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An adjoint-functor theorem over topoi

B.J. Day

The usual statements of the classical adjoint-functor theorems contain the hypothesis that the codomain category should admit arbitrary intersections of families of monomorphisms with a common codomain. The aim of this article is to formulate an adjoint-functor theorem which refers, in a similar manner, to arbitrary internal intersections of "families of monomorphisms" in the case where the categories under consideration are suitably defined relative to a fixed elementary base topos (in the usual sense of Lawvere and Tierney).

Introduction

The aim of this article is to formulate a suitable context in which to establish the adjoint-functor theorem based on internal intersection in an elementary topos. This is done in Section 1, and the theorem proved in Section 2 generalises a form of the adjoint-functor theorem ([*1*], Theorem 2.1) which, under additional completeness hypotheses, contains Freyd's original adjoint functor theorems (as given in [*6*], Chapter V, 6-8). It is closely related to an extension of the adjoint-functor theorem due to Mikkelsen which serves to describe the free E-locale on an object in an elementary topos E.

The references for basic theory and notation are Eilenberg and Kelly [*2*], Lawvere [*5*], and Mac Lane [*6*].

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1. Categories over a topos

Throughout this section we suppose that E is a fixed elementary topos with subobject representor Ω and that all categorical algebra is *relative* to E. We denote by $\hat{E}$ the category of ordered objects in E (see [4], 1.2).

A 2-category $E\text{-}Cat$ is constructed as follows. A 0-*cell* of $E\text{-}Cat$ is a category C together with a functor $M : C^{op} \to \hat{E}$ and a natural transformation $\phi : M \to \Omega : C^{op} \to \hat{E}$ called "factorisation". By the representation theorem the components of $\phi_C : MC \to \Omega$ of ϕ yield a natural transformation:

$$C(C, D) \times MD \to \Omega \, .$$

Thus we obtain a family:

$$\Phi = \Phi_{CD} \rightarrowtail C(C, D) \times MD$$

of monomorphisms in E ; the "elements" of Φ_{CD} are thought of as "pairs" (f, m) such that f factors through m .

PROPOSITION 1.1. *If* $\gamma : C(C, D) \times MD \to \Omega$ *denotes the canonical transformation*

$$C(C, D) \times MD \to MC \xrightarrow{\phi_C} \Omega \, ,$$

then the diagram

$$\begin{array}{ccc} \Phi & \longrightarrow & C(C,D)\times MD \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \gamma} \\ 1 & \underset{t}{\longrightarrow} & \Omega \end{array}$$

is a pullback.

Proof. This is immediate from the definition of Φ and the representation theorem. //

A 1-*cell* of $E\text{-}Cat$ from (C, M, ϕ) to (B, N, ψ) is a functor $T : C \to B$ together with a structure transformation

$$\tau : M \Rightarrow NT^{op} : C^{op} \to \hat{E} \, .$$

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A *2-cell* $\alpha : (T, \tau) \Rightarrow (S, \sigma)$ is a natural transformation $\alpha : T \Rightarrow S$ such that $\tau = (N\alpha^{\mathrm{op}})\cdot\sigma$.

With these definitions we see that the topos $\mathcal{E}$ is itself a 0-cell with $MC = [C, \Omega]$ and $\phi_C = \forall u_C : [C, \Omega] \to \Omega$. It then follows that, for each 0-cell $(\mathcal{C}, M, \phi)$, each representable functor $\mathcal{C}(K, -) : \mathcal{C} \to \mathcal{E}$ is a 1-cell with structure

$$MD \to [\mathcal{C}(K, D), \Omega]$$

derived from

$$\mathcal{C}(K, D) \times MD \to \Omega$$

by adjunction.

We also note that the constant functor $\mathcal{C} \to \mathcal{E}$ which sends C to 1 is a 1-cell with structure $\phi_C : MC \to \Omega \cong [1, \Omega]$.

Let $E = \mathcal{E}_0(1, -) : \mathcal{E} \to \mathrm{Ens}$. An $\mathcal{E}$-category $(\mathcal{C}, M, \phi)$ is said to be MR (mono representable) if there exists a *subcategory* M_0 of $\mathcal{E}$-monomorphisms in $\mathcal{C}$ such that:

MR1. There is a natural *bijection* between morphisms $1 \to MD$ ("global sections" of MD = elements of EMD) and M_0-monomorphisms $B \rightarrowtail D$; strictly speaking of course the bijection is with equivalence classes of M_0-monomorphisms with codomain D .

MR2. Each diagram

$$\begin{array}{ccc} M(f)(m) & \dashrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[f]{} & D \end{array},$$

with $m \in M_0$, admits completion to a *pullback* diagram in $\mathcal{C}$; that is, $M(f)(m) = f^{-1}m$.

A 1-cell $T : (\mathcal{C}, M, \phi) \to (\mathcal{B}, N, \psi)$ between MR-categories $\mathcal{C}$ and $\mathcal{B}$ is called MR if the transformation $E\tau : EMC \to ENTC$ is induced by T .

An $\mathcal{E}$-category $(\mathcal{C}, M, \phi)$ is called CMR (completely mono representable) if it is MR and it satisfies the following conditions for each $C \in \mathcal{C}$:

CMR1. each MC is a complete lattice in $\hat{\mathcal{E}}$;

CMR2. the square

$$\begin{array}{ccc} 1 & \xrightarrow{1} & 1 \\ {\scriptstyle 1_C}\downarrow & & \downarrow{\scriptstyle t} \\ MC & \xrightarrow[\phi_C]{} & \Omega \end{array}$$

is a pullback;

CMR3. given any monomorphism $i : B \rightarrowtail MC$ and morphism $f : 1 \to \mathcal{C}(C, D)$, if there exists a factorisation

$$\begin{array}{ccc} & & B \\ & \swarrow & \downarrow{\scriptstyle f\times i} \\ \Phi & \rightarrowtail & \mathcal{C}(C,D)\times MD \end{array}$$

then there exists a factorisation

$$\begin{array}{ccc} & & 1 \\ & \swarrow & \downarrow{\scriptstyle f\times \inf B} \\ \Phi & \rightarrowtail & \mathcal{C}(C,D)\times MD \ . \end{array}$$

PROPOSITION 1.2. *If* $(\mathcal{C}, M, \phi)$ *is CMR then the set map* $E\Phi \to E\mathcal{C}(C, D) \times EMD$ *is a bijection onto the set of all pairs* (f, m) *such that* $m \in M_0$ *and* f *factors through* m .

Proof. Because E is representable the diagram

$$\begin{array}{ccc} E\Phi & \longrightarrow & E\,(C,D)\times EMD \\ \downarrow & & \downarrow{\scriptstyle E\gamma} \\ * & \xrightarrow[1_C]{} & EMC \end{array}$$

is a pullback by Proposition 1.1 and CMR2. Thus $E\Phi$ is equivalent to the set of all pairs (f, m) , $m \in M_0$, such that $M(f)(m) = 1_C$. But, by MR2,

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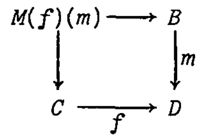
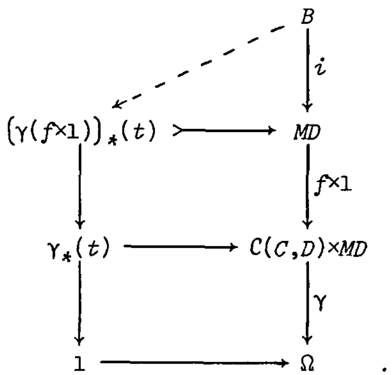


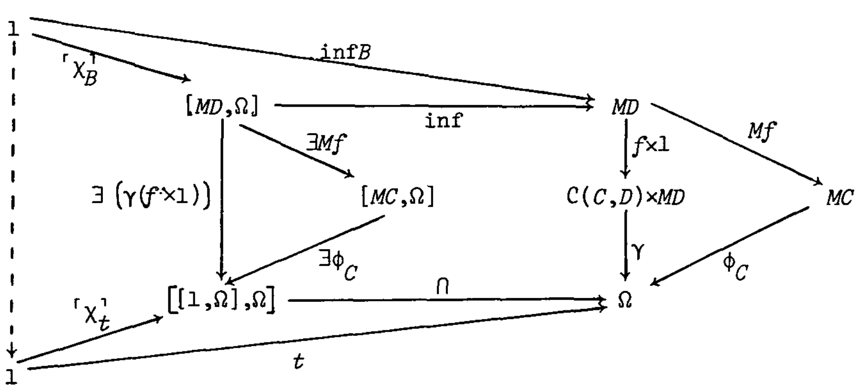
is a pullback so $M(f)(m) = 1_C$ if and only if f factors through m . //

PROPOSITION 1.3. *If* (C, M, ϕ) *is MR and satisfies* CMR1 *then* CMR3 *is satisfied if* $Mf : MC \to MD$ *preserves* inf *for all* $f \in C_0(C, D)$ *and* $\phi_C : MC \to \Omega$ *preserves* inf *for all* $C \in C$.

Proof. By Proposition 1.1, $\Phi = \gamma_*(t)$.



Thus, if $B \le \gamma_*(t)$ then $B \le (\gamma(f \times 1))_*(t)$ so $\exists(\gamma(f \times 1))(B) \le t$. The proof then follows from considering the diagram:



in which $\cap \ulcorner \chi_t \urcorner = t$ since $\cap\{\cdot\} = 1$. //

COROLLARY 1.4. *The elementary topos* E *is itself CMR.*

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Proof. Each object $[X, \Omega]$, $X \in E$, is a complete lattice object in $\hat{E}$ and

$$\begin{array}{ccc} 1 & \xrightarrow{1} & 1 \\ {\scriptstyle \ulcorner\chi\urcorner_X}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ [X,\Omega] & \xrightarrow[\forall u_X]{} & \Omega \end{array}$$

is a pullback diagram. Moreover $\phi_X = \forall u_X$ is inf preserving since $u_* \dashv \forall u$ and each $[f, \Omega]$ has left adjoint $\exists f$ hence is inf preserving. Thus the result follows from Proposition 1.3. //

2. The adjoint-functor theorem

This section is devoted to the proof of the main theorem. Again we suppose that, unless otherwise stipulated, the categorical algebra is *relative* to a fixed elementary topos E .

We suppose that $T : (C, M, \phi) \to (B, N, \psi)$ is an MR E-functor and that B is CMR. Furthermore, we suppose that there exists a "bounding" family $\{\beta_B : B \to TC(B)\}$ of morphisms in B_0 such that for all $C \in C$ and $f \in B_0(B, TC)$ there exists a commuting square:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & TC(B) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Tg} \\ TC & \xrightarrow[Tm]{} & TD \end{array}$$

with $m \in M_0$.

THEOREM 2.1. *Under the above hypotheses on* T *the functor* $T_0 : C_0 \to B_0$ *has an ordinary (*Ens*-based) left adjoint if* C *is M-complete in the sense that*

(a) MC *is a complete lattice for each* $C \in C$,

(b) C_0 *has pullbacks of* M_0*-subobjects and they lie in* M_0 ,

(c) C_0 *has equalisers and they lie in* M_0 , *and* T *is*

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M-continuous in the sense that T_0 preserves pullbacks of M_0-subobjects and equalisers and E applied to

$$\begin{array}{ccc} [MC,\Omega] & \xrightarrow{\text{inf}} & MC \\ {\scriptstyle \exists\tau_C}\downarrow & & \downarrow{\scriptstyle \tau_C} \\ [NTC,\Omega] & \xrightarrow{\text{inf}} & NTC \end{array}$$

commutes.

Proof. First form the pullback

$$\begin{array}{ccc} P(\beta) & \longrightarrow & 1\times MC(B) \\ \downarrow & (*) & \downarrow{\scriptstyle \ulcorner\beta\urcorner\times\tau_{C(B)}} \\ \Phi & \longrightarrow & \mathcal{B}\big(B,TC(B)\big)\times NTC(b) \end{array}$$

for each β_B in the bounding family. An M_0-monomorphism $i : SB \to C(B)$ is then defined by

$$\begin{array}{ccc} & 1 & \\ {\scriptstyle \ulcorner\chi_P\urcorner}\swarrow & & \searrow{\scriptstyle \text{inf}P(\beta)} \\ [MC(B),\Omega] & \xrightarrow{\text{inf}} & MC(B) \end{array} .$$

LEMMA 2.2. *If*

$$\begin{array}{ccc} P & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[h]{} & C \end{array}$$

is a pullback diagram in $\mathcal{E}$ *then there exists a factorisation*

$$\begin{array}{ccc} & & D \\ & \swarrow & \downarrow{\scriptstyle \exists f\ulcorner\chi_P\urcorner} \\ Q & \xrightarrow[h]{} & C \end{array} .$$

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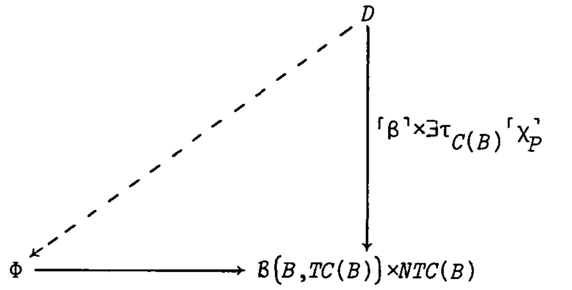
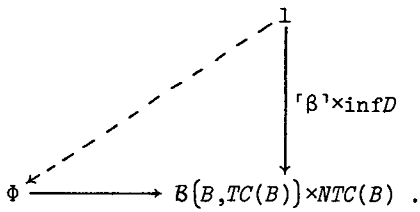
Proof. $\exists f \dashv f_*$ so $\exists f \ulcorner \chi_P \urcorner \leq Q$ if and only if $P \leq f_* Q$. But $P = f_* Q$; hence $\exists f \ulcorner \chi_P \urcorner \leq Q$. //

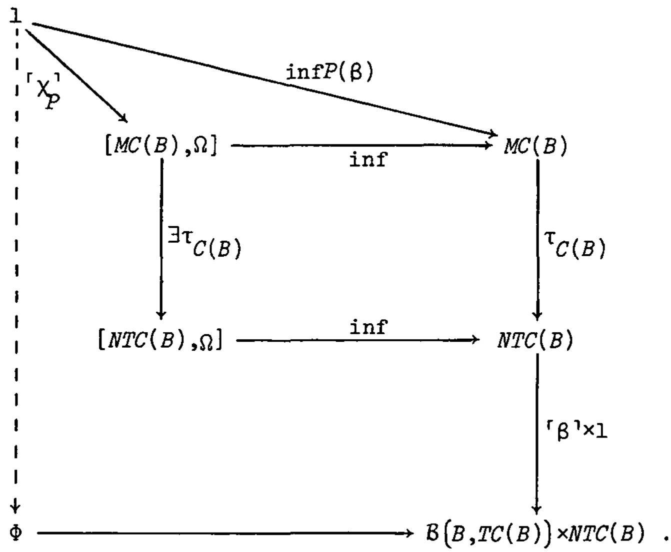
From the pullback diagram (*) we obtain, by Lemma 2.2, a factorisation



Because $\mathcal{B}$ is assumed CMR this factorisation gives



Because T is M-continuous this gives



Thus we obtain a factorisation

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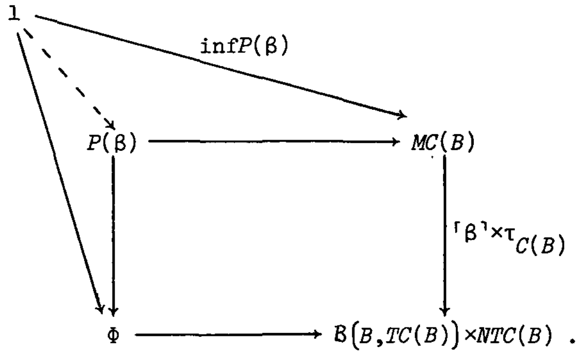
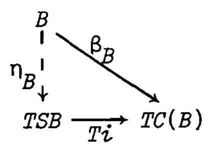
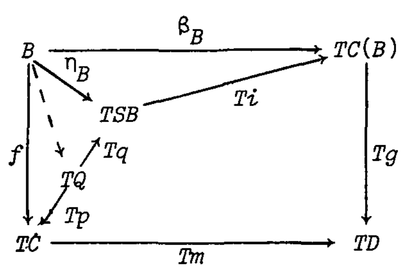


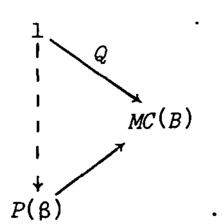
By Proposition 1.2 this implies that, on applying E we obtain a factorisation



where Ti is a monomorphism because T is MR, so η_B is well defined. Finally, to verify that $\eta_B : B \to TSB$ is the required universal arrow, we consider



with $m \in M_0$. Let Q be the pullback of m along gi . Clearly $Q \leq SB$ in $EMC(B)$. Also $Q \in EP(\beta)$; thus $Q \geq SB$:



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Hence $Q \cong SB$ as subobjects of $C(B)$. Similarly, equalisers can be used to show that factorisation of the required type through η_B is unique. //

REMARK 2.3. The adjunction $S \dashv T_0 : C_0 \to B_0$ can be enriched to an E-adjunction if C has cotensoring over E and T preserves this cotensoring (see [3]).

3. Examples

EXAMPLE 3.1 (Mikkelsen). Let C be the E-category of E-locales and let $U : C \to E$ be the underlying-E-object functor. Then there exists a "bounding" functor $R : E \to C$ given by

$$RX = \big[[X, \Omega], \Omega\big]$$

with bounding unit

$$\beta_X : X \to U\big[[X, \Omega], \Omega\big]$$

the canonical "evaluation" transformation. If $X = UA$ where A is an E-locale then

$$\beta_{UA} : UA \to U\big[[UA, \Omega], \Omega\big]$$

is $Um : UA \to U\big[[UA, \Omega], \Omega\big]$ where m has the left-exact left adjoint $\sup : \big[[UA, \Omega], \Omega\big] \to UA$. Thus, by Theorem 2.1, U has a left adjoint. This left adjoint describes the free E-locale on each object $x \in E$.

EXAMPLE 3.2. Suppose E and E' are elementary topoi and $T : E \to E'$ is a functor which preserves finite limits and which, as a closed functor $T = \big(T, \hat{T}, T^0\big)$ is normal in the sense that the canonical transformation $E \Rightarrow E'T$ is an isomorphism.

We can consider T_*E as an E'-category with

$$|T_*E| = |E| ,$$

$$T_*E(X, Y) = T[X, Y] .$$

Moreover, T_*E is an E'-category with

$$M : (T_*E)^{op} \to E'$$

given by $MX = T[X, \Omega]$ and $\phi : M \to \Omega'$ given by

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$$\phi_X : T[X, \Omega] \xrightarrow{T\chi_X} T\Omega \xrightarrow{\chi_{Tt}} \Omega' .$$

The functor $T : T_*E \to E'$ is an E'-functor with $\tau_X : MX \to NTX$ given by

$$\tau_X : T[X, \Omega] \xrightarrow{\hat{T}} [TX, T\Omega] \xrightarrow{[1,\chi_{Tt}]} [TX, \Omega'] .$$

Both T_*E and E' are CMR and (T, τ) is MR by normality of T and the fact that T is assumed to preserve finite limits

$$\begin{array}{ccccc} TY & \longrightarrow & T1 & \xrightarrow{\cong} & 1' \\ \downarrow & & \downarrow Tt & & \downarrow t' \\ TX & \xrightarrow[T\chi_Y]{} & T\Omega & \xrightarrow[\chi_{Tt}]{} & \Omega' \end{array} .$$

The functor $T : T_*E \to E'$ then has a left adjoint if E' applied to

$$\begin{array}{ccc} [T[X,\Omega],\Omega'] & \xrightarrow{\inf} & T[X,\Omega] \\ \exists\tau_X \downarrow & & \downarrow \tau_X \\ [[TX,\Omega'],\Omega'] & \xrightarrow[\inf]{} & [TX,\Omega'] \end{array}$$

commutes and T has a bounding family of morphisms. It has a left-adjoint E'-functor if T_*E is cotensored and T preserves this cotensoring. In particular T_*E is cotensored if T is the left-adjoint part of a geometrical morphism of topoi (see [3], 5).

EXAMPLE 3.3. Suppose $S \dashv T : E' \to E$ is a geometrical morphism of topoi. Then, as in Example 3.2, we obtain T_*E' as an E-category and we obtain, by Kelly [3], 5, an E-adjunction

$$(\varepsilon, \eta) : S \dashv T : T_*E' \to E .$$

The E-category T_*E' is cotensored over E by Kelly [3], 5.1, with $[X, X'] = [SX, X']'$ and to say that the induced E-adjoint $S : E \to T_*E'$ preserves this cotensoring is to say that $S[X, Y] \cong [SX, SY]'$; note that S is not necessarily a normal closed functor, so this does not always imply that $S_0 : E_0 \to (T_*E')_0$ is a full embedding.

The E-category T_*E' is a CMR E-category with $NX' = T[X', \Omega']'$. Moreover S has structure

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$$\sigma : [X, \Omega] \to T[SX, \Omega']' \cong [X, T\Omega']$$

given by

$$\Omega \xrightarrow{\eta_\Omega} TS\Omega \xrightarrow{T\chi_{St}} T\Omega' .$$

To see that (S, σ) is an MR-functor let Y be an arbitrary subobject of X and note that the diagram

$$\begin{array}{ccccccc}
Y & \longrightarrow & 1 & \longrightarrow & & \longrightarrow & T1' \\
\downarrow & & \downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle Tt'} \\
X & \xrightarrow[\chi_Y]{} & \Omega & \xrightarrow[\eta_\Omega]{} & TS\Omega & \xrightarrow[T\chi_{St}]{} & T\Omega'
\end{array}$$

transforms to

$$\begin{array}{ccccccccc}
SY & \longrightarrow & S1 & \longrightarrow & & \longrightarrow & & \longrightarrow & 1' \\
\downarrow & & \downarrow{\scriptstyle St} & & & & & & \downarrow{\scriptstyle t'} \\
SX & \xrightarrow[S\chi_Y]{} & S\Omega & \xrightarrow[S\eta_\Omega]{} & STS\Omega & \xrightarrow[ST\chi_{St}]{} & ST\Omega' & \xrightarrow[\varepsilon_{\Omega'}]{} & \Omega'
\end{array}$$

which becomes

$$\begin{array}{ccccc}
SY & \longrightarrow & S1 & \longrightarrow & 1' \\
\downarrow & & \downarrow{\scriptstyle St} & & \downarrow{\scriptstyle t'} \\
SX & \xrightarrow[S\chi_Y]{} & S\Omega & \xrightarrow[\chi_{St}]{} & \Omega'
\end{array} ,$$

and use the fact that S preserves finite limits.

Because E has Ω as a strong E-cogenerator we obtain a bounding family of morphisms

$$\beta_{X'} : X' \to [T_* E'(X', S\Omega), S\Omega] = [ST[X', S\Omega]', S\Omega]'$$

for S with the property that if S preserves cotensoring, the diagram

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$$\begin{array}{ccc}
X' & \xrightarrow{\beta_{X'}} & [ST[X',S\Omega]',S\Omega]' \\
 & & \wr\| \\
 & & S[T[X',S\Omega]',\Omega] \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle S[T[f,1]',1]} \\
 & & S[T[SX,S\Omega'],\Omega] \\
 & & \downarrow{\scriptstyle S[S,1]} \\
SX & \xrightarrow[Sm]{} & S[[X,\Omega],\Omega]
\end{array}$$

commutes for all $f \in E_0'(X', SX)$, where $m : X \to [[X, \Omega], \Omega]$ is the canonical monomorphism in E.

This gives us the result that $S : E \to T_* E'$ has a left E-adjoint if S preserves cotensoring and E applied to the diagram

$$\begin{array}{ccc}
[[X,\Omega],\Omega] & \xrightarrow{\text{inf}} & [X,\Omega] \\
{\scriptstyle \exists\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\
[[X,T\Omega'],\Omega] & \xrightarrow{\text{inf}} & [X,T\Omega']
\end{array}$$

commutes.

References

[1] Brian Day, "On adjoint-functor factorisation", *Category Seminar*, 1-19 (Proc. Sydney Category Theory Seminar 1972/1973. Lecture Notes in Mathematics, 420. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[2] Samuel Eilenberg and G. Max Kelly, "Closed categories", *Proc. Conf. Categorical Algebra*, 421-562 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[3] G.M. Kelly, "Adjunction for enriched categories", *Reports of the Midwest Category Seminar* III, 166-177 (Lecture Notes in Mathematics, 106. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

https://doi.org/10.1017/S0004972700022814 Published online by Cambridge University Press

[4] G.M. Kelly and Ross Street, "The elements of topoi", *Abstracts of the Sydney Category Theory Seminar* 1972, 2nd Edition, 6-65 (Department of Mathematics, University of Sydney, Sydney; School of Mathematics and Physics, Macquarie University, North Ryde; 1972).

[5] F. William Lawvere, "Introduction", *Toposes, algebraic geometry and logic*, 1-12 (Lecture Notes in Mathematics, 274. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

[6] S. Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics, 5. Springer-Verlag, New York, Heidelberg, Berlin, 1971).

Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales.

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