Then $|V(G)| = m + n - 4$ and $\Delta(\overline{G}) = n - 3$. Clearly, $G_m$ is not a subgraph of $G$ and $T_n'$ is not a subgraph of $\overline{G}$. Thus $r(G_m, T_n') > m + n - 4$. Since $m - 1 \mid n - 3$, $n \ge m + 2$ and so $4 \le m - 2 \le n - 4$. Hence, using Lemma 2.5,

$$\begin{aligned}\mathrm{ex}(m + n - 3; T_n') &= \left\lfloor \frac{(n - 2)(m + n - 4) - (m - 1)}{2} \right\rfloor \\ &< \frac{(n - 2)(m + n - 3) - (m + n - 4)}{2}.\end{aligned}$$

Therefore

$$\mathrm{ex}(m + n - 3; G_m) + \mathrm{ex}(m + n - 3; T_n') < \binom{m + n - 3}{2}.$$

Applying Lemma 2.1, we see that $r(G_m, T_n') \le m + n - 3$, so the result follows. □

LEMMA 5.2. *Let $m, n \in \mathbb{N}, n > m \ge 6$ and $m - 1 \nmid n - 3$. Suppose that $G_m$ is a connected graph of order $m$ satisfying $\mathrm{ex}(m + n - 4; G_m) < (m - 2)(m + n - 4)/2$ or $G_m \in \{T_m^*, P_m\}$. Then $r(G_m, T_n') \le m + n - 4$.*

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Proof. Since $m-1 \nmid n-3$, $m-1 \nmid m+n-4$. Thus, applying Lemma 2.9,

$$\mathrm{ex}(m+n-4; T_m^*) \le (m-2)(m+n-5)/2$$

and

$$\mathrm{ex}(m+n-4; P_m) \le (m-2)(m+n-5)/2.$$

As $n > m$, $3 \le m-3 \le n-4$. By Lemma 2.5,

$$\begin{aligned}\mathrm{ex}(m+n-4; T_n') &= \left\lfloor \frac{(n-2)(m+n-5)-(m-2)}{2} \right\rfloor \\ &\le \frac{(n-2)(m+n-5)-(m-2)}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{ex}(m+n-4; G_m) + \mathrm{ex}(m+n-4; T_n') &< \frac{(m-2+n-2)(m+n-5)}{2} \\ &= \binom{m+n-4}{2}.\end{aligned}$$

This, together with Lemma 2.1, yields the result. □

Theorem 5.3. *Let $m, n \in \mathbb{N}$, $n > m \ge 6$ and $m-1 \nmid n-3$. Then $r(T_m^*, T_{m+1}') = 2m-3$ and $r(T_m^*, T_n') = m+n-4$ or $m+n-5$ for $n \ge m+3$. Suppose that $G_m$ is a connected graph of order $m$ satisfying $\mathrm{ex}(m+n-4; G_m) < (m-2)(m+n-4)/2$ or $G_m \in \{T_m^*, P_m\}$. If $m+n-5 = (m-1)x + (m-2)y$ for some nonnegative integers $x$ and $y$, then $r(G_m, T_n') = m+n-4$.*

Proof. By Lemma 2.3,

$$r(T_m^*, T_{m+1}') \ge 2(m-1)-1 = 2m-3$$

and

$$r(T_m^*, T_n') \ge m-3+n-2 \quad \text{for } n \ge m+3.$$

By Lemma 5.2, $r(G_m, T_n') \le m+n-4$. Thus, $r(T_m^*, T_{m+1}') = 2m-3$. Applying Lemma 2.10, we deduce the remaining result. □

From Theorem 5.3 and the proof of Theorem 4.4 we deduce the following result.

Theorem 5.4. *Suppose that $m, n \in \mathbb{N}$, $n > m \ge 6$, $n = k(m-1)+b = q(m-2)+a$, $k, q \in \mathbb{N}$, $a \in \{0, 1, \ldots, m-3\}$ and $b \in \{0, 1, \ldots, m-2\} - \{3\}$. Let $G_m$ be a connected graph of order $m$ such that $\mathrm{ex}(m+n-4; G_m) < (m-2)(m+n-4)/2$ or $G_m \in \{P_m, T_m^*\}$. If one of the conditions:*

(i) $b \in \{1, 2, 4\}$;
(ii) $b = 0$ *and* $k \ge 3$;
(iii) $n \ge (m-3)^2 + 2$;
(iv) $n \ge m^2 - 1 - b(m-2)$; *or*
(v) $a \ge 3$ *and* $n \ge (a-4)(m-1)+4$

*holds, then $r(G_m, T_n') = m+n-4$.*

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## 6. The Ramsey number $r(T_m, K_{1,n-1})$ for $m < n$

The following two propositions are known.

Proposition 6.1 [1]. *Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $m-1 | n-2$. Let $T_m$ be a tree on $m$ vertices. Then $r(T_m, K_{1,n-1}) = m+n-2$.*

Proposition 6.2 [6, Theorem 3.1]. *Let $m, n \in \mathbb{N}$, $m \geq 3$ and $n = k(m-1)+b$ with $k \in \mathbb{N}$ and $b \in \{0, 1, \ldots, m-2\} - \{2\}$. Let $T_m \neq K_{1,m-1}$ be a tree on $m$ vertices. Then $r(T_m, K_{1,n-1}) \leq m+n-3$. Moreover, if $k \geq m-b$, then $r(T_m, K_{1,n-1}) = m+n-3$.*

Theorem 6.3. *Let $m, n \in \mathbb{N}$, $n \geq m \geq 3$, $m-1 \nmid n-2$, $n = q(m-2)+a$, $q \in \mathbb{N}$ and $a \in \{2, 3, \ldots, m-3\}$. Let $T_m \neq K_{1,m-1}$ be a tree on $m$ vertices. If $n \geq (a-3)(m-1)+3$, then $r(T_m, K_{1,n-1}) = m+n-3$.*

Proof. Since $q(m-2) = n-a \geq (a-3)(m-2)$, $q \geq a-3$. Set $G = (a-2)K_{m-1} \cup (q-(a-3))K_{m-2}$. Then

$$|V(G)| = (a-2)(m-1) + (q-(a-3))(m-2) = m+n-4 \quad \text{and} \quad \Delta(\overline{G}) \leq n-2.$$

Clearly, $T_m$ is not a subgraph of $G$ and $K_{1,n-1}$ is not a subgraph of $\overline{G}$. Thus $r(T_m, K_{1,n-1}) > |V(G)| = m+n-4$. By Proposition 6.2, $r(T_m, K_{1,n-1}) \leq m+n-3$. So $r(T_m, K_{1,n-1}) = m+n-3$. This proves the theorem. □

Theorem 6.4. *Let $m, n \in \mathbb{N}$ with $n > m \geq 5$ and $m-1 \nmid n-2$. Then $r(T_m^*, K_{1,n-1}) = m+n-3$ or $m+n-4$. Moreover, if $m+n-4 = (m-1)x + (m-2)y + 2(m-3)z$ for some nonnegative integers $x$, $y$ and $z$, then $r(T_m^*, K_{1,n-1}) = m+n-3$.*

Proof. By Proposition 6.2, $r(T_m^*, K_{1,n-1}) \leq m+n-3$. By Lemma 2.3, $r(T_m^*, K_{1,n-1}) \geq m+n-4$. If $m+n-4 = (m-1)x + (m-2)y + 2(m-3)z$ for some nonnegative integers $x$, $y$ and $z$, setting $G = xK_{m-1} \cup yK_{m-2} \cup zK_{m-3,m-3}$ we find that $\Delta(\overline{G}) \leq n-2$. Clearly, $G$ does not contain any copies of $T_m^*$, and $\overline{G}$ does not contain any copies of $K_{1,n-1}$. Thus, $r(T_m^*, K_{1,n-1}) > |V(G)| = m+n-4$ and so $r(T_m^*, K_{1,n-1}) = m+n-3$. This proves the theorem. □

## References


[1] S. A. Burr, 'Generalized Ramsey theory for graphs—a survey', in: *Graphs and Combinatorics*, Lecture Notes in Mathematics, 406 (eds. R.A. Bari and F. Harary) (Springer, Berlin–New York, 1974), pp. 52–75.

[2] S. A. Burr and J. A. Roberts, 'On Ramsey numbers for stars', *Util. Math.* **4** (1973), 217–220.

[3] P. Erdős and T. Gallai, 'On maximal paths and circuits in graphs', *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356.

[4] G. H. Fan and L. L. Sun, 'The Erdős–Sós conjecture for spiders', *Discrete Math.* **307** (2007), 3055–3062.

[5] R. J. Faudree and R. H. Schelp, 'Path Ramsey numbers in multicolorings', *J. Combin. Theory Ser. B* **19** (1975), 150–160.

[6] Y. B. Guo and L. Volkmann, 'Tree-Ramsey numbers', *Aust. J. Combin.* **11** (1995), 169–175.

[7] L. K. Hua, *Introduction to Number Theory* (Springer, Berlin, 1982).
[8] S. P. Radziszowski, 'Small Ramsey numbers', *Dynamic Surveys of Electronic J. Combinatorics* (2011), DS1.13, 84 pp.
[9] A. F. Sidorenko, 'Asymptotic solution for a new class of forbidden $r$-graphs', *Combinatorica* **9** (1989), 207–215.
[10] Z. H. Sun and L. L. Wang, 'Turán's problem for trees', *J. Combin. Number Theory* **3** (2011), 51–69.
[11] M. Woźniak, 'On the Erdős–Sós conjecture', *J. Graph Theory* **21** (1996), 229–234.

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