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Multiple Lattice Tilings in Euclidean Spaces

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Dedicated to Professor Dr. Christian Buchta on the occasion of his 60th birthday

Abstract. In 1885, Fedorov discovered that a convex domain can form a lattice tiling of the Euclidean plane if and only if it is a parallelogram or a centrally symmetric hexagon. This paper proves the following results. Except for parallelograms and centrally symmetric hexagons, there are no other convex domains that can form two-, three- or four-fold lattice tilings in the Euclidean plane. However, there are both octagons and decagons that can form five-fold lattice tilings. Whenever $n \geqslant 3$, there are non-parallelohedral polytopes that can form five-fold lattice tilings in the n-dimensional Euclidean space.

1 Introduction

Planar tiling is an ancient subject in our civilization. It has been considered in the arts by craftsmen since antiquity. It is still an active research field in mathematics and some basic problems remain unsolved. In 1885, Fedorov [11] discovered that there are only two types of two-dimensional lattice tiles: parallelograms and centrally symmetric hexagons. In 1917, Bieberbach suggested that Reinhardt determine all the two-dimensional convex congruent tiles (see [25]). However, to complete the list turns out to be challenging and dramatic. Over the years, the list has been successively extended by Reinhardt, Kershner, James, Rice, Stein, Mann, McLoud-Mann, and Von Derau (see [32, 21]); its completeness has been mistakenly announced several times. In 2017, M. Rao [24] announced a completeness proof based on computer checks.

The three-dimensional case was also studied in ancient times. More than 2,300 years ago, Aristotle claimed that both identical regular tetrahedra and identical cubes can fill the whole space without gap. His cube case is obvious. However, his tetrahedron case is wrong; such a tiling is impossible [20].

Let K be a convex body with (relative) interior $\mathrm{int}(K)$, (relative) boundary $\partial(K)$ and volume $\mathrm{vol}(K)$, and let X be a discrete set, both in $\mathbb{E}^n$. We call $K + X$ a *translative tiling* of $\mathbb{E}^n$ and K a *translative tile* if $K + X = \mathbb{E}^n$ and the translates $\mathrm{int}(K) + \mathbf{x}_i$, and $\mathbf{x}_i \in X$, are pairwise disjoint, *i.e.*, if $K + X$ is both a packing and a covering in $\mathbb{E}^n$. In particular, we call $K + \Lambda$ a *lattice tiling* of $\mathbb{E}^n$ and K a *lattice tile* if Λ is an n-dimensional lattice. Apparently, a translative tile must be a convex polytope. Usually, a lattice tile is called a *parallelohedron*.

Fedorov [11] also characterized the three-dimensional lattice tiles: A three-dimensional lattice tile must be a parallelotope, a hexagonal prism, a rhombic

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dodecahedron, an elongated dodecahedron, or a truncated octahedron. The situations in higher dimensions turn out to be very complicated. Through the works of Delone [7], Stogrin [27], and Engel [10], we know that there are exactly 52 combinatorially different types of parallelohedra in $\mathbb{E}^4$. A computer classification for the five-dimensional parallelohedra was announced by Dutour Sikirić, Garber, Schürmann, and Waldmann [9] only in 2015.

Let Λ be an n-dimensional lattice. The *Dirichlet–Voronoi cell* of Λ is defined by $C = \{\mathbf{x} : \mathbf{x} \in \mathbb{E}^n, \|\mathbf{x}, \mathbf{o}\| \leq \|\mathbf{x}, \Lambda\|\}$, where $\|X, Y\|$ denotes the Euclidean distance between X and Y. Clearly, $C + \Lambda$ is a lattice tiling and the Dirichlet–Voronoi cell C is a parallelohedron. In 1908, Voronoi [29] conjectured that every parallelohedron is a linear transformation image of the Dirichlet–Voronoi cell of a suitable lattice. In $\mathbb{E}^2$, $\mathbb{E}^3$, and $\mathbb{E}^4$, this conjecture was confirmed by Delone [7] in 1929. In higher dimensions, it is still open.

To characterize the translative tiles is another fascinating problem. In 1897, Minkowski [23] showed that every translative tile must be centrally symmetric. In 1954, Venkov [28] proved that every translative tile must be a lattice tile (see [1] for generalizations). Later, McMullen [22] independently discovered a new proof for this beautiful result.

Let X be a discrete multiset in $\mathbb{E}^n$ and let k be a positive integer. We call $K + X$ a *k-fold translative tiling* of $\mathbb{E}^n$ and K a *k-fold translative tile* if every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at least k translates of K in $K + X$ and every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at most k translates of $\mathrm{int}(K)$ in $\mathrm{int}(K) + X$. In other words, $K + X$ is both a k-fold packing and a k-fold covering in $\mathbb{E}^n$. In particular, we call $K + \Lambda$ a k-fold lattice tiling of $\mathbb{E}^n$ and K a *k-fold lattice tile* if Λ is an n-dimensional lattice. Apparently, a k-fold translative tile must be a convex polytope. In fact, as shown by Gravin, Robins, and Shiryaev [14], a k-fold translative tile must be a centrally symmetric polytope with centrally symmetric facets.

Multiple tilings were first investigated by Furtwängler [13] in 1936 as a generalization of Minkowski's conjecture on cube tilings. Let C denote the n-dimensional unit cube. Furtwängler conjectured that every k-fold lattice tiling $C + \Lambda$ has twin cubes, *i.e.*, every multiple lattice tiling $C + \Lambda$ has two cubes sharing a whole facet. In the same paper, he proved the two- and three-dimensional cases. Unfortunately, when $n \geq 4$, Hajós [18] disproved this beautiful conjecture in 1941. In 1979, Robinson [26] determined all the integer pairs $\{n, k\}$ for which Furtwängler's conjecture is false. We refer to Zong [30, 31] for an introduction and a detailed account, respectively, of this fascinating problem, and to pp. 82–84 of Gruber and Lekkerkerker [17] for some generalizations.

In 1994, Bolle [5] proved that every centrally symmetric lattice polygon is a multiple lattice tile. Let Λ denote the two-dimensional integer lattice and let D_8 denote the octagon with vertices $(1,0)$, $(2,0)$, $(3,1)$, $(3,2)$, $(2,3)$, $(1,3)$, $(0,2)$ and $(0,1)$. As a particular example of Bolle's theorem, Gravin, Robins, and Shiryaev [14] discovered that $D_8 + \Lambda$ is a seven-fold lattice tiling of $\mathbb{E}^2$. Apparently, the octagon D_8 is not a lattice tile. Based on this example and McMullen's criterion on parallelohedra (see Lemma 4 in Section 3), one can easily deduce that whenever $n \geq 2$, there is a non-parallelohedral polytope that can form a seven-fold lattice tiling in $\mathbb{E}^n$.

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In 2000, Kolountzakis [19] proved that if D is a two-dimensional convex domain that is not a parallelogram and $D + X$ is a multiple tiling in $\mathbb{E}^2$, then X must be a finite union of translated two-dimensional lattices. In 2013, Gravin, Kolountzakis, Robins, and Shiryaev [15] discovered a similar result in $\mathbb{E}^3$.

Let P denote an n-dimensional centrally symmetric convex polytope, let $\tau(P)$ denote the smallest integer k such that P is a k-fold translative tile, and let $\tau^*(P)$ denote the smallest integer k such that P is a k-fold lattice tile. For convenience, we define $\tau(P) = \infty$ (or $\tau^*(P) = \infty$) if P cannot form a translative tiling (or a lattice tiling) of any multiplicity. Clearly, for every centrally symmetric convex polytope we have $\tau(P) \le \tau^*(P)$.

It is a basic and natural problem to study the distribution of the integers $\tau(P)$ or $\tau^*(P)$ when P runs over all n-dimensional polytopes. In particular, is there an n-dimensional polytope P satisfying $\tau(P) = 2$ or 3? Is there an n-dimensional polytope P satisfying $\tau(P) \ne \tau^*(P)$? Is there a convex domain D satisfying $2 \le \tau^*(D) \le 6$?

In this paper, we will prove the following results.

Theorem 1 *If D is a two-dimensional centrally symmetric convex domain that is neither a parallelogram nor a centrally symmetric hexagon, then we have $\tau^*(D) \ge 5$, where the equality holds when D is a suitable octagon or a suitable decagon.*

Theorem 2 *Whenever $n \ge 3$, there are n-dimensional convex polytopes P such that* $2 \le \tau^*(P) \le 5$.

2 Proof of Theorem 1

In 1994, Bolle [5] proved the following criterion for the two-dimensional multiple lattice tilings.

Lemma 1 *A convex polygon is a k-fold lattice tile for a lattice Λ and some positive integer k if and only if the following conditions are satisfied.*

(i) *It is centrally symmetric.*
(ii) *When it is centered at the origin, in the relative interior of each edge G there is a point of $\frac{1}{2}\Lambda$.*
(iii) *If the midpoint of G is not in $\frac{1}{2}\Lambda$, then G is a lattice vector of Λ.*

Let D denote a two-dimensional centrally symmetric compact convex domain, let $\delta^*(D)$ denote the density of the densest lattice packings of D, and let $\theta^*(D)$ denote the density of the thinnest lattice coverings of D. Then the two-dimensional case of Fedorov's discovery implies the following statement.

Lemma 2 *For every two-dimensional centrally symmetric compact convex domain D we have $\delta^*(D) \le 1$, where the equality holds if and only if D is a parallelogram or a centrally symmetric hexagon.*

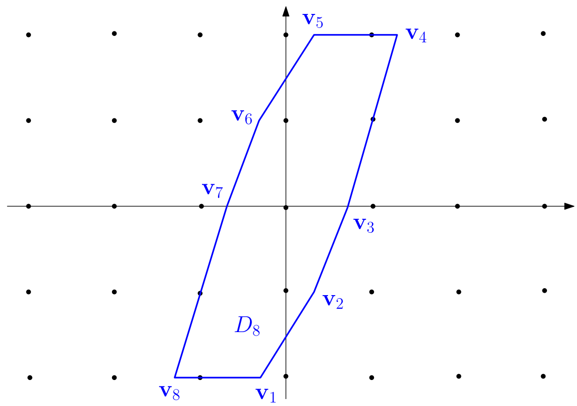

Figure 1: An octagon five-fold lattice tile

Let $\delta_k^*(D)$ denote the density of the densest k-fold lattice packings of D and let $\theta_k^*(D)$ denote the density of the thinnest k-fold lattice coverings of D (see [32]). Clearly, we have $\delta_k^*(D) \geq k \cdot \delta^*(D)$ and $\theta_k^*(D) \leq k \cdot \theta^*(D)$.

For smaller k, Dumir and Hans-Gill [8] and G. Fejes Tóth [12] proved the following explicit result. In fact, Dumir and Hans-Gill proved the $k = 2$ case, and G. Fejes Tóth proved the $k = 3$ and $k = 4$ cases.

Lemma 3 *If $k = 2$, 3, or 4, then $\delta_k^*(D) = k \cdot \delta^*(D)$ holds for every two-dimensional centrally symmetric convex domain D.*

Proof of Theorem 1 Let k be a positive integer satisfying $k \leq 4$. If D is a two-dimensional centrally symmetric convex domain that can form a k-fold lattice tiling in the Euclidean plane, then we have $\delta_k^*(D) = k$. By Lemma 3 it follows that $\delta^*(D) = \frac{\delta_k^*(D)}{k} = 1$. Then it follows by Lemma 2 that D must be a parallelogram or a centrally symmetric hexagon. In other words, if D is neither a parallelogram nor a centrally symmetric hexagon, then we have

$$\tau^*(D) \geq 5. \tag{1}$$

We take $\Lambda = \mathbb{Z}^2$. Let D_8 denote the octagon with vertices

$$\begin{aligned} \mathbf{v}_1 &= \left(-\tfrac{3}{10}, -2\right), & \mathbf{v}_2 &= \left(\tfrac{3}{10}, -1\right), \\ \mathbf{v}_3 &= \left(\tfrac{7}{10}, 0\right), & \mathbf{v}_4 &= \left(\tfrac{13}{10}, 2\right), \\ \mathbf{v}_5 &= \left(\tfrac{3}{10}, 2\right), & \mathbf{v}_6 &= \left(-\tfrac{3}{10}, 1\right), \\ \mathbf{v}_7 &= \left(-\tfrac{7}{10}, 0\right), & \mathbf{v}_8 &= \left(-\tfrac{13}{10}, -2\right), \end{aligned}$$

as shown in Figure 1. It can be easily verified that

$$\mathbf{u}_i = \tfrac{1}{2}(\mathbf{v}_i + \mathbf{v}_{i+1}) \in \tfrac{1}{2}\Lambda, \quad i = 1, 2, \ldots, 8,$$

where $\mathbf{v}_9 = \mathbf{v}_1$, and $\mathrm{vol}(D_8) = 5$. It follows from Lemma 1 that $D_8 + \Lambda$ is a five-fold lattice tiling. Combined with (1), it can be deduced that $\tau^*(D_8) = 5$.

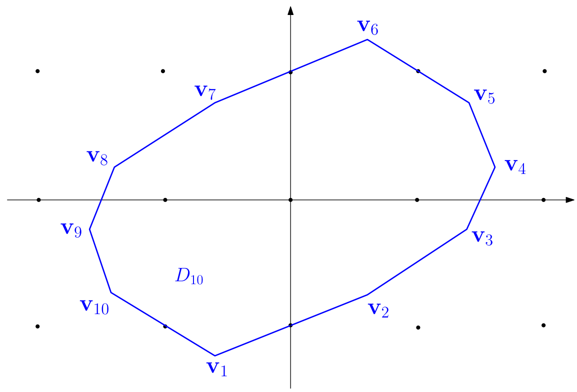

Figure 2: A decagon five-fold lattice tile

Similarly, let D_{10} denote the decagon with vertices

$$\begin{aligned} \mathbf{v}_1 &= \left(-\tfrac{3}{5}, -\tfrac{5}{4}\right), & \mathbf{v}_2 &= \left(\tfrac{3}{5}, -\tfrac{3}{4}\right), \\ \mathbf{v}_3 &= \left(\tfrac{7}{5}, -\tfrac{1}{4}\right), & \mathbf{v}_4 &= \left(\tfrac{8}{5}, \tfrac{1}{4}\right), \\ \mathbf{v}_5 &= \left(\tfrac{7}{5}, \tfrac{3}{4}\right), & \mathbf{v}_6 &= \left(\tfrac{3}{5}, \tfrac{5}{4}\right), \\ \mathbf{v}_7 &= \left(-\tfrac{3}{5}, \tfrac{3}{4}\right), & \mathbf{v}_8 &= \left(-\tfrac{7}{5}, \tfrac{1}{4}\right), \\ \mathbf{v}_9 &= \left(-\tfrac{8}{5}, -\tfrac{1}{4}\right), & \mathbf{v}_{10} &= \left(-\tfrac{7}{5}, -\tfrac{3}{4}\right), \end{aligned}$$

as shown in Figure 2. It can be easily verified that

$$\mathbf{u}_i = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_{i+1}) \in \frac{1}{2}\Lambda, \quad i = 1, 2, \dots, 10,$$

where $\mathbf{v}_{11} = \mathbf{v}_1$, and $\mathrm{vol}(D_{10}) = 5$. It follows from Lemma 1 that $D_{10} + \Lambda$ is a five-fold lattice tiling. Combined with (1), it can be deduced that $\tau^*(D_{10}) = 5$. ∎

3 Comparisons and Generalizations

It is interesting to make comparisons with multiple packings and multiple coverings (see [32] for a detailed survey). Let O denote the unit circular disk. Blunden [2,3] discovered that $\delta_k^*(O) = k \cdot \delta^*(O)$ is no longer true when $k \geq 5$, and $\theta_k^*(O) = k \cdot \theta^*(O)$ is no longer true when $k \geq 3$. So the packing case is rather similar to tilings, while the covering case is much different.

On the other hand, for every two-dimensional convex domain D, Cohn [6], Bolle [4], and Groemer [16] proved that

$$\lim_{k\to\infty} \frac{\delta_k^*(D)}{k} = \lim_{k\to\infty} \frac{\theta_k^*(D)}{k} = 1.$$

In other words, from the density point of view, when the multiplicity is big, there is not much difference among packing, covering, and tiling.

Let P denote an n-dimensional centrally symmetric convex polytope with centrally symmetric facets and let V denote an $(n-2)$-dimensional face of P. We call the collection of all those facets of P that contain a translate of V as a subface a *belt* of P.

In 1980, P. McMullen [22] proved the following criterion for parallelohedra.

Lemma 4 *A convex body K is a parallelohedron if and only if it is a centrally symmetric polytope with centrally symmetric facets and each belt contains four or six facets.*

Proof of Theorem 2 For convenience, we write $\mathbb{E}^n = \mathbb{E}^2 \times \mathbb{E}^{n-2}$. Let P_{2m} be a centrally symmetric $2m$-gon ($m \geq 4$) such that $P_{2m} + \mathbb{Z}^2$ is a k-fold lattice tiling of $\mathbb{E}^2$, let I^{n-2} denote the unit cube $\{(x_3, x_4, \dots, x_n) : |x_i| \leq \frac{1}{2}\}$ in $\mathbb{E}^{n-2}$, and define $P = P_{2m} \times I^{n-2}$. It is easy to see that $P + \mathbb{Z}^n$ is a k-fold lattice tiling of $\mathbb{E}^n$.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{2m}$ be the $2m$ vertices of P_{2m}, let $G_1, G_2, \dots, G_{2m}$ denote the $2m$ edges of P_{2m}, and define $V = \mathbf{v}_1 \times I^{n-2}$ and $F_i = G_i \times I^{n-2}$. Clearly, $\{F_1, F_2, \dots, F_{2m}\}$ is a belt of P with $2m$ facets. Therefore, by McMullen's criterion it follows that P is not a parallelohedron in $\mathbb{E}^n$. In particular, the octagon D_8 and the decagon D_{10} defined in the proof of Theorem 1 produce non-parallelohedral five-fold lattice tiles $D_8 \times I^{n-2}$ and $D_{10} \times I^{n-2}$, respectively. Thus, we have both

$$2 \leq \tau^*(D_8 \times I^{n-2}) \leq 5 \quad \text{and} \quad 2 \leq \tau^*(D_{10} \times I^{n-2}) \leq 5. \qquad \blacksquare$$

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