ON LATTICE PATHS WITH DIAGONAL STEPS

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In [1] L. Moser and W. Zayachkowski considered lattice paths from (0, 0) to (x, y) where the possible moves are of three types: (1) a horizontal step, (2) a vertical step, and (3) a diagonal step. They obtained an expression for the number of paths from (0, 0) to (n, n) lying below the main diagonal except at the terminal points. In this note we extend their results to cover any point (m, n) lying below the main diagonal. We use the notation of [1] and recall that

$$f(m,n) = \text{total number of paths from } (0,0) \text{ to } (m,n)$$

$$= \sum_{r=0}^{\min(m,n)} \frac{(m+n-r)!}{(m-r)!\,(n-r)!\,r!},$$

and

Q_n = number of paths from (0, 0) to (n, n) lying below the main diagonal except at end points

$$= \sum_{r=0}^{n-2} \frac{1}{n-r-1} \frac{(2n-r-2)!}{(n-r)!\,(n-r-2)!\,r!} + 1 .$$

Our method consists of counting all paths from (0, 0) to (m, n) (with $m > n$) which touch or cross the main diagonal. Any path from (0, 0) to (m, n) that touches or crosses the main diagonal must do so for the first time in any one of the following mutually exclusive ways:

(1) the path may touch or cross the main diagonal for the first time vertically (and hence from below) at the point (i, i), $i = 1, \ldots, n$, or

(2) the path may touch or cross the main diagonal for the first time horizontally (and hence from above) at the point (i, i), $i = 1, \ldots, n$, or

(3) the path may touch the main diagonal for the first time

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diagonally in which case it must start with a diagonal step at (0, 0).

It is now easy to see that the total number of paths in the first two categories is

$$2f(m-1, n-1) + 2 \sum_{i=2}^{n} Q_i f(m-i, n-i)$$

and that in the third is $\sum_{i=1}^{n} f(m-i, n-i)$.

Thus the number of paths lying below the main diagonal is given by

$$(1) \qquad Q_{m,n} = f(m,n) - 2f(m-1, n-1) - 2 \sum_{i=2}^{n} Q_i f(m-i, n-i) - \sum_{i=1}^{n} f(m-i, n-i) .$$

The number of paths which may touch but not cross the diagonal, $Q'_{m,n}$ is obtained by replacing m by $m+1$ in (1). That is,

$$(2) \qquad Q'_{m,n} = f(m+1, n) - 2f(m, n-1) - 2 \sum_{i=2}^{n} Q_i f(m+1-i, n-i) - \sum_{i=1}^{n} f(m+1-i, n-i) .$$

Finally, if $n > m$, we may ask the following question: What is the number of paths from $(0,0)$ to (m,n) that cross the main diagonal vertically? The answer obviously is

$$f(m,n) - Q'_{n,m} .$$

REFERENCE

1. L. Moser and W. Zayachkowski, Lattice paths with diagonal steps, Scripta Mathematica, Vol. XXVI, No. 3, pp. 223-229.

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